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# Nonlinear analysis in $p$ -vector spaces for single-valued 1-set contractive mappings

George Xianzhi Yuan<sup>1,2,3,4\*</sup>

\*Correspondence:

[george\\_yuan99@yahoo.com](mailto:george_yuan99@yahoo.com)

<sup>1</sup>Business School, Chengdu University, Chengdu 610601, China

<sup>2</sup>School of Mathematics, Sichuan University, Chengdu 610065, China  
Full list of author information is available at the end of the article

## Abstract

The goal of this paper is to develop some fundamental and important nonlinear analysis for single-valued mappings under the framework of  $p$ -vector spaces, in particular, for locally  $p$ -convex spaces for  $0 < p \leq 1$ . More precisely, based on the fixed point theorem of single-valued continuous condensing mappings in locally  $p$ -convex spaces as the starting point, we first establish best approximation results for (single-valued) continuous condensing mappings, which are then used to develop new results for three classes of nonlinear mappings consisting of 1) condensing; 2) 1-set contractive; and 3) semiclosed 1-set contractive mappings in locally  $p$ -convex spaces. Next they are used to establish the general principle for nonlinear alternative, Leray–Schauder alternative, fixed points for nonself mappings with different boundary conditions for nonlinear mappings from locally  $p$ -convex spaces, to nonexpansive mappings in uniformly convex Banach spaces, or locally convex spaces with the Opial condition. The results given by this paper not only include the corresponding ones in the existing literature as special cases, but are also expected to be useful tools for the development of new theory in nonlinear functional analysis and applications to the study of related nonlinear problems arising from practice under the general framework of  $p$ -vector spaces for  $0 < p \leq 1$ .

Finally, the work presented by this paper focuses on the development of nonlinear analysis for single-valued (instead of set-valued) mappings for locally  $p$ -convex spaces. Essentially, it is indeed the continuation of the associated work given recently by Yuan (*Fixed Point Theory Algorithms Sci. Eng.* 2022:20, 2022); therein, the attention is given to the study of nonlinear analysis for set-valued mappings in locally  $p$ -convex spaces for  $0 < p \leq 1$ .

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## 1 Introduction

It is known that the class of  $p$ -seminorm spaces ( $0 < p \leq 1$ ) is an important generalization of usual normed spaces with rich topological and geometrical structures, and related study has received a lot of attention, e.g., see work by Alghamdi et al. [4], Balachandran [6], Bay-

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oumi [7], Bayoumi et al. [8], Bernuées and Pena [10], Ding [29], Ennassik and Taoudi [32], Ennassik et al. [31], Gal and Goldstein [38], Gholizadeh et al. [39], Jarchow [52], Kalton [53, 54], Kalton et al. [55], Machrafi and Oubbi [72], Park [89], Qiu and Rolewicz [98], Rolewicz [102], Silva et al. [111], Simons [112], Tabor et al. [115], Tan [116], Wang [119], Xiao and Lu [122], Xiao and Zhu [123, 124], Yuan [134], and many others. However, to the best of our knowledge, the corresponding basic tools and associated results in the category of nonlinear functional analysis for  $p$ -vector spaces have not been well developed, in particular for the three classes of (single-valued) continuous nonlinear mappings, which are: 1) condensing; 2) 1-set contractive; and 3) semiclosed 1-set contractive operators under locally  $p$ -convex spaces. Our goal in this paper is to develop some fundamental and important nonlinear analysis for single-valued mappings under the framework of  $p$ -vector spaces, in particular, for locally  $p$ -convex spaces for  $0 < p \leq 1$ . More precisely, based on the fixed point theorem of single-valued continuous condensing mappings in locally  $p$ -convex spaces as the starting point, we first establish best approximation results for (single-valued) continuous condensing mappings, which are then used to develop new results for three classes of nonlinear mappings, which are 1): condensing; 2): 1-set contractive; and 3): semiclosed 1-set contractive in locally  $p$ -convex spaces. Then these new results are used to establish the general principle for nonlinear alternative, Leray–Schauder alternative, fixed points for nonself mappings with different boundary conditions for nonlinear mappings from locally  $p$ -convex spaces, to nonexpansive mappings in uniformly convex Banach spaces, or locally convex spaces with the Opial condition. The results given by this paper not only include the corresponding results in the existing literature as special cases, but are also expected to be useful tools for the development of new theory in nonlinear functional analysis and applications to the study of related nonlinear problems arising from practice under the general framework of  $p$ -vector spaces for  $0 < p \leq 1$ .

In addition, we would like to point out that the work presented by this paper focuses on the development of nonlinear analysis for single-valued (instead of set-valued) mappings for locally  $p$ -convex spaces; essentially, it is very important. It is also the continuation of the work given recently by Yuan [134]; therein, the attention was given to establishing new results on fixed points, the principle of nonlinear alternative for nonlinear mappings mainly on set-valued (instead of single-valued) mappings developed in locally  $p$ -convex spaces for  $0 < p \leq 1$ . Although some new results for set-valued mappings in locally  $p$ -convex spaces have been developed (see Gholizadeh et al. [39], Park [89], Qiu and Rolewicz [98], Xiao and Zhu [123, 124], Yuan [134], and others), we still would like to emphasize that results obtained for set-valued mappings for  $p$ -vector spaces may face some challenges in dealing with true nonlinear problems. One example is that the assumption used for “set-valued mappings with closed  $p$ -convex values” seems too strong, as it always means that the zero element is a trivial fixed point of the set-valued mappings, and this was also discussed in pp. 40–41 by Yuan [134] for  $0 < p \leq 1$ .

On the development since 1920s and, in particular, on the fixed points for nonself mappings, best approximation method, and on some key aspects of nonlinear analysis related to Birkhoff–Kellogg problems, nonlinear alternative, Leray–Schauder alternative, KKM principle, best approximation, and related topics, readers can find some most important contributions by Birkhoff and Kellogg [11] in 1920s, Leray and Schauder [65] in 1934, Fan [34] in 1969; plus the related comprehensive references given by Agarwal et al. [1], Bernstein [9], Chang et al. [22], Granas and Dugundji [46], Isac [51], Park [87], Singh et al. [113],

Zeidler [136]; and also see work contributed by Agarwal and O'Regan [2, 3], Furi and Pera [37], Park [87], O'Regan [80], O'Regan and Precup [82], Poincare [96], Rothe [103, 104], Yuan [132–134], Zeidler [136].

It is well known that the best approximation is one of very important aspects for the study of nonlinear problems related to the problems on their solvability for partial differential equations, dynamic systems, optimization, mathematical program, operation research; and in particular, it is the one approach well accepted for studying nonlinear problems in optimization, complementarity problems, variational inequality problems, and so on, strongly based on the so-called Fan's best approximation theorem given by Fan [33–36] in 1969, which acts as a very powerful tool in nonlinear analysis (see also the book of Singh et al. [113] for the related discussion and study on the fixed point theory and best approximation with the KKM-map principle). Among them, the related tools are Rothe type and the principle of Leray–Schauder alternative in topological vector spaces (TVS) and locally convex topological vector spaces (LCS), which are comprehensively studied by Chang et al. [22], Chang et al. [23–25], Carbone and Conti [18], Ennassik and Taoudi [32], Ennassik et al. [31], Isac [51], Granas and Dugundji [46], Kirk and Shahzad [58], Liu [70], Park [90], Rothe [103, 104], Shahzad [109, 110], Xu [126], Yuan [132–134], Zeidler [136], and the references therein.

On the other hand, the celebrated KKM principle established in 1929 in [60] was based on the celebrated Sperner combinatorial lemma and first applied to a simple proof of the Brouwer fixed point theorem. Later it became clear that these three theorems are mutually equivalent, and they were regarded as a sort of mathematical trinity (Park [90]). Since Fan extended the classical KKM theorem to infinite-dimensional spaces in 1961 [34–36], there have been a number of generalizations and applications in numerous areas of nonlinear analysis, and fixed points in TVS and LCS as developed by Browder [12–17] and related references therein. Among them, Schauder's fixed point theorem [106] in normed spaces is one of the powerful tools in dealing with nonlinear problems in analysis. Most notably, it has played a major role in the development of fixed point theory and related nonlinear analysis and mathematical theory of partial and differential equations and others. A generalization of Schauder's theorem from normed spaces to general topological vector spaces is an old conjecture in fixed point theory, which is explained by Problem 54 of the book "The Scottish Book" by Mauldin [74] stated as Schauder's conjecture: "Every nonempty compact convex set in a topological vector space has the fixed point property, or in its analytic statement, does a continuous function defined on a compact convex subset of a topological vector space to itself have a fixed point?" Recently, this question has been answered by the work of Ennassik and Taoudi [32] by using  $p$ -seminorm methods under locally  $p$ -convex spaces! See also the related work in this direction given by Askoura and Godet-Thobie [5], Cauty [19, 20], Chang [21], Chang et al. [22], Chen [27], Dobrowolski [30], Gholizadeh et al. [39], Górniewicz [44], Górniewicz et al. [45], Isac [51], Li [68], Li et al. [67], Liu [70], Nhu [76], Okon [78], Park [89–91], Reich [99], Smart [114], Weber [120, 121], Xiao and Lu [122], Xiao and Zhu [123, 124], Xu [129], Xu et al. [130], Yuan [132–134], and the related references therein under the general framework of  $p$ -vector spaces, in particular, locally  $p$ -convex spaces for nonself mappings with various boundary conditions for  $0 < p \leq 1$ .

The goal of this paper is to establish the general new tools of nonlinear analysis under the framework of general locally  $p$ -convex space ( $p$ -seminorm spaces) for general condensing

mappings, 1-set contractive mappings, and semiclosed mappings (here  $0 < p \leq 1$ ), and we do wish these new results such as best approximation, theorems of Birkhoff–Kellogg type, nonlinear alternative, fixed point theorems for nonself (single-valued) continuous operators with various boundary conditions, Rothe, Petryshyn type, Altman type, Leray–Schauder types, and other related nonlinear problems would play important roles for the nonlinear analysis of  $p$ -seminorm spaces for  $0 < p \leq 1$ . In addition, our results also show that fixed point theorem for condensing continuous mappings for closed  $p$ -convex subsets provides solutions for Schauder’s conjecture since 1930s in the affirmative way under the general setting of  $p$ -vector spaces (which may not be locally convex, see the related study given by Ennassik and Taoudi [32], Kalton [53, 54], Kalton et al. [55], Jarchow [52], Roloewicz [102] in this direction).

The paper has ten sections. Section 1 is the introduction. Section 2 describes general concepts for the  $p$ -convex subsets of topological vector spaces ( $0 < p \leq 1$ ). In Sect. 3, some basic results of the KKM principle related to abstract convex spaces are given. In Sect. 4, as the application of the KKM principle in abstract convex spaces, which includes  $p$ -convex vector spaces as a special class ( $0 < p \leq 1$ ), by combining the embedding lemma for compact  $p$ -convex subsets from topological vector spaces into locally  $p$ -convex spaces, we provide general fixed point theorems for condensing continuous mappings for both a single-valued version in topological vector spaces and an upper semicontinuous set-valued version in locally convex spaces defined on closed  $p$ -convex subsets for  $0 < p \leq 1$ . Sections 5, 6, and 7 mainly focus on the study of nonlinear analysis for 1-set contractive (single-valued) continuous mappings in locally  $p$ -convex vector spaces to establish general existence theorems for solutions of the Birkhoff–Kellogg (problem) alternative, the general principle of nonlinear alternative, including Leray–Schauder alternative, Rothe type, Altman type associated with different boundary conditions. Sections 8, 9, and 10 mainly focus on the study of new results based on semiclosed 1-set contractive (single-valued) continuous mappings related to nonlinear alternative principles, Birkhoff–Kellogg theorems, Leray–Schauder alternative, and nonself operations from general locally  $p$ -convex spaces to uniformly convex Banach spaces for nonexpansive mappings, or locally convex topological spaces with the Opial condition.

For the convenience of our discussion, throughout this paper, we always assume that all  $p$ -vector spaces are Hausdorff for  $0 < p \leq 1$  unless specified otherwise; and we also denote by  $\mathbb{N}$  the set of all positive integers, i.e.,  $\mathbb{N} := \{1, 2, \dots\}$ .

## 2 Some basic results for $p$ -vector spaces

For the convenience of self-reading, we recall some notions and definitions for  $p$ -convex vector spaces below as summarized by Yuan [134] (see also Balachandran [6], Bayoumi [7], Jarchow [52], Kalton [53], Rolewicz [102], Gholizadeh et al. [39], Ennassik and Taoudi [32], Ennassik et al. [31], Xiao and Lu [122], Xiao and Zhu [124], and the references therein).

**Definition 2.1** A set  $A$  in a vector space  $X$  is said to be  $p$ -convex for  $0 < p \leq 1$  if for any  $x, y \in A$ ,  $0 \leq s, t \leq 1$  with  $s^p + t^p = 1$ , we have  $s^{1/p}x + t^{1/p}y \in A$ ; and if  $A$  is 1-convex, it is simply called convex (for  $p = 1$ ) in general vector spaces; the set  $A$  is said to be absolutely  $p$ -convex if  $s^{1/p}x + t^{1/p}y \in A$  for  $0 \leq |s|, |t| \leq 1$  with  $|s|^p + |t|^p \leq 1$ .

**Definition 2.2** If  $A$  is a subset of a topological vector space  $X$ , the closure of  $A$  is denoted by  $\overline{A}$ , then the  $p$ -convex hull of  $A$  and its closed  $p$ -convex hull are denoted by  $C_p(A)$  and

$\overline{C}_p(A)$ , respectively, which are the smallest  $p$ -convex set containing  $A$  and the smallest closed  $p$ -convex set containing  $A$ , respectively.

**Definition 2.3** Let  $A$  be  $p$ -convex and  $x_1, \dots, x_n \in A$ , and  $t_i \geq 0$ ,  $\sum_1^n t_i^p = 1$ . Then  $\sum_1^n t_i x_i$  is called a  $p$ -convex combination of  $\{x_i\}$  for  $i = 1, 2, \dots, n$ . If  $\sum_1^n |t_i|^p \leq 1$ , then  $\sum_1^n t_i x_i$  is called an absolutely  $p$ -convex combination. It is easy to see that  $\sum_1^n t_i x_i \in A$  for a  $p$ -convex set  $A$ .

**Definition 2.4** A subset  $A$  of a vector space  $X$  is called circled (or balanced) if  $\lambda A \subset A$  holds for all scalars  $\lambda$  satisfying  $|\lambda| \leq 1$ . We say that  $A$  is absorbing if for each  $x \in X$ , there is a real number  $\rho_x > 0$  such that  $\lambda x \in A$  for all  $\lambda > 0$  with  $|\lambda| \leq \rho_x$ .

By Definition 2.4, it is easy to see that the system of all circled subsets of  $X$  is easily seen to be closed under the formation of linear combinations, arbitrary unions, and arbitrary intersections. In particular, every set  $A \subset X$  determines the smallest circled subset  $\hat{A}$  of  $X$  in which it is contained:  $\hat{A}$  is called the circled hull of  $A$ . It is clear that  $\hat{A} = \bigcup_{|\lambda| \leq 1} \lambda A$  holds, so that  $A$  is circled if and only if (in short, iff)  $\hat{A} = A$ . We use  $\overline{\hat{A}}$  to denote the closed circled hull of  $A \subset X$ .

In addition, if  $X$  is a topological vector space, we use the  $\text{int}(A)$  to denote the (relative topological) interior of set  $A \subset X$  and if  $0 \in \text{int}(A)$ , then  $\text{int}(A)$  is also circled, and we use  $\partial A$  to denote the (relative topological) boundary of  $A$  in  $X$  unless specified otherwise.

**Definition 2.5** A topological vector space is said to be locally  $p$ -convex if the origin has a fundamental set of absolutely  $p$ -convex 0-neighborhoods. This topology can be determined by  $p$ -seminorms which are defined in the obvious way (see p. 52 of Bayoumi [7], Jarchow [52], or Rolewicz [102]).

**Definition 2.6** Let  $X$  be a vector space and  $\mathbb{R}^+$  be a nonnegative part of a real line  $\mathbb{R}$ . Then a mapping  $P: X \rightarrow \mathbb{R}^+$  is said to be a  $p$ -seminorm if it satisfies the requirements for  $(0 < p \leq 1)$ :

- (i)  $P(x) \geq 0$  for all  $x \in X$ ;
- (ii)  $P(\lambda x) = |\lambda|^p P(x)$  for all  $x \in X$  and  $\lambda \in \mathbb{R}$ ;
- (iii)  $P(x + y) \leq P(x) + P(y)$  for all  $x, y \in X$ .

A  $p$ -seminorm  $P$  is called a  $p$ -norm if  $x = 0$  whenever  $P(x) = 0$ , so a vector space with a specific  $p$ -norm is called a  $p$ -normed space, and of course if  $p = 1$ ,  $X$  is a normed space as discussed before (e.g., see Jarchow [52]).

By Lemma 3.2.5 of Balachandran [6], the following proposition gives a necessary and sufficient condition for a  $p$ -seminorm to be continuous.

**Proposition 2.1** Let  $X$  be a topological vector space,  $P$  be a  $p$ -seminorm on  $X$ , and  $V := \{x \in X : P(x) < 1\}$ . Then  $P$  is continuous if and only if  $0 \in \text{int}(V)$ , where  $\text{int}(V)$  is the interior of  $V$ .

Now, given a  $p$ -seminorm  $P$ , the  $p$ -seminorm topology determined by  $P$  (in short, the  $p$ -topology) is the class of unions of open balls  $B(x, \epsilon) := \{y \in X : P(y - x) < \epsilon\}$  for  $x \in X$  and  $\epsilon > 0$ .

**Definition 2.7** A topological vector space  $X$  is said to be locally  $p$ -convex if it has a 0-basis consisting of  $p$ -convex neighborhoods for  $(0 < p \leq 1)$ . If  $p = 1$ , then  $X$  is a usual locally convex space.

We also need the following notion for the so-called  $p$ -gauge (see Balachandran [6]).

**Definition 2.8** Let  $A$  be an absorbing subset of a vector space  $X$ . For  $x \in X$  and  $0 < p \leq 1$ , set  $P_A = \inf\{\alpha > 0 : x \in \alpha^{\frac{1}{p}}A\}$ , then the nonnegative real-valued function  $P_A$  is called the  $p$ -gauge (gauge if  $p = 1$ ). The  $p$ -gauge of  $A$  is also known as the Minkowski  $p$ -functional.

By Proposition 4.1.10 of Balachandran [6], we have the following proposition.

**Proposition 2.2** *Let  $A$  be an absorbing subset of  $X$ . Then the  $p$ -gauge  $P_A$  has the following properties:*

- (i)  $P_A(0) = 0$ ;
- (ii)  $P_A(\lambda x) = |\lambda|^p P_A(x)$  if  $\lambda \geq 0$ ;
- (iii)  $P_A(\lambda x) = |\lambda|^p P_A(x)$  for all  $\lambda \in \mathbb{R}$  provided  $A$  is circled;
- (iv)  $P_A(x + y) \leq P_A(x) + P_A(y)$  for all  $x, y \in A$  provided  $A$  is  $p$ -convex.

*In particular,  $P_A$  is a  $p$ -seminorm if  $A$  is absolutely  $p$ -convex (and also absorbing).*

As mentioned above, a given  $p$ -seminorm is said to be a  $p$ -norm if  $x = 0$  whenever  $P(x) = 0$ . A vector space with a specific  $p$ -norm is called a  $p$ -normed space. The  $p$ -norm of an element  $x \in E$  will usually be denoted by  $\|x\|_p$ . If  $p = 1$ , then  $X$  is a usual normed space. If  $X$  is a  $p$ -normed space, then  $(X, d_p)$  is a metric linear space with a translation invariant metric  $d_p$  such that  $d_p = d_p(x, y) = \|x - y\|_p$  for  $x, y \in X$ . We point out that  $p$ -normed spaces are very important in the theory of topological vector spaces. Specifically, a Hausdorff topological vector space is locally bounded if and only if it is a  $p$ -normed space for some  $p$ -norm  $\|\cdot\|_p$ , where  $0 < p \leq 1$  (see p. 114 of Jarchow [52]). We also note that examples of  $p$ -normed spaces include  $L^p(\mu)$ -spaces and Hardy spaces  $H_p$ ,  $0 < p < 1$ , endowed with their usual  $p$ -norms.

**Remark 2.1** We would like to make the following two important points:

- (1) First, by the fact that (e.g., see Kalton et al. [55] or Ding [29]), there is no open convex nonvoid subset in  $L^p[0, 1]$  (for  $0 < p < 1$ ) except  $L^p[0, 1]$  itself. This means that  $p$ -normed spaces with  $0 < p < 1$  are not necessarily locally convex. Moreover, we know that every  $p$ -normed space is locally  $p$ -convex; and incorporating Lemma 2.3, it seems that a  $p$ -vector space (for  $0 < p \leq 1$ ) is a nicer space as we can use the  $p$ -vector space to approximate (Hausdorff) topological vector spaces (TVS) in terms of Lemma 2.1 (ii) for the convex subsets in TVS by using bigger  $p$ -convex subsets in  $p$ -vector spaces for  $p \in (0, 1)$  by also considering Lemma 2.3. In this way,  $p$ -vector spaces seem to have better properties in terms of  $p$ -convexity than the usually (1-)convex subsets used in TVS with  $p = 1$ .
- (2) Second, it is worthwhile noting that a 0-neighborhood in a topological vector space is always absorbing by Lemma 2.1.16 of Balachandran [6] or Proposition 2.2.3 of Jarchow [52].

Now, by Proposition 4.1.12 of Balachandran [6], we also have the following Proposition 2.3 and Remark 2.2 (which is Remark 2.3 of Ennassik and Taoudi [32]).



**Proposition 2.3** *Let  $A$  be a subset of a vector space  $X$ , which is absolutely  $p$ -convex ( $0 < p \leq 1$ ) and absorbing. Then, we have that*

- (i) *The  $p$ -gauge  $P_A$  is a  $p$ -seminorm such that if  $B_1 := \{x \in X : P_A(x) < 1\}$  and  $\overline{B_1} = \{x \in X : P_A(x) \leq 1\}$ , then  $B_1 \subset A \subset \overline{B_1}$ ; in particular,  $\ker P_A \subset A$ , where  $\ker P_A := \{x \in X : P_A(x) = 0\}$ .*
- (ii)  *$A = B_1$  or  $\overline{B_1}$  according to whether  $A$  is open or closed in the  $P_A$ -topology.*

**Remark 2.2** Let  $X$  be a topological vector space, and let  $U$  be an open absolutely  $p$ -convex neighborhood of the origin, and let  $\epsilon$  be given. If  $y \in \epsilon^{\frac{1}{p}} U$ , then  $y = \epsilon^{\frac{1}{p}} u$  for some  $u \in U$  and  $P_U(y) = P_U(\epsilon^{\frac{1}{p}} u) = \epsilon P_U(u) \leq \epsilon$  (as  $u \in U$  implies that  $P_U(u) \leq 1$ ). Thus,  $P_U$  is continuous at zero, and therefore,  $P_U$  is continuous everywhere. Moreover, we have  $U = \{x \in X : P_U(x) < 1\}$ .

Indeed, since  $U$  is open and the scalar multiplication is continuous, we have that, for any  $x \in U$ , there exists  $0 < t < 1$  such that  $x \in t^{\frac{1}{p}} U$ , and so  $P_U(x) \leq t < 1$ . This shows that  $U \subset \{x \in X : P_U(x) < 1\}$ . The conclusion follows by Proposition 2.3.

The following result is a very important and useful result, which allows us to make the approximation for convex subsets in topological vector spaces by  $p$ -convex subsets in  $p$ -convex vector spaces. For the readers self-contained in reading, we provide a sketch of proof below (see also Lemma 2.1 of Ennassik and Taoudi [31], Remark 2.1 of Qiu and Rolewicz [98]).

**Lemma 2.1** *Let  $A$  be a subset of a vector space  $X$ , then we have*

- (i) *If  $A$  is  $p$ -convex with  $0 < p < 1$ , then  $\alpha x \in A$  for any  $x \in A$  and any  $0 < \alpha \leq 1$ ;*
- (ii) *If  $A$  is convex and  $0 \in A$ , then  $A$  is  $p$ -convex for any  $p \in (0, 1]$ ;*
- (iii) *If  $A$  is  $p$ -convex for some  $p \in (0, 1)$ , then  $A$  is  $s$ -convex for any  $s \in (0, p]$ .*

**Proof** (i) As  $r \leq 1$ , the fact that “for all  $x \in A$  and all  $\alpha \in [2^{(n+1)(1-\frac{1}{p})}, 2^{n(1-\frac{1}{p})}]$ , we have  $\alpha x \in A$ ” is true for all integer  $n \geq 0$ . Taking into account the fact that  $(0, 1] = \bigcup_{n \geq 0} [2^{(n+1)(1-\frac{1}{p})}, 2^{n(1-\frac{1}{p})}]$ , we obtain the result.

(ii) Assume that  $A$  is a convex subset of  $X$  with  $0 \in A$  and take a real number  $s \in (0, 1]$ . We show that  $A$  is  $s$ -convex. Indeed, let  $x, y \in A$  and  $\alpha, \beta > 0$  with  $\alpha^s + \beta^s = 1$ . Since  $A$  is convex, then  $\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y \in A$ . Keeping in mind that  $0 < \alpha + \beta < \alpha^p + \beta^p = 1$ , it follows that  $\alpha x + \beta y = (\alpha + \beta)(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y) + (1 - \alpha - \beta)0 \in A$ .

(iii) Now, assume that  $A$  is  $r$ -convex for some  $p \in (0, 1)$  and pick up any real  $s \in (0, p]$ . We show that  $A$  is  $s$ -convex. To see this, let  $x, y \in A$  and  $\alpha, \beta > 0$  such that  $\alpha^s + \beta^s = 1$ . First notice that  $0 < \alpha^{\frac{p-s}{p}} \leq 1$  and  $0 < \beta^{\frac{p-s}{p}} \leq 1$ , which imply that  $\alpha^{\frac{p-s}{p}}x \in A$  and  $\beta^{\frac{p-s}{p}}y \in A$ . By the  $p$ -convexity of  $A$  and the equality  $(\alpha^{\frac{s}{p}})^p + (\beta^{\frac{s}{p}})^p = 1$ , it follows that  $\alpha x + \beta y = \alpha^{\frac{s}{p}}(\alpha^{\frac{p-s}{p}}x) + \beta^{\frac{s}{p}}(\beta^{\frac{p-s}{p}}y) \in A$ . This completes the sketch of the proof.  $\square$

**Remark 2.3** We would like to point out that results (i) and (iii) of Lemma 2.1 do not hold for  $p = 1$ . Indeed, any singleton  $\{x\} \subset X$  is convex in topological vector spaces; but if  $x \neq 0$ , then it is not  $p$ -convex for any  $p \in (0, 1)$ .

We also need the following proposition, which is Proposition 6.7.2 of Jarchow [52].

**Proposition 2.4** *Let  $K$  be compact in a topological vector  $X$  and  $(1 < p \leq 1)$ . Then the closure  $\overline{C}_p(K)$  of a  $p$ -convex hull and the closure  $\overline{AC}_p(K)$  of an absolutely  $p$ -convex hull of  $K$  are compact if and only if  $\overline{C}_p(K)$  and  $\overline{AC}_p(K)$  are complete, respectively.*

We also need the following fact, which is a special case of Lemma 2.4 of Xiao and Zhu [124].

**Lemma 2.2** *Let  $C$  be a bounded closed  $p$ -convex subset of  $p$ -seminorm  $X$  with  $0 \in \text{int } C$ , where  $(0 < p \leq 1)$ . For every  $x \in X$ , define an operator by  $r(x) := \frac{x}{\max\{1, (P_C(x))^{\frac{1}{p}}\}}$ , where  $P_C$  is the Minkowski  $p$ -functional of  $C$ . Then  $C$  is a retract of  $X$  and  $r : X \rightarrow C$  is continuous such that*

- (1) *if  $x \in C$ , then  $r(x) = x$ ;*
- (2) *if  $x \notin C$ , then  $r(x) \in \partial C$ ;*
- (3) *if  $x \notin C$ , then the Minkowski  $p$ -functional  $P_C(x) > 1$ .*

*Proof* Taking  $s = p$  in Lemma 2.4 of Xiao and Zhu [124], Proposition 2.3, and Remark 2.2, we complete the proof.  $\square$

**Remark 2.4** As discussed by Remark 2.2, Lemma 2.2 still holds if “the bounded closed  $p$ -convex subset  $C$  of the  $p$ -normed space  $(X, \|\cdot\|_p)$ ” is replaced by “ $X$  is a  $p$ -seminorm vector space and  $C$  is a bounded closed absorbing  $p$ -convex subset with  $0 \in \text{int } C$  of  $X$ ”.

Before we close this section, we would like to point out that the structure of  $p$ -convexity when  $p \in (0, 1)$  is really different from what we normally have for the concept of “convexity” used in topological vector spaces (TVS). In particular, maybe the following fact is one of the reasons for us to use better ( $p$ -convex) structures in  $p$ -vector spaces to approximate the corresponding structure of the convexity used in TVS (i.e., the  $p$ -vector space when  $p = 1$ ). Based on the discussion in p. 1740 of Xiao and Zhu [124] (see also Bernués and Pena [10] and Sezer et al. [107]), we have the following fact, which indicates that each  $p$ -convex subset is “bigger” than the convex subset in topological vector spaces for  $0 < p < 1$ .

**Lemma 2.3** *Let  $x$  be a point of  $p$ -vector space  $E$ , where assume  $0 < p < 1$ , then the  $p$ -convex hull and the closure of  $\{x\}$  are given by*

$$C_p(\{x\}) = \begin{cases} \{tx : t \in (0, 1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0; \end{cases} \quad (1)$$

and

$$\overline{C_p(\{x\})} = \begin{cases} \{tx : t \in [0, 1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases} \quad (2)$$

But note that if  $x$  is a given one point in  $p$ -vector space  $E$ , when  $p = 1$ , we have that  $\overline{C_1(\{x\})} = C_1(\{x\}) = \{x\}$ . This shows significantly different for the structure of  $p$ -convexity between  $p = 1$  and  $p \neq 1$ !



As an application of Lemma 2.3, we have the following fact for (set-valued) mappings with nonempty closed  $p$ -convex values in  $p$ -vector spaces for  $p \in (0, 1)$ , which are truly different from any (set-valued) mappings defined in topological vector spaces (i.e., for a  $p$ -vector space with  $p = 1$ ).

**Lemma 2.4** *Let  $U$  be a nonempty subset of a  $p$ -vector space  $E$  (where  $0 < p < 1$ ) with zero  $0 \in U$ , and assume that a (set-valued) mapping  $T : U \rightarrow 2^E$  is with nonempty closed  $p$ -convex values. Then  $T$  has at least one fixed point in  $U$ , which is the element zero, i.e.,  $0 \in \bigcap_{x \in U} T(x) \neq \emptyset$ .*

*Proof* For each  $x \in U$ , as  $T(x)$  is nonempty closed  $p$ -convex, by Lemma 2.3, we have at least  $0 \in T(x)$ . It implies that  $0 \in \bigcap_{x \in U} T(x)$ , and thus zero of  $E$  is a fixed point of  $T$ . This completes the proof.  $\square$

**Remark 2.5** We would like to point out that Lemma 2.4 shows that any set-valued mapping with closed  $p$ -convex values in  $p$ -spaces for  $0 < p < 1$  has the zero point as its trivial fixed point, thus it is very important to study the fixed point and related principle of nonlinear analysis for single-valued (instead of set-valued) mappings for  $p$ -vector spaces (for  $0 < p < 1$ ), as pointed out in the discussion given in pp. 40–41 by Yuan [134]. Thus the newest results established in this paper are for the three classes of (single-valued) continuous mappings, which are: 1) condensing; 2) 1-set contractive; and 3) semiclosed 1-set contractive mappings. This is a key difference from those results obtained by Yuan [134] recently for the study of set-valued mappings in  $p$ -vector spaces for  $0 < p \leq 1$ .

By following Definitions 2.5 and 2.6, the discussion given by Proposition 2.3, and remarks thereafter, each given (open)  $p$ -convex subset  $U$  in a  $p$ -vector space  $E$  with the zero  $0 \in \text{int}(U)$  always corresponds to a  $p$ -seminorm  $P_U$ , which is indeed the Minkowski  $p$ -functional of  $U$  in  $E$ , and  $P_U$  is continuous in  $E$ . In particular, a topological vector space is said to be locally  $p$ -convex if the origin  $0$  of  $E$  has a fundamental set (denoted by)  $\mathcal{U}$ , which is a family of absolutely  $p$ -convex  $0$ -neighborhoods (each denoted by  $U$ ). This topology can be determined by  $p$ -seminorm  $P_U$ , which is indeed the family  $\{P_U\}_{U \in \mathcal{U}}$ , where  $P_U$  is just the Minkowski  $p$ -functional for each  $U \in \mathcal{U}$  in  $E$  (see also p. 52 of Bayoumi [7], Jarchow [52], or Rolewicz [102]).

Throughout this paper, by following Remark 2.5, without loss of generality, unless specified otherwise, for a given  $p$ -vector space  $E$ , where  $p \in (0, 1]$ , we always denote by  $\mathcal{U}$  the base of the  $p$ -vector space  $E$ 's topology structure, which is the family of its  $0$ -neighborhoods. For each  $U \in \mathcal{U}$ , its corresponding  $P$ -seminorm  $P_U$  is the Minkowski  $p$ -functional of  $U$  in  $E$ . For a given point  $x \in E$  and a subset  $C \subset E$ , we denote by  $d_{P_U}(x, C) := \inf\{P_U(x - y) : y \in C\}$  the distance of  $x$  and  $C$  by the seminorm  $P_U$ , where  $P_U$  is the Minkowski  $p$ -functional for each  $U \in \mathcal{U}$  in  $E$ .

### 3 The KKM principle in convex vector spaces

Since Knaster, Kuratowski, and Mazurkiewicz (in short, KKM) [60] in 1929 obtained the so-called KKM principle (theorem) to give a new proof for the Brouwer fixed point theorem in finite dimensional spaces, and later in 1961, Fan [36] (see also Fan [35]) extended the KKM principle (theorem) to any topological vector spaces and applied it to various results including the Schauder fixed point theorem, there have appeared a large number

of works devoted to applications of the KKM principle (theorem). In 1992, such research field was called the KKM theory for the first time by Park [84]. Then the KKM theory has been extended to general abstract convex spaces by Park [88] (see also Park [89] and [90]), which actually include locally  $p$ -convex spaces ( $0 < p \leq 1$ ) as a special class. Same as in the last section, for the convenience of self-reading, we recall some notions and definitions for the KKM principle in convex vector spaces, which include  $p$ -vector spaces as a special class, as summarized by Yuan [134] below.

Here we first give some notions and definitions on the abstract convex spaces which play an important role in the development of the KKM principle and related applications. Once again, for the corresponding comprehensive discussion on KKM theory and its various applications to nonlinear analysis and related topics, we refer to Mauldin [74], Granas and Dugundji [46], Park [90] and [91], Yuan [133, 134], and related comprehensive references therein.

Let  $\langle D \rangle$  denote the set of all nonempty finite subsets of a given nonempty set  $D$ , and let  $2^D$  denote the family of all subsets of  $D$ . We have the following definition for abstract convex spaces essentially by Park [88].

**Definition 3.1** An abstract convex space  $(E, D; \Gamma)$  consists of a topological space  $E$ , a nonempty set  $D$ , and a set-valued mapping  $\Gamma : \langle D \rangle \rightarrow 2^E$  with nonempty values  $\Gamma_A := \Gamma(A)$  for each  $A \in \langle D \rangle$ , such that the  $\Gamma$ -convex hull of any  $D' \subset D$  is denoted and defined by  $\text{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E$ .

A subset  $X$  of  $E$  is said to be a  $\Gamma$ -convex subset of  $(E, D; \Gamma)$  relative to  $D'$  if for any  $N \in \langle D' \rangle$ , we have  $\Gamma_N \subseteq X$ , that is,  $\text{co}_\Gamma D' \subset X$ . For the convenience of our discussion, in the case  $E = D$ , the space  $(E, E; \Gamma)$  is simply denoted by  $(E; \Gamma)$  unless specified otherwise.

**Definition 3.2** Let  $(E, D; \Gamma)$  be an abstract convex space and  $Z$  be a topological space. For a set-valued mapping (or, say, multimap)  $F : E \rightarrow 2^Z$  with nonempty values, if a set-valued mapping  $G : D \rightarrow 2^Z$  satisfies  $F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y)$  for all  $A \in \langle D \rangle$ , then  $G$  is called a KKM mapping with respect to  $F$ . A KKM mapping  $G : D \rightarrow 2^E$  is a KKM mapping with respect to the identity map  $1_E$ .

**Definition 3.3** The partial KKM principle for an abstract convex space  $(E, D; \Gamma)$  is that, for any closed-valued KKM mapping  $G : D \rightarrow 2^E$ , the family  $\{G(y)\}_{y \in D}$  has the finite intersection property. The KKM principle is that the same property also holds for any open-valued KKM mapping.

An abstract convex space is called a (partial) KKM space if it satisfies the (partial) KKM principle (resp.). We now give some known examples of (partial) KKM spaces (see Park [88] and also [89]) as follows.

**Definition 3.4** A  $\phi_A$ -space  $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$  consists of a topological space  $X$ , a nonempty set  $D$ , and a family of continuous functions  $\phi_A : \Delta_n \rightarrow 2^X$  (that is, singular  $n$ -simplices) for  $A \in \langle D \rangle$  with  $|A| = n + 1$ . By putting  $\Gamma_A := \phi_A(\Delta_n)$  for each  $A \in \langle D \rangle$ , the triple  $(X, D; \Gamma)$  becomes an abstract convex space.

*Remark 3.1* For a  $\phi_A$ -space  $(X, D; \{\phi_A\})$ , we see easily that any set-valued mapping  $G : D \rightarrow 2^X$  satisfying  $\phi_A(\Delta_J) \subset G(J)$  for each  $A \in \langle D \rangle$  and  $J \in \langle A \rangle$  is a KKM mapping.

By the definition, it is clear that every  $\phi_A$ -space is a KKM space, thus we have the following fact (see Lemma 1 of Park [89]).

**Lemma 3.1** *Let  $(X, D; \Gamma)$  be a  $\phi_A$ -space and  $G : D \rightarrow 2^X$  be a set-valued (multimap) with nonempty closed [resp. open] values. Suppose that  $G$  is a KKM mapping, then  $\{G(a)\}_{a \in D}$  has the finite intersection property.*

By following Definition 2.7, we recall that a topological vector space is said to be locally  $p$ -convex if the origin has a fundamental set of absolutely  $p$ -convex 0-neighborhoods. This topology can be determined by  $p$ -seminorms which are defined in the obvious way (see Jarchow [52] or p. 52 of Bayoumi [7]).

Now we have a new KKM space as follows inducted by the concept of  $p$ -convexity (see Lemma 2 of Park [89]).

**Lemma 3.2** *Suppose that  $X$  is a subset of topological vector space  $E$  and  $p \in (0, 1]$ , and  $D$  is a nonempty subset of  $X$  such that  $C_p(D) \subset X$ . Let  $\Gamma_N := C_p(N)$  for each  $N \in \langle D \rangle$ . Then  $(X, D; \Gamma)$  is a  $\phi_A$ -space.*

*Proof* Since  $C_p(D) \subset X$ ,  $\Gamma_N$  is well defined. For each  $N = \{x_0, x_1, \dots, x_n\} \subset D$ , we define  $\phi_N : \Delta_n \rightarrow \Gamma_N$  by  $\sum_{i=0}^n t_i e_i \mapsto \sum_{i=0}^n (t_i)^{\frac{1}{p}} x_i$ . Then, clearly,  $(X, D; \Gamma)$  is a  $\phi_A$ -space. This completes the proof.  $\square$

#### 4 Fixed point theorems for condensing mappings in locally $p$ -convex vector spaces

In this section, we establish fixed point theorems for upper semicontinuous, single-valued continuous condensing mappings for  $p$ -convex subsets under the general framework of  $p$ -vector spaces, which will be a tool used in Sects. 5 and 6 to establish the best approximation, fixed points, the principle of nonlinear alternative, Birkhoff–Kellogg problems, Leray–Schauder alternative, which would be useful tools in nonlinear analysis for the study of nonlinear problems arising from theory to practice. Here, we first gather together necessary definitions, notations, and known facts needed in this section.

**Definition 4.1** Let  $X$  and  $Y$  be two topological spaces. A set-valued mapping (also saying, multifunction)  $T : X \rightarrow 2^Y$  is a point to set function such that, for each  $x \in X$ ,  $T(x)$  is a subset of  $Y$ . The mapping  $T$  is said to be upper semicontinuous (USC) if the subset  $T^{-1}(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$  (resp., the set  $\{x \in X : T(x) \subset B\}$ ) is closed (resp., open) for any closed (resp., open) subset  $B$  in  $Y$ . The function  $T : X \rightarrow 2^Y$  is said to be lower semicontinuous (LSC) if the set  $T^{-1}(A)$  is open for any open subset  $A$  in  $Y$ .

As an application of the KKM principle for general abstract convex spaces with the help of embedding lemma for Hausdorff compact  $p$ -convex subsets from topological vector spaces (TVS) into locally  $p$ -convex vector spaces, we have the following general existence result for the “approximation” of fixed points for upper and lower semicontinuous set-valued mappings in  $p$ -convex vector spaces for  $0 < p \leq 1$  (see the corresponding related results given by Theorem 2.7 of Gholizadeh et al. [39], Theorem 5 of Park [89], and related discussion therein).

The following result was originally given by Yuan [134]; here we provide the sketch of its proof for the purpose of self-contained reading.

**Theorem 4.1** *Let  $A$  be a  $p$ -convex compact subset of a locally  $p$ -convex vector space  $X$ , where  $0 < p \leq 1$ . Suppose that  $T : A \rightarrow 2^A$  is lower (resp. upper) semicontinuous with nonempty  $p$ -convex values. Then, for any given  $U$  which is a  $p$ -convex neighborhood of zero in  $X$ , there exists  $x_U \in A$  such that  $T(x_U) \cap (x_U + U) \neq \emptyset$ .*

*Proof* Suppose that  $U$  is any given element of  $\mathcal{U}$ , there is a symmetric open neighborhood  $V$  of zero for which  $\overline{V} + \overline{V} \subset U$  in locally  $p$ -convex neighborhood of zero. We prove the results by two cases:  $T$  is lower semicontinuous (LSC) and upper semicontinuous (USC).

Case 1, by assuming  $T$  is lower semicontinuous: As  $X$  is a locally  $p$ -convex vector space, suppose that  $\mathcal{U}$  is the family of neighborhoods of 0 in  $X$ . For any element  $U$  of  $\mathcal{U}$ , there is a symmetric open neighborhood  $V$  of zero for which  $\overline{V} + \overline{V} \subset U$ . Since  $A$  is compact, so there exist  $x_0, x_1, \dots, x_n$  in  $A$  such that  $A \subset \bigcup_{i=0}^n (x_i + V)$ . By using the fact that  $A$  is  $p$ -convex, we find  $D := \{b_0, b_2, \dots, b_n\} \subset A$  for which  $b_i - x_i \in V$  for all  $i \in \{0, 1, \dots, n\}$ , and we define  $C$  by  $C := C_p(D) \subset A$ . By the fact that  $T$  is LSC, it follows that the subset  $F(b_i) := \{c \in C : T(c) \cap (x_i + V) = \emptyset\}$  is closed in  $C$  (as the set  $x_i + V$  is open) for each  $i \in \{0, 1, \dots, n\}$ . For any  $c \in C$ , we have  $\emptyset \neq T(c) \cap A \subset T(c) \cap \bigcup_{i=0}^n (x_i + V)$ , it follows that  $\bigcap_{i=0}^n F(b_i) = \emptyset$ . Now, applying Lemma 3.1 and Lemma 3.2 implies that there is  $N := \{b_{i_0}, b_{i_1}, \dots, b_{i_k}\} \in \langle D \rangle$  and  $x_U \in C_p(N) \subset A$  for which  $x_U \notin F(N)$ , and so  $T(x_U) \cap (x_{i_j} + V) \neq \emptyset$  for all  $j \in \{0, 1, \dots, k\}$ . As  $b_i - x_i \in V$  and  $\overline{V} + \overline{V} \subset U$ , which imply that  $x_{i_j} + \overline{V} \subset b_{i_j} + U$ , which means that  $T(x_U) \cap (b_{i_j} + U) \neq \emptyset$ , it follows that  $N \subset \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$ . By the fact that the subsets  $C$ ,  $T(x_U)$  and  $U$  are  $p$ -convex, we have that  $x_U \in \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$ , which means that  $T(x_U) \cap (x_U + U) \neq \emptyset$ .

Case 2, by assuming  $T$  is upper semicontinuous: We define  $F(b_i) := \{c \in C : T(c) \cap (x_i + \overline{V}) = \emptyset\}$ , which is then open in  $C$  (as the subset  $x_i + \overline{V}$  is closed) for each  $i = 0, 1, \dots, n$ . Then the argument is similar to the proof for the case  $T$  is USC, and by applying Lemma 3.1 and Lemma 3.2 again, it follows that there exists  $x_U \in A$  such that  $T(x_U) \cap (x_U + U) \neq \emptyset$ . This completes the proof.  $\square$

By Theorem 4.1, we have the following Fan–Glicksberg fixed point theorems (Fan [33]) in locally  $p$ -convex vector spaces for ( $0 < p \leq 1$ ), which also improve or generalize the corresponding results given by Yuan [133], Xiao and Lu [122], Xiao and Zhu [123, 124] into locally  $p$ -convex vector spaces.

**Theorem 4.2** *Let  $A$  be a  $p$ -convex compact subset of a locally  $p$ -convex vector space  $X$ , where  $0 < p \leq 1$ . Suppose that  $T : A \rightarrow 2^A$  is upper semicontinuous with nonempty  $p$ -convex closed values. Then  $T$  has at least one fixed point.*

*Proof* Assume that  $\mathcal{U}$  is the family of open  $p$ -convex neighborhoods of 0 in  $X$ , and  $U \in \mathcal{U}$ , by Theorem 4.1, there exists  $x_U \in A$  such that  $T(x_U) \cap (x_U + U) \neq \emptyset$ . Then there exist  $a_U, b_U \in A$  for which  $b_U \in T(a_U)$  and  $b_U \in a_U + U$ . Now, two nets  $\{a_U\}$  and  $\{b_U\}$  in  $\text{Graph}(T)$ , which is a compact graph of mapping  $T$  as  $A$  is compact and  $T$  is semicontinuous, we may assume that  $a_U$  has a subnet converging to  $a$ , and  $\{b_U\}$  has a subnet converging to  $b$ . As  $\mathcal{U}$  is the family of neighborhoods for 0, we should have  $a = b$  (e.g., by the Hausdorff separation property) and  $a = b \in T(b)$  due to the fact that  $\text{Graph}(T)$  is close (e.g., see Lemma 3.1.1 in p. 40 of Yuan [132]), thus the proof is complete.  $\square$

For a given set  $A$  in vector space  $X$ , we denote by “ $\text{lin}(A)$ ” the “linear hull” of  $A$  in  $X$ .

**Definition 4.2** Let  $A$  be a subset of a topological vector space  $X$ , and let  $Y$  be another topological vector space. We shall say that  $A$  can be linearly embedded in  $Y$  if there is a linear map  $L : \text{lin}(A) \rightarrow Y$  (not necessarily continuous) whose restriction to  $A$  is a homeomorphism.

The following embedded Lemma 4.1 is a significant result due to Theorem 1 of Kalton [53], which says that although not every compact convex set can be linearly embedded in a locally convex space (e.g., see Kalton et al. [55], and Roberts [100]), but when  $0 < p < 1$ , each compact  $p$ -convex set in topological vector spaces can be considered as a subset of a locally  $p$ -convex vector space, hence every such set has sufficiently many  $p$ -extreme points.

Secondly, by property (ii) of Lemma 2.1, each convex subset of a topological vector space containing zero is always  $p$ -convex for  $0 < p \leq 1$ . Thus it is possible for us to transfer the problem involving  $p$ -convex subsets from topological vector spaces into the locally  $p$ -convex vector spaces, which indeed allows us to establish the existence of fixed points for upper semicontinuous set-valued mappings for compact  $p$ -convex subsets in locally convex spaces for  $0 < p \leq 1$ . But we note that by Lemma 2.4 any set-valued mapping with closed  $p$ -convex values in  $p$ -spaces for  $0 < p < 1$  has the zero point as its trivial fixed point, thus it is essential to study the fixed point and related principle of nonlinear analysis for single-valued (instead of set-valued) mappings in  $p$ -vector spaces as pointed out by Remark 2.5 (see also the discussion in pp. 40–41 given by Yuan [134]).

Indeed, a fixed point theorem for a topological vector space for (single-valued) continuous and condensing mappings given by Theorem 4.5, which will be proved below (also see Theorem 4.3 essentially due to Ennassik and Taoudi [32]), provides the answer for Schauder's conjecture in the affirmative.

**Lemma 4.1** *Let  $K$  be a compact  $p$ -convex subset ( $0 < p < 1$ ) of a topological vector space  $X$ . Then  $K$  can be linearly embedded in a locally  $p$ -convex topological vector space.*

*Proof* It is Theorem 1 of Kalton [53], which completes the proof.  $\square$

**Remark 4.1** At this point, it is important to note that Lemma 4.1 does not hold for  $p = 1$ . By Theorem 9.6 of Kalton et al. [55], it was shown that the spaces  $L_p = L_p(0, 1)$ , where  $0 < p < 1$ , contain compact convex sets with no extreme points, which thus cannot be linearly embedded in a locally convex space, see also Roberts [100].

Now we give the following fixed point theorem for single-valued continuity mappings, which are essentially Theorem 3.1 and Theorem 3.3 given first by Ennassik and Taoudi [32]. Here we include the argument for the second part of the conclusions below only.

**Theorem 4.3** *If  $K$  is a nonempty compact  $p$ -convex subset of a locally  $p$ -convex space  $E$  for  $0 < p \leq 1$ , then the (single-valued) continuous mapping  $T : K \rightarrow K$  has at least a fixed point. Secondly, if  $K$  is a nonempty compact  $p$ -convex subset of a Hausdorff topological vector space  $E$ , then the (single-valued) continuous mapping  $T : K \rightarrow K$  has at least a fixed point.*

*Proof* The first part is Theorem 3.1 of Ennassik and Taoudi [32], and the second part is indeed Theorem 3.3 of Ennassik and Taoudi [32], but here we include their very smart proof as follows.

Case 1: For  $0 < p < 1$ ,  $K$  is a nonempty compact  $p$ -convex subset of a topological vector space  $X$  for  $0 < p < 1$ . By Lemma 4.1, it follows that  $K$  can be linearly embedded in a locally  $p$ -convex space  $E$ , which means that there exists a linear map  $L : \text{lin}(K) \rightarrow E$  whose restriction to  $K$  is a homeomorphism. Define the mapping  $S : L(K) \rightarrow L(K)$  by  $S(x) := L(Tx)$  for  $x \in X$ . This mapping is easily checked to be well defined. The mapping  $S$  is continuous since  $L$  is a (continuous) homeomorphism and  $T$  is continuous on  $K$ . Furthermore, the set  $L(K)$  is compact, being the image of a compact set under a continuous mapping  $L$ . It is also  $p$ -convex since it is the image of a  $p$ -convex set under a linear mapping. Then, by the conclusion in the first part (see also Theorem 3.1 in [32]), there exists  $x \in K$  such that  $Lx = S(Lx) = L(Tx)$ , thus it implies that  $x = T(x)$  since  $L$  is a homeomorphism, which is the fixed point of  $T$ .

Case 2: For  $p = 1$ , taking any point  $x_0 \in K$ , let  $K_0 := K - \{x_0\}$ . Now define a new mapping  $T_0 : K_0 \rightarrow K_0$  by  $T_0(x) = T(x) - x_0$  for each  $x \in K_0$ . By the fact that now  $K_0$  is  $p$ -convex for any  $0 < p < 1$  by Lemma 2.1(ii), the  $T_0$  has a fixed point in  $K_0$  by the proof in Case 1, so  $T$  has a fixed point in  $K$ . The proof is complete.  $\square$

**Remark 4.2** Theorem 4.3 is indeed the result of Theorem 3.1 and Theorem 3.3 (of Ennassik and Taoudi [32]) for  $0 < p \leq 1$  which provides an answer to Schauder's conjecture under the TVS. Here we also mention a number of related works and discussion by authors in this direction, see Mauldin [74], Granas and Dugundji [46], Park [90, 91], and the references therein.

We recall that for two given topological spaces  $X$  and  $Y$ , a set-valued mapping  $T : X \rightarrow 2^Y$  is said to be compact if there is a compact subset set  $C$  in  $Y$  such that  $F(X) (= \{y \in F(X), x \in X\})$  is contained in  $C$ , i.e.,  $F(X) \subset C$ . Now we have the following noncompact version of fixed point theorems for compact set-valued mappings defined on a general  $p$ -convex subset in  $p$ -vector spaces for  $0 < p \leq 1$ .

As an immediate consequence of Theorem 4.2 for  $p = 1$ , we have following result for an upper semicontinuous version in locally convex spaces (LCS).

**Theorem 4.4** *If  $K$  is a nonempty compact convex subset of a locally convex space  $X$ , then any upper semicontinuous set-valued mapping  $T : K \rightarrow 2^K$  with nonempty closed convex values has at least a fixed point.*

*Proof* Apply Theorem 4.2 with  $p = 1$ , this completes the proof.  $\square$

Theorem 4.4 also improves or unifies corresponding results given by Askoura and Godet-Thobie [5], Cauty [19], Cauty [20], Chen [27], Isac [51], Li [68], Nhu [76], Okon [78], Park [91], Reich [99], Smart [114], Yuan [133], Theorem 3.14 of Gholizadeh et al. [39], Xiao and Lu [122], Xiao and Zhu [123, 124] under the framework of LCS for set-valued mappings instead of single-valued functions.

In order to establish fixed point theorems for the classes of 1-set contractive and condensing mappings in  $p$ -vector spaces by using the concept of the measure of noncompactness (or the noncompactness measures), which was introduced and widely accepted in mathematical community by Kuratowski [63], Darbo [28], and related references therein, we first need to have a brief introduction for the concept of noncompactness measures for the so-called Kuratowski or Hausdorff measures of noncompactness in normed spaces



(see Alghamdi et al. [4], Machrafi and Oubbi [72], Nussbaum [77], Sadovskii [105], Silva et al. [111], Xiao and Lu [122] for the general concepts under the framework of  $p$ -seminorm or just for locally convex  $p$ -convex settings for  $0 < p \leq 1$ , which will be discussed below, too).

For a given metric space  $(X, d)$  (or a  $p$ -normed space  $(X, \|\cdot\|_p)$ ), we recall the notions of completeness, boundedness, relative compactness, and compactness as follows. Let  $(X, d)$  and  $(Y, d)$  be two metric spaces and  $T : X \rightarrow Y$  be a mapping (or operator). Then: 1)  $T$  is said to be bounded if for each bounded set  $A \subset X$ ,  $T(A)$  is bounded set of  $Y$ ; 2)  $T$  is said to be continuous if for every  $x \in X$ , the  $\lim_{n \rightarrow \infty} x_n = x$  implies that  $\lim_{n \rightarrow \infty} T(x_n) = T(x)$ ; and 3)  $T$  is said to be completely continuous if  $T$  is continuous and  $T(A)$  is relatively compact for each bounded subset  $A$  of  $X$ .

Let  $A_1, A_2 \subset X$  be bounded of a metric space  $(X, d)$ , we also recall that the Hausdorff metric  $d_H(A_1, A_2)$  between  $A_1$  and  $A_2$  is defined by

$$d_H(A_1, A_2) := \max \left\{ \sup_{x \in A_1} \inf_{y \in A_2} d(x, y), \sup_{y \in A_2} \inf_{x \in A_1} d(x, y) \right\}.$$

The Hausdorff and Kuratowski measures of noncompactness (denoted by  $\beta_H$  and  $\beta_K$ , respectively) for a nonempty bounded subset  $D$  in  $X$  are the nonnegative real numbers  $\beta_H(D)$  and  $\beta_K(D)$  defined by

$$\beta_H(D) := \inf \{ \epsilon > 0 : D \text{ has a finite } \epsilon\text{-net} \},$$

and

$$\beta_K(D) := \inf \left\{ \epsilon > 0 : D \subset \bigcup_{i=1}^n D_i, \text{ where } D_i \text{ is bounded and } \text{diam } D_i \leq \epsilon, \right. \\ \left. n \text{ is an integer} \right\}.$$

Here  $\text{diam } D_i$  means the diameter of the set  $D_i$ , and it is well known that  $\beta_H \leq \beta_K \leq 2\beta_H$ . We also point out that the notions above can be well defined under the framework of  $p$ -seminorm spaces  $(E, \|\cdot\|_p)_{p \in \mathbb{R}}$  by following the similar idea and method used by Chen and Singh [26], Ko and Tasi [61], Kozlov et al. [62] (see the references therein for more details).

Let  $T$  be a mapping from  $D \subset X$  to  $X$ . Then we have that: 1)  $T$  is said to be a  $k$ -set contraction with respect to  $\beta_K$  (or  $\beta_H$ ) if there is a number  $k \in (0, 1]$  such that  $\beta_K(T(A)) \leq k\beta_K(A)$  (or  $\beta_H(T(A)) \leq k\beta_H(A)$ ) for all bounded sets  $A$  in  $D$ ; and 2)  $T$  is said to be  $\beta_K$ -condensing (or  $\beta_H$ -condensing) if  $(\beta_K(T(A)) < \beta_K(A))$  (or  $\beta_H(T(A)) < \beta_H(A)$ ) for all bounded sets  $A$  in  $D$  with  $\beta_K(A) > 0$  (or  $\beta_H(A) > 0$ ).

For the convenience of our discussion, throughout the rest of this paper, if a mapping “is  $\beta_K$ -condensing (or  $\beta_H$ -condensing)”, we simply say it is “a condensing mapping” unless specified otherwise.

Moreover, it is easy to see that: 1) if  $T$  is a compact operator, then  $T$  is a  $k$ -set contraction; and 2) if  $T$  is a  $k$ -set contraction for  $k \in (0, 1)$ , then  $T$  is condensing.

In order to establish the fixed points of set-valued condensing mappings in  $p$ -vector spaces for  $0 < p \leq 1$ , we need to recall some notions introduced by Machrafi and Oubbi

[72] for the measure of noncompactness in locally  $p$ -convex vector spaces, which also satisfies some necessary (common) properties of the classical measures of noncompactness such as  $\beta_K$  and  $\beta_H$  mentioned above introduced by Kuratowski [63], Sadovskii [105] (see also related discussion by Alghamdi et al. [4], Nussbaum [77], Silva et al. [111], Xiao and Lu [122], and the references therein). In particular, the measures of noncompactness in locally  $p$ -vector spaces (for  $0 < p \leq 1$ ) should have the stable property, which means the measure of noncompactness  $A$  is the same by transition to the (closure) for the  $p$ -convex hull of subset  $A$ .

For the convenience of discussion, we follow up to use  $\alpha$  and  $\beta$  to denote the Kuratowski and the Hausdorff measures of noncompactness in topological vector spaces, respectively (see the same way used by Machrafi and Oubbi [72]), unless otherwise stated. The  $E$  is used to denote a Hausdorff topological vector space over the field  $\mathbb{K} \in \{\mathbb{R}, \mathbb{Q}\}$ , here  $\mathbb{R}$  denotes all real numbers and  $\mathbb{Q}$  all complex numbers, and  $p \in (0, 1]$ . Here, the base set of family of all balanced zero neighborhoods in  $E$  is denoted by  $\mathfrak{V}_0$ .

We recall that  $U \in \mathfrak{V}_0$  is said to be shrinkable if it is absorbing, balanced, and  $rU \subset U$  for all  $r \in (0, 1)$ , and we know that any topological vector space admits a local base at zero consisting of shrinkable sets (see Klee [59] or Jarchow [52] for details).

Recall again that a topological vector space  $E$  is said to be a locally  $p$ -convex space if  $E$  has a local base at zero consisting of  $p$ -convex sets. The topology of a locally  $p$ -convex space is always given by an upward directed family  $P$  of  $p$ -seminorms, where a  $p$ -seminorm on  $E$  is any nonnegative real-valued and subadditive functional  $\|\cdot\|_p$  on  $E$  such that  $\|\lambda x\|_p = |\lambda|^p \|x\|_p$  for each  $x \in E$  and  $\lambda \in \mathbb{R}$  (i.e., the real number line). When  $E$  is Hausdorff, then for every  $x \neq 0$ , there is some  $p \in P$  such that  $P(x) \neq 0$ . Whenever the family  $P$  is reduced to a singleton, one says that  $(E, \|\cdot\|)$  is a  $p$ -seminormed space. A  $p$ -normed space is a Hausdorff  $p$ -seminormed space, and when  $p = 1$ , it is the usual locally convex case. Furthermore, a  $p$ -normed space is a metric vector space with the translation invariant metric  $d_p(x, y) := \|x - y\|_p$  for all  $x, y \in E$ , which is the same notation used above.

By Remark 2.5, if  $P$  is a continuous  $p$ -seminorm on  $E$ , then the ball  $B_p(0, s) := \{x \in E : P(x) < s\}$  is shrinkable for each  $s > 0$ . Indeed, if  $r \in (0, 1)$  and  $x \in \overline{rB_p(0, s)}$ , then there exists a net  $(x_i)_{i \in I} \subset B_p(0, s)$  such that  $rx_i$  converges to  $x$ . By the continuity of  $P$ , we get  $P(x) \leq r^p s < s$ , which means that  $\overline{rB_p(0, s)} \subset B_p(0, s)$ . In general, it can be shown that every  $p$ -convex  $U \in \mathfrak{V}_0$  is shrinkable.

We recall that given such neighborhood  $U$ , a subset  $A \subset E$  is said to be  $U$ -small if  $A - A \subset U$  (or small of order  $U$  by Robertson [101]). Now, by following the idea of Kaniok [56] in the setting of a topological vector space  $E$ , we use zero neighborhoods in  $E$  instead of seminorms to define the measure of noncompactness in (local convex)  $p$ -vector spaces ( $0 < p \leq 1$ ) as follows: For each  $A \subset E$ , the  $U$ -measures of noncompactness  $\alpha_U(A)$  and  $\beta_U(A)$  for  $A$  are defined by:

$$\alpha_U(A) := \inf\{r > 0 : A \text{ is covered by a finite number of } rU\text{-small sets } A_i \\ \text{for } i = 1, 2, \dots, n\}$$

and

$$\beta_U(A) := \inf\left\{r > 0 : \text{there exists } x_1, \dots, x_n \in E \text{ such that } A \subset \bigcup_{i=1}^n (x_i + rU)\right\},$$

here we set  $\inf \emptyset := \infty$ .

By the definition above, it is clear that when  $E$  is a normed space and  $U$  is the closed unit ball of  $E$ ,  $\alpha_U$  and  $\beta_U$  are nothing else but the Kuratowski measure  $\beta_K$  and Hausdorff measure  $\beta_H$  of noncompactness, respectively. Thus, if  $\mathfrak{U}$  denotes a fundamental system of balanced and closed zero neighborhoods in  $E$  and  $\mathfrak{F}_{\mathfrak{U}}$  is the space of all functions  $\phi : \mathfrak{U} \rightarrow \mathbb{R}$ , endowed with the pointwise ordering, then the  $\alpha_U$  (resp.,  $\beta_U$ ) measures for noncompactness introduced by Kaniok [56] can be expressed by the Kuratowski (resp., the Hausdorff) measure of noncompact  $\alpha(A)$  (resp.,  $\beta(A)$ ) for a subset  $A$  of  $E$  as the function defined from  $\mathfrak{U}$  into  $[0, \infty)$  by

$$\alpha(A)(U) := \alpha_U(A) \quad (\text{resp., } \beta(A)(U) := \beta_U(A)).$$

By following Machrafi and Oubbi [72], in order to define the measure of noncompactness in (locally convex)  $p$ -vector space  $E$ , we need the following notions of basic and sufficient collections for zero neighborhoods in a topological vector space. To do this, let us introduce an equivalence relation on  $V_0$  by saying that  $U$  is related to  $V$ , written  $U \mathfrak{R} V$ , if and only if there exist  $r, s > 0$  such that  $rU \subset V \subset sU$ . We now have the following definition.

**Definition 4.3** (BCZN) We say that  $\mathfrak{B} \subset \mathfrak{V}_0$  is a basic collection of zero neighborhoods (in short, BCZN) if it contains at most one representative member from each equivalence class with respect to  $\mathfrak{R}$ . It will be said to be sufficient (in short, SCZN) if it is basic and, for every  $V \in \mathfrak{V}_0$ , there exist some  $U \in \mathfrak{B}$  and some  $r > 0$  such that  $rU \subset V$ .

*Remark 4.3* By Remark 2.5, it follows that for a locally  $p$ -convex space  $E$ , its base set  $\mathfrak{U}$ , the family of all open  $p$ -convex subsets for 0 is BCZB. We also note that: 1) In the case  $E$  is a normed space, if  $f$  is a continuous functional on  $E$ ,  $U := \{x \in E : |f(x)| < 1\}$ , and  $V$  is the open unit ball of  $E$ , then  $\{U\}$  is basic but not sufficient, but  $\{V\}$  is sufficient; 2) Secondly, if  $(E, \tau)$  is a locally convex space whose topology is given by an upward directed family  $P$  of seminorms, so that no two of them are equivalent, the collection  $(B_p)_{p \in P}$  is an SCZN, where  $B_p$  is the open unit ball of  $p$ . Further, if  $\mathfrak{W}$  is a fundamental system of zero neighborhoods in a topological vector space  $E$ , then there exists an SCZN consisting of  $\mathfrak{W}$  members; and 3) By following Oubbi [83], we recall that a subset  $A$  of  $E$  is called uniformly bounded with respect to a sufficient collection  $\mathfrak{B}$  of zero neighborhoods if there exists  $r > 0$  such that  $A \subset rV$  for all  $V \in \mathfrak{B}$ . Note that in the locally convex space  $C_c(X) := C_c(X, \mathbb{K})$ , the set  $B_\infty := \{f \in C(X) : \|f\|_\infty \leq 1\}$  is uniformly bounded with respect to the SCZN  $\{B_k, k \in \mathbb{K}\}$ , where  $B_k$  is the (closed or) open unit ball of the seminorm  $P_k$ , where  $k \in \mathbb{K}$ .

Now we are ready to give the definition for the measure of noncompactness in a (locally  $p$ -convex) topological vector space  $E$  as follows.

**Definition 4.4** Let  $\mathfrak{B}$  be an SCZN in  $E$ . For each  $A \subset E$ , we define the measure of noncompactness of  $A$  with respect to  $\mathfrak{B}$  by  $\alpha_{\mathfrak{B}}(A) := \sup_{U \in \mathfrak{B}} \alpha_U(A)$ .

By the definition above, it is clear that: 1) The measure of noncompactness  $\mathfrak{B}$  holding the semi-additivity, i.e.,  $\alpha_{\mathfrak{B}}(A \cup B) = \max\{\alpha_{\mathfrak{B}}(A), \alpha_{\mathfrak{B}}(B)\}$ ; and 2)  $\alpha_{\mathfrak{B}}(A) = 0$  if and only if

$A$  is a precompact subset of  $E$  (for more properties in detail, see Proposition 1 and the related discussion by Machraf and Oubbi [83]).

As we know, under the normed spaces (and even seminormed spaces), Kuratowski [63], Darbo [28], and Sadovskii [105] introduced the notions of  $k$ -set-contractions for  $k \in (0, 1)$  and the condensing mappings to establish fixed point theorems in the setting of Banach spaces, normed, or seminormed spaces. By following the same idea, if  $E$  is a Hausdorff locally  $p$ -convex space, we have the following definition for general (nonlinear) mappings.

**Definition 4.5** A mapping  $T : C \rightarrow 2^C$  is said to be a  $k$ -set contraction (resp., condensing) if there is some SCZN  $\mathfrak{B}$  in  $E$  consisting of  $p$ -convex sets, such that (resp., condensing) for any  $U \in \mathfrak{B}$ , there exists  $k \in (0, 1)$  (resp., condensing) such that  $\alpha_U(T(A)) \leq k\alpha_U(A)$  for  $A \subset C$  (resp.,  $\alpha_U(T(A)) < \alpha_U(A)$  for each  $A \subset C$  with  $\alpha_U(A) > 0$ ).

It is clear that a contraction mapping on  $C$  is a  $k$ -set contraction mapping (where we always mean  $k \in (0, 1)$ ), and a  $k$ -set contraction mapping on  $C$  is condensing; and they all reduce to the usual cases by the definitions for  $\beta_K$  and  $\beta_H$  which are the Kuratowski measure and the Hausdorff measure of noncompactness, respectively, in normed spaces (see Kuratowski [63]).

From now on, we denote by  $\mathfrak{V}_0$  the set of all shrinkable zero neighborhoods in  $E$ , we have the following result, which is Theorem 1 of Machrafi and Oubbi [72], saying that in the general setting of locally  $p$ -convex spaces, the measure of noncompactness  $\alpha$  for  $U$  given by Definition 4.4 is stable from  $U$  to its  $p$ -convex hull  $C_p(A)$  of the subset  $A$  in  $E$ , which is key for us to establish the fixed points for condensing mappings in locally  $p$ -convex spaces for  $0 < p \leq 1$ . This also means that it is the key property for the measures due to the Kuratowski and Hausdorff measures of noncompactness in normed (or  $p$ -seminorm) spaces, which also holds for the measure of noncompactness by Definition 4.4 in the setting of locally  $p$ -convex spaces with  $(0 < p \leq 1)$  (see more similar and related discussion in detail by Alghamdi et al. [4] and Silva et al. [111]).

**Lemma 4.2** *If  $U \in \mathfrak{V}_0$  is  $p$ -convex for some  $0 < p \leq 1$ , then  $\alpha(C_p(A)) = \alpha(A)$  for every  $A \subset E$ .*

*Proof* It is Theorem 1 of Machrafi and Oubbi [72]. The proof is complete.  $\square$

Now, based on the definition for the measure of noncompactness given by Definition 4.4 (originally from Machrafi and Oubbi [72]), we have the following general extension version of Schauder, Darbo, and Sadovskii type fixed point theorems in the context of locally  $p$ -convex vector spaces for condensing mappings.

**Theorem 4.5** (Schauder fixed point theorem for single-valued condensing mappings) *Let  $C \subset E$  be a complete  $p$ -convex subset of a Hausdorff locally  $p$ -convex or Hausdorff topological vector space  $E$  with  $0 < p \leq 1$ . If  $T : C \rightarrow C$  is continuous and  $(\alpha)$  condensing, then  $T$  has a fixed point in  $C$  and the set of fixed points of  $T$  is compact.*

*Proof* We first prove the conclusion by assuming  $E$  is a locally  $p$ -convex space, then we prove the conclusion when  $E$  is a topological vector space.

Case A: Assuming  $E$  is locally  $p$ -convex. In this case, let  $\mathfrak{B}$  be a sufficient collection of  $p$ -convex zero neighborhoods in  $E$  with respect to which  $T$  is condensing and for any given

$U \in \mathfrak{B}$ . We choose some  $x_0 \in C$  and let  $\mathfrak{F}$  be the family of all closed  $p$ -convex subsets  $A$  of  $C$  with  $x_0 \in A$  and  $T(A) \subset A$ . Note that  $\mathfrak{F}$  is not empty since  $C \in \mathfrak{F}$ . Let  $A_0 = \bigcap_{A \in \mathfrak{F}} A$ . Then  $A_0$  is a nonempty closed  $p$ -convex subset of  $C$  such that  $T(A_0) \subset A_0$ , and then the conclusion follows by Theorem 4.3 for the continuous mapping  $T$  from  $A_0$  to  $A_0$  to show that  $A_0$  is compact. Now we prove  $A_0$  is compact. Indeed, let  $A_1 = \overline{C_p(T(A_0) \cup \{x_0\})}$ . Since  $T(A_0) \subset A_0$  and  $A_0$  is closed and  $p$ -convex,  $A_1 \subset A_0$ . Hence,  $T(A_1) \subset T(A_0) \subset A_1$ . It follows that  $A_1 \in \mathfrak{F}$  and therefore  $A_1 = A_0$ . Now, by Proposition 1 of Machrafi and Oubbi [72] and Lemma 4.2 above (i.e., Theorem 1 and Theorem 2 in [72]), we get  $\alpha_U(T(A_0)) = \alpha_U(A)$ . Our assumption on  $T$  shows that  $\alpha_U(A_0) = 0$  since  $T$  is condensing. As  $U$  is arbitrary from the family  $\mathfrak{B}$ , thus  $A_0$  is  $p$ -convex and compact (see Proposition 4 in [72]). Now, the conclusion follows by Theorem 4.3. Secondly, let  $C_0$  be the set of fixed points of  $T$  in  $C$ . Then it follows that  $C_0 \subset T(C_0)$ , and the upper semicontinuity of  $T$  implies that its graph is closed, so is the set  $C_0$ . As  $T$  is condensing, we have  $\alpha_U(T(C_0)) \leq \alpha_U(C_0)$ , which implies that  $\alpha_U(C_0) = 0$ . As  $U$  is arbitrary from the family  $\mathfrak{B}$ , it implies that  $C_0$  is compact (by Proposition 4 in [72] again).

Case B: We now prove the conclusion by assuming  $E$  is a topological vector space. Based on the argument in Case A's proof above, when  $T$  is condensing, there exists a nonempty compact  $p$ -convex subset  $A_0$  such that  $T : A_0 \rightarrow A_0$ . We prove the conclusion by considering two situations: (1)  $0 < p < 1$  and (2)  $p = 1$ .

Now, for case (1)  $0 < p < 1$ : By the proof above,  $A_0$  is a nonempty compact  $p$ -convex subset of a topological vector space  $E$ . By Lemma 4.1, it follows that  $A_0$  can be linearly embedded in a locally  $p$ -convex space  $X$ , which means that there exists a linear mapping  $L : \text{lin}(A_0) \rightarrow X$  whose restriction to  $A_0$  is a homeomorphism. Define the mapping  $S : L(A_0) \rightarrow L(A_0)$  by  $S(x) := L(Tx)$  for  $x \in A_0$ . This mapping is easily checked to be well defined. The mapping  $S$  is continuous (and condensing) since  $L$  is a (continuous) homeomorphism and  $T$  is continuous (and condensing) on  $A_0$ . Furthermore, the set  $L(A_0)$  is compact, being the image of a compact set under a continuous mapping  $L$ . It is also  $p$ -convex as it is the image of a  $p$ -convex set under a linear mapping. Then, by the conclusion in the first part above for  $S$  on  $A_0$ , there exists  $x \in A_0$  such that  $Lx = S(Lx) = L(Tx)$ , thus it implies that  $x = T(x)$  since  $L$  is a homeomorphism, which means  $x$  is the fixed point of  $T$ .

Now, for case (2)  $p = 1$ : take any point  $x_0 \in A_0$ , and let  $K_0 := A_0 - \{x_0\}$ . Now define a new mapping  $T_0 : K_0 \rightarrow K_0$  by  $T_0(x) = T(x) - x_0$  for each  $x \in A_0$ . By the fact that now  $K_0$  is  $p$ -convex for any  $0 < p < 1$  by Lemma 2.1(ii), the  $T_0$  has a fixed point in  $K_0$  by the proof above for case (1) when  $0 < p < 1$ , so  $T_0$  has a fixed point in  $K_0$  implies that  $T$  has a fixed point in  $A_0$ .

This completes the proof.  $\square$

**Remark 4.4** We first note that Theorem 4.5 improves Theorem 4.5 of Yuan [134]. Secondly, as pointed out by Remark 2.2 (for Theorem 3.1 and Theorem 3.3 given by Ennassik and Taoudi [32]), Theorem 4.5 above provides an answer to Schauder's conjecture in the affirmative way under the general framework of closed  $p$ -convex subsets in topological vector spaces for  $0 < p \leq 1$  of (single-valued) continuous condensing mappings. Here we also mention a number of related works and discussion by authors in this direction, see Mauldin [74], Granas and Dugundji [46], Park [90, 91], and the references therein.

Following the argument used by Theorem 4.5, we have the following results for upper semicontinuous set-valued mappings in locally convex spaces as an application of Theorem 4.2.

**Theorem 4.6** (Schauder fixed point theorem for upper semicontinuous condensing mappings) *Let  $C$  be a convex subset of a locally convex space  $E$ . If  $T : C \rightarrow 2^C$  is upper semicontinuous,  $(\alpha)$  condensing with closed convex values, then  $T$  has a fixed point in  $C$  and the set of fixed points of  $T$  is compact.*

*Proof* By the same argument as in Theorem 4.5 by applying Theorem 4.4.  $\square$

As applications of Theorem 4.5, we have the following fixed points for condensing mappings in locally  $p$ -convex or topological vector spaces for  $0 < p \leq 1$ .

**Corollary 4.1** (Darbo type fixed point theorem) *Let  $C$  be a complete  $p$ -convex subset of a Hausdorff locally  $p$ -convex space or topological vector space  $E$  with  $0 < p \leq 1$ . If  $T : C \rightarrow C$  is a  $(k)$ -set-contraction (where  $k \in (0, 1)$ ), then  $T$  has a fixed point.*

**Corollary 4.2** (Sadovskii type fixed point theorem) *Let  $(E, \|\cdot\|)$  be a complete  $p$ -normed space and  $C$  be a bounded, closed, and  $p$ -convex subset of  $E$ , where  $0 < p \leq 1$ . Then every continuous and condensing mapping  $T : C \rightarrow C$  has a fixed point.*

*Proof* In Theorem 4.5, let  $\mathfrak{B} := \{B_p(0, 1)\}$ , where  $B_p(0, 1)$  stands for the closed unit ball of  $E$ , and by the fact that it is clear that  $\alpha(A) = (\alpha_{\mathfrak{B}}(A))^p$  for each  $A \subset E$ . Then  $T$  satisfies all conditions of Theorem 4.5. This completes the proof.  $\square$

**Corollary 4.3** (Darbo type) *Let  $(E, \|\cdot\|)$  be a complete  $p$ -normed space and  $C$  be a bounded, closed, and  $p$ -convex subset of  $E$ , where  $0 < p \leq 1$ . Then each single-valued mapping  $T : C \rightarrow C$  has a fixed point.*

Theorems 4.5 and 4.6 improve Theorem 5 of Machrafi and Oubbi [72] for general condensing mappings and also unify corresponding the results in the existing literature, e.g., see Alghamdi et al. [4], Górniewicz [44], Górniewicz et al. [45], Nussbaum [77], Silva et al. [111], Xiao and Lu [122], Xiao and Zhu [123, 124], and the references therein.

Before ending this section, we would also like to remark that by comparing with the topological method or related arguments used by Askoura et al. [5], Cauty [19, 20], Nhu [76], Reich [99], the fixed points given in this section improve or unify the corresponding ones given by Alghamdi et al. [4], Darbo [28], Liu [70], Machrafi and Oubbi [72], Sadovskii [105], Silva et al. [111], Xiao and Lu [122], and those from references therein.

## 5 Best approximation for the class of single and set-valued 1-set contractive mappings in locally $p$ -convex spaces

The goal of this section is first to establish one general best approximation result for the classes of single-valued 1-set continuous and hemicompact (see the definition below) non-self mappings, which in turn are used as a tool to derive the general principle for the existence of solutions for Birkhoff–Kellogg problems (see Birkhoff and Kellogg [11]), fixed points for nonself 1-set contractive mappings.



Here, we recall that since the Birkhoff–Kellogg theorem was first introduced and proved by Birkhoff and Kellogg [11] in 1922 in discussing the existence of solutions for the equation  $x = \lambda F(x)$ , where  $\lambda$  is a real parameter and  $F$  is a general nonlinear nonself mapping defined on an open convex subset  $U$  of a topological vector space  $E$ , now the general form of the Birkhoff–Kellogg problem is to find the so-called invariant direction for the nonlinear single-valued or set-valued mappings  $F$ , i.e., to find  $x_0 \in \overline{U}$  (or  $x_0 \in \partial \overline{U}$ ) and  $\lambda > 0$  such that  $\lambda x_0 = F(x_0)$  or  $\lambda x_0 \in F(x_0)$ . But the current paper focuses on the study for single-valued mappings for  $p$ -vector spaces for  $0 < 1 \leq 1$ .

Since the Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920s, the study on the Birkhoff–Kellogg problem has received a lot of scholars' attention; for example, one of the fundamental results in nonlinear functional analysis, called the Leray–Schauder alternative by Leray and Schauder [65] in 1934, was established via topological degree. Thereafter, certain other types of Leray–Schauder alternatives were proved using different techniques other than topological degree, see work given by Granas and Dugundji [46], Furi and Pera [37] in the Banach space setting and applications to the boundary value problems for ordinary differential equations, and a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces, and also Birkhoff–Kellogg type theorems for general class mappings in TVS by Agarwal et al. [1], Agarwal and O'Regan [2, 3], Park [87]; in particular, recently O'Regan [80] used the Leray–Schauder type coincidence theory to establish some Birkhoff–Kellogg problem, Furi–Pera type results for a general class of single-valued or set-valued mappings, too.

In this section, one best approximation result for 1-set contractive mappings in locally  $p$ -convex spaces is first established, it is then used to establish the solution principle for Birkhoff–Kellogg problems and related nonlinear alternatives. These new results allow us to give a general principle for Leray–Schauder type and related fixed point theorems of nonself mappings in locally  $p$ -convex spaces for  $(0 < p \leq 1)$ . The new results given in this part not only include the corresponding results in the existing literature as special cases, but also would be expected to play the fundamental role in the development of nonlinear problems arising from theory to practice for 1-set contractive mappings under the framework of  $p$ -vector spaces, which include the general topological vector spaces as a special class.

We also note that the general principles for nonlinear alternative related to Leray–Schauder alternative and other types under the framework of locally  $p$ -convex spaces for  $(0 < p \leq 1)$  given in this section would be useful tools for the study of nonlinear problems. In addition, we also note that the corresponding results in the existing literature for Birkhoff–Kellogg problems and the Leray–Schauder alternatives have been studied comprehensively by Granas and Dugundji [46], Isac [51], Kim et al. [57], Park [88–90], Carbone and Conti [18], Chang et al. [23, 24], Chang and Yen [25], Shahzad [109, 110], Singh [113]; and in particular, many general forms have been recently obtained by O'Regan [81] and Yuan [134] (see also the references therein).

In order to study the general existence of fixed points for nonself mappings in locally  $p$ -convex spaces, we need some definitions and notations given below.

**Definition 5.1** (Inward and outward sets in  $p$ -vector spaces) Let  $C$  be a subset of a  $p$ -vector space  $E$  and  $x \in E$  for  $0 < p \leq 1$ . Then the  $p$ -inward set  $I_C^p(x)$  and the  $p$ -outward set

$O_C^p(x)$  are defined by

$$I_C^p(x) := \left\{ x + r(y - x) : y \in C \text{ for any } r \geq 0 \text{ (1) if } 0 \leq r \leq 1 \text{ with } (1 - r)^p + r^p = 1; \right. \\ \left. \text{or (2) if } r \geq 1 \text{ with } \left(\frac{1}{r}\right)^p + \left(1 - \frac{1}{r}\right)^p = 1 \right\}; \quad \text{and} \\ O_C^p(x) := \left\{ x + r(y - x) : y \in C \text{ for any } r \leq 0 \text{ (1) if } 0 \leq |r| \leq 1 \text{ with } (1 - |r|)^p + |r|^p = 1; \right. \\ \left. \text{or (2) if } |r| \geq 1 \text{ with } \left(\frac{1}{|r|}\right)^p + \left(1 - \frac{1}{|r|}\right)^p = 1 \right\}.$$

From the definition, it is obvious that when  $p = 1$ , both inward and outward sets  $I_C^p(x)$ ,  $O_C^p(x)$  are reduced to the definition for the inward set  $I_C(x)$  and the outward set  $O_C(x)$ , respectively, in topological vector spaces introduced by Halpern and Bergman [47] and used for the study of nonself mappings related to nonlinear functional analysis in the literature. In this paper, we mainly focus on the study of the  $p$ -inward set  $I_U^p(x)$  for the best approximation and related to the boundary condition for the existence of fixed points in locally  $p$ -convex spaces. By the special property of  $p$ -convex concept for  $p \in (0, 1)$  and  $p = 1$ , we have the following fact.

**Lemma 5.1** *Let  $C$  be a subset of a  $p$ -vector space  $E$  and  $x \in E$  for  $0 < p \leq 1$ . Then, for both  $p$ -inward and outward sets  $I_C^p(x)$  and  $O_C^p(x)$  defined above, we have*

- (I) *when  $p \in (0, 1)$ ,  $I_C^p(x) = [\{x\} \cup C]$  and  $O_C^p(x) = [\{x\} \cup \{2x\} \cup -C]$ ,*
- (II) *when  $p = 1$ , in general  $[\{x\} \cup C] \subset I_C^p(x)$  and  $[\{x\} \cup \{2x\} \cup -C] \subset O_C^p(x)$ .*

*Proof* First, when  $p \in (0, 1)$ , by the definitions of  $I_C^p(x)$ , the only real number  $r \geq 0$  satisfying the equation  $(1 - r)^p + r^p = 1$  for  $r \in [0, 1]$  is  $r = 0$  or  $r = 1$ , and when  $r \geq 1$ , the equation  $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1$  implies that  $r = 1$ . The same reason for  $O_C^p(x)$ , it follows that  $r = 0$  and  $r = -1$ .

Secondly, when  $p = 1$ , all  $r \geq 0$  and all  $r \leq 0$  satisfy the requirement of definition for  $I_C^p(x)$  and  $O_C^p(x)$ , respectively, thus the proof is complete.  $\square$

By following the original idea by Tan and Yuan [117] for hemicompact mappings in metric spaces, we introduce the following definition for a mapping being hemicompact in  $p$ -seminorm spaces for  $p \in (0, 1]$ , which is indeed the “(H) condition” used in Theorem 5.1 to prove the existence of best approximation results for 1-set contractive mappings in locally  $p$ -convex spaces for  $p \in (0, 1]$ .

**Definition 5.2** (Hemicompact mapping) Let  $E$  be a locally  $p$ -convex space for  $1 < p \leq 1$ . For a given bounded (closed) subset  $D$  in  $E$ , a mapping  $F : D \rightarrow 2^E$  is said to be hemicompact if each sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $D$  has a convergent subsequence with limit  $x_0$  such that  $x_0 \in F(x_0)$ , whenever  $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$  for each  $U \in \mathcal{U}$ , where  $d_{P_U}(x, C) := \inf\{P_U(x - y) : y \in C\}$  is the distance of a single point  $x$  with the subset  $C$  in  $E$  based on  $P_U$ ,  $P_U$  is the Minkowski  $p$ -functional in  $E$  for  $U \in \mathcal{U}$ , which is the base of the family consisting of all open  $p$ -convex subsets for 0-neighborhoods in  $E$ .

**Remark 5.1** We would like to point out that Definition 5.2 is indeed an extension for a “hemicompact mapping” defined from a metric space to a (locally)  $p$ -convex space with the  $p$ -seminorm, where  $p \in (0, 1]$  (see Tan and Yuan [117]). By the monotonicity of Minkowski  $p$ -functionals, i.e., the bigger 0-neighborhoods, the smaller Minkowski  $p$ -functionals’ values (see also p. 178 of Balachandran [6]). Definition 5.2 describes the converge for the distance between  $x_n$  and  $F(x_n)$  by using the language of seminorms in terms of Minkowski  $p$ -functionals for each 0-neighborhood in  $\mathfrak{U}$  (the base), which is the family consisting of its open  $p$ -convex 0-neighborhoods in a  $p$ -vector space  $E$ .

Now we have the following Schauder fixed point theorem for 1-set contractive mappings in locally  $p$ -convex spaces for  $p \in (0, 1]$ .

**Theorem 5.1** (Schauder fixed point theorem for single-valued 1-set contractive mappings) *Let  $U$  be a nonempty bounded open subset of a (Hausdorff) locally  $p$ -convex space  $E$  and its zero  $0 \in U$ , and  $C \subset E$  be a closed  $p$ -convex subset of  $E$  such that  $0 \in C$  with  $0 < p \leq 1$ . If  $F : C \cap \overline{U} \rightarrow C \cap \overline{U}$  is a continuous and 1-set contractive single-valued mapping satisfying the following (H) or (H1) condition:*

(H) condition: *The sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $\overline{U}$  has a convergent subsequence with limit  $x_0 \in \overline{U}$  such that  $x_0 \in F(x_0)$ , whenever  $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$ , where,  $d_{P_U}(x_n, F(x_n)) := P_U(x_n - F(x_n))$ , where  $P_U$  is the Minkowski  $p$ -functional for any  $U \in \mathfrak{U}$ , which is the family of all nonempty open  $p$ -convex subsets of zero in  $E$ .*

(H1) condition: *There exists  $x_0 \in \overline{U}$  with  $x_0 = F(x_0)$  if there exists  $\{x_n\}_{n \in \mathbb{N}}$  in  $\overline{U}$  such that  $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$ , where  $P_U$  is the Minkowski  $p$ -functional for any  $U \in \mathfrak{U}$ , which is the family of all nonempty open  $p$ -convex subsets of zero in  $E$ .*

*Then  $F$  has at least one fixed point in  $C \cap \overline{U}$ .*

**Proof** Let  $U$  be any element in  $\mathfrak{U}$ , which is the family of all nonempty open  $p$ -convex subsets for zero in  $E$ . As the mapping  $T$  is 1-set contractive, take an increasing sequence  $\{\lambda_n\}$  such that  $0 < \lambda_n < 1$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ , where  $n \in \mathbb{N}$ . Now we define a mapping  $F_n : C \rightarrow C$  by  $F_n(x) := \lambda_n F(x)$  for each  $x \in C$  and  $n \in \mathbb{N}$ . Then it follows that  $F_n$  is a  $\lambda_n$ -set-contractive mapping with  $0 < \lambda_n < 1$ . By Theorem 4.5 on the condensing mapping  $F_n$  in a  $p$ -vector space with  $p$ -seminorm  $P_U$  for each  $n \in \mathbb{N}$ , there exists  $x_n \in C$  such that  $x_n \in F_n(x_n) = \lambda_n F(x_n)$ . As  $P_U$  is the Minkowski  $p$ -functional of  $U$  in  $E$ , it follows that  $P_U$  is continuous as  $0 \in \text{int}(U) = U$ . Note that for each  $n \in \mathbb{N}$ ,  $\lambda_n x_n \in \overline{U} \cap C$ , which implies that  $x_n = r(\lambda_n F(x_n)) = \lambda_n F(x_n)$ , thus  $P_U(\lambda_n F(x_n)) \leq 1$  by Lemma 2.2. Note that

$$\begin{aligned} P_U(F(x_n) - x_n) &= P_U(F(x_n) - \lambda_n F(x_n)) = P_U\left(\frac{(1 - \lambda_n)\lambda_n F(x_n)}{\lambda_n}\right) \\ &\leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n F(x_n)) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that  $\lim_{n \rightarrow \infty} P_U(F(x_n) - x_n) = 0$  for all  $U \in \mathfrak{U}$ .

Now (1) if  $F$  satisfies the (H) condition, it implies that the consequence  $\{x_n\}_{n \in \mathbb{N}}$  has a convergent subsequence which converges to  $x_0$  such that  $x_0 = F(x_0)$ . Without loss of generality, we assume that  $\lim_{n \rightarrow \infty} x_n = x_0$  is with  $x_n = \lambda_n F(x_n)$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ . It implies that  $x_0 = \lim_{n \rightarrow \infty} (\lambda_n F(x_n))$ , which means  $\lim_{n \rightarrow \infty} F(x_n) = x_0$ .

(ii) If  $F$  satisfies the (H1) condition, then by the (H1) condition it follows that there exists  $x_0 \in \overline{U}$  such that  $x_0 = F(x_0)$ , which is a fixed point of  $F$ . We complete the proof.  $\square$

**Theorem 5.2** (Best approximation for single-valued 1-set-contractive mappings) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  ( $0 \leq p \leq 1$ ) with zero  $0 \in U$ , and  $C$  be a (bounded) closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ . Assume that  $F : \overline{U} \cap C \rightarrow C$  is (single-valued) 1-set contractive, and for each  $x \in \partial_C U$  with  $F(x) \in C \setminus \overline{U}$ ,  $(P_U^{\frac{1}{p}}(F(x)) - 1)^p \leq P_U(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ). In addition, if  $F$  satisfies one of the following conditions:*

(H) condition: *The sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $\overline{U}$  has a convergent subsequence with limit  $x_0 \in \overline{U}$  such that  $x_0 = F(x_0)$ , whenever  $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$ , where  $d_{P_U}(x_n, F(x_n)) := \inf\{P_U(x_n - F(x_n))\}$ , where  $P_U$  is the Minkowski  $p$ -functional for any  $U \in \mathfrak{U}$ , which is the family of all nonempty open  $p$ -convex subsets containing the zero in  $E$ .*

(H1) condition: *There exists  $x_0 \in \overline{U}$  with  $x_0 = F(x_0)$  if there exists  $\{x_n\}_{n \in \mathbb{N}}$  in  $\overline{U}$  such that  $\lim_{n \rightarrow \infty} d_{P_U}(x_n, F(x_n)) = 0$ , where  $P_U$  is the Minkowski  $p$ -functional for any  $U \in \mathfrak{U}$ , which is the family of all nonempty open  $p$ -convex subsets containing the zero in  $E$ .*

*Then we have that there exists  $x_0 \in C \cap \overline{U}$  such that*

$$P_U(F(x_0) - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C),$$

*where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . More precisely, we have that either (I) or (II) holds:*

(I)  *$F$  has a fixed point  $x_0 \in \overline{U} \cap C$ , i.e.,*

$$0 = P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C),$$

(II) *There exist  $x_0 \in \partial_C(U)$  and  $F(x_0) \notin \overline{U}$  with*

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(F(x_0)) - 1)^p > 0.$$

*Proof* As  $E$  is a locally  $p$ -convex space, it suffices to prove that for each open  $p$ -convex subset  $U$  in  $\mathfrak{U}$  (which is the family of all nonempty open  $p$ -convex subsets containing the zero in  $E$ ), there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\overline{U}$  such that  $\lim_{n \rightarrow \infty} P_U(F(x_n) - x_n) = 0$ , and the conclusion follows by applying the (H) condition.

Let  $r : E \rightarrow U$  be a retraction mapping defined by  $r(x) := \frac{x}{\max\{1, (P_U(x))^{\frac{1}{p}}\}}$  for each  $x \in E$ , where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . Since the space  $E$ 's zero  $0 \in U$  ( $= \text{int } U$  as  $U$  is open), it follows that  $r$  is continuous by Lemma 2.2. As the mapping  $F$  is 1-set contractive, take an increasing sequence  $\{\lambda_n\}$  such that  $0 < \lambda_n < 1$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ , where  $n \in \mathbb{N}$ . Now, for each  $n \in \mathbb{N}$ , we define a mapping  $F_n : C \cap \overline{U} \rightarrow C$  by  $F_n(x) := \lambda_n F \circ r(x)$  for each  $x \in C \cap \overline{U}$ . By the fact that  $C$  and  $\overline{U}$  are  $p$ -convex, it follows that  $r(C) \subset C$  and  $r(\overline{U}) \subset \overline{U}$ , thus  $r(C \cap \overline{U}) \subset C \cap \overline{U}$ . Therefore  $F_n$  is a mapping from  $\overline{U} \cap C$  to itself. For each  $n \in \mathbb{N}$ , by the fact that  $F_n$  is a  $\lambda_n$ -set-contractive mapping with  $0 < \lambda_n < 1$ , it follows by Theorem 4.5 for the condensing mapping that there exists  $z_n \in C \cap \overline{U}$  such that  $F_n(z_n) = \lambda_n F \circ r(z_n)$ . As  $r(C \cap \overline{U}) \subset C \cap \overline{U}$ , let  $x_n = r(z_n)$ . Then we have that  $x_n \in C \cap \overline{U}$  and with  $x_n = r(\lambda_n F_n(x_n))$  such that the following (1) or (2) holds for each  $n \in \mathbb{N}$ : (1)  $\lambda_n F_n(x_n) \in C \cap \overline{U}$  or (2)  $\lambda_n F_n(x_n) \in C \setminus \overline{U}$ .

Now we prove the conclusion by considering the following two cases under the (H) condition and (H1) condition:

Case (I) For each  $n \in \mathbb{N}$ ,  $\lambda_n F(x_n) \in C \cap \overline{U}$ ; or

Case (II) There exists a positive integer  $n$  such that  $\lambda_n F(x_n) \in C \setminus \overline{U}$ .

First, by case (I), for each  $n \in \mathbb{N}$ ,  $\lambda_n F(x_n) \in \overline{U} \cap C$ , which implies that  $x_n = r(\lambda_n F(x_n)) = \lambda_n F(x_n)$ , thus  $P_U(\lambda_n F(x_n)) \leq 1$  by Lemma 2.2. Note that

$$\begin{aligned} P_U(F(x_n) - x_n) &= P_U(F(x_n) - x_n) = P_U(F(x_n) - \lambda_n F(x_n)) = P_U\left(\frac{(1 - \lambda_n)\lambda_n F(x_n)}{\lambda_n}\right) \\ &\leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n F(x_n)) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that  $\lim_{n \rightarrow \infty} P_U(F(x_n) - x_n) = 0$ . Now, for any  $V \in \mathbb{U}$ , without loss of generality, let  $U_0 = V \cap U$ . Then we have the following conclusion:

$$\begin{aligned} P_{U_0}(F(x_n) - x_n) &= P_{U_0}(F(x_n) - x_n) = P_{U_0}(F(x_n) - \lambda_n F(x_n)) = P_{U_0}\left(\frac{(1 - \lambda_n)\lambda_n F(x_n)}{\lambda_n}\right) \\ &\leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_{U_0}(\lambda_n F(x_n)) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that  $\lim_{n \rightarrow \infty} P_{U_0}(F(x_n) - x_n) = 0$ , where  $P_{U_0}$  is the Minkowski  $p$ -functional of  $U_0$  in  $E$ .

Now, if  $F$  satisfies the (H) condition, it follows that the consequence  $\{x_n\}_{n \in \mathbb{N}}$  has a convergent subsequence, which converges to  $x_0$  such that  $x_0 = F(x_0)$ . Without loss of generality, we assume that  $\lim_{n \rightarrow \infty} x_n = x_0$ ,  $x_n = \lambda_n y_n$ , and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ , and as  $x_0 = \lim_{n \rightarrow \infty} (\lambda_n F(x_n))$ , which implies that  $F(x_0) = \lim_{n \rightarrow \infty} F(x_n) = x_0$ . Thus there exists  $x_0 = F(x_0)$ , thus we have  $0 = d_p(x_0, F(x_0)) = d(y_0, \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p}(x_0) \cap C)$  as indeed  $x_0 = F(x_0) \in \overline{U} \cap C \subset \overline{I_U^p}(x_0) \cap C$ .

If  $F$  satisfies the (H1) condition, it follows that there exists  $x_0 \in \overline{U} \cap C$  with  $x_0 = F(x_0)$ . Then we have  $0 = P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p}(x_0) \cap C)$ .

Second, by case (II) there exists a positive integer  $n$  such that  $\lambda_n F(x_n) \in C \setminus \overline{U}$ . Then we have that  $P_U(\lambda_n F(x_n)) > 1$ , and also  $P_U(F(x_n)) > 1$  as  $\lambda_n < 1$ . As  $x_n = r(\lambda_n F(x_n)) = \frac{\lambda_n F(x_n)}{(P_U(\lambda_n F(x_n)))^{\frac{1}{p}}}$ , which implies that  $P_U(x_n) = 1$ , thus  $x_n \in \partial_C(U)$ . Note that

$$P_U(F(x_n) - x_n) = P_U\left(\frac{(P_U(F(x_n)))^{\frac{1}{p}} - 1}{P_U(F(x_n))^{\frac{1}{p}}} F(x_n)\right) = (P_U^{\frac{1}{p}}(F(x_n)) - 1)^p.$$

By the assumption, we have  $(P_U^{\frac{1}{p}}(F(x_n)) - 1)^p \leq P_U(F(x_n) - x)$  for  $x \in C \cap \partial \overline{U}$ , it follows that

$$\begin{aligned} P_U(F(x_n)) - 1 &\leq P_U(F(x_n)) - \sup\{P_U(z) : z \in C \cap \overline{U}\} \\ &\leq \inf\{P_U(F(x_n) - z) : z \in C \cap \overline{U}\} = d_p(F(x_n), C \cap \overline{U}). \end{aligned}$$

Thus we have the best approximation:  $P_U(F(x_n) - x_n) = d_p(y_n, \overline{U} \cap C) = (P_U^{\frac{1}{p}}(F(x_n)) - 1)^p > 0$ .

Now we want to show that  $P_U(y_n - x_n) = d_p(F(x_n), \overline{U} \cap C) = d_p(F(x_n), \overline{I_U^p}(x_0) \cap C) > 0$ .

By the fact that  $(\overline{U} \cap C) \subset \overline{I_U^p}(x_n) \cap C$ , let  $z \in \overline{I_U^p}(x_n) \cap C \setminus (\overline{U} \cap C)$ , we first claim that  $P_U(F(x_n) - x_n) \leq P_U(F(x_n) - z)$ . If not, we have  $P_U(F(x_n) - x_n) > P_U(F(x_n) - z)$ . As  $z \in \overline{I_U^p}(x_n) \cap C \setminus (\overline{U} \cap C)$ , there exists  $y \in \overline{U}$  and a nonnegative number  $c$  (actually  $c \geq 1$  as shown soon

below) with  $z = x_n + c(y - x_n)$ . Since  $z \in C$ , but  $z \notin \overline{U} \cap C$ , it implies that  $z \notin \overline{U}$ . By the fact that  $x_n \in \overline{U}$  and  $y \in \overline{U}$ , we must have the constant  $c \geq 1$ ; otherwise, it implies that  $z = (1 - c)x_n + cy \in \overline{U}$ , this is impossible by our assumption, i.e.,  $z \notin \overline{U}$ . Thus we have that  $c \geq 1$ , which implies that  $y = \frac{1}{c}z + (1 - \frac{1}{c})x_n \in C$  (as both  $x_n \in C$  and  $z \in C$ ). On the other hand, as  $z \in I_U^p(x_n) \cap C \setminus (\overline{U} \cap C)$  and  $c \geq 1$  with  $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$ , combining with our assumption that for each  $x \in \partial_C \overline{U}$  and  $y \in F(x_n) \setminus \overline{U}$ ,  $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$  for  $0 < p \leq 1$ , it then follows that

$$\begin{aligned} P_U(F(x_n) - y) &= P_U\left[\frac{1}{c}(F(x_n) - z) + \left(1 - \frac{1}{c}\right)(F(x_n) - x_n)\right] \\ &\leq \left[\left(\frac{1}{c}\right)^p P_U(F(x_n) - z) + \left(1 - \frac{1}{c}\right)^p P_U(F(x_n) - x_n)\right] \\ &< P_U(F(x_n) - x_n), \end{aligned}$$

which contradicts that  $P_U(F(x_n) - x_n) = d_p(F(x_n), \overline{U} \cap C)$  as shown above, we know that  $y \in \overline{U} \cap C$ , we should have  $P_U(F(x_n) - x_n) \leq P_U(F(x_n) - y)$ ! This helps us to complete the claim:  $P_U(F(x_n) - x_n) \leq P_U(F(x_n) - z)$  for any  $z \in I_U^p(x_n) \cap C \setminus (\overline{U} \cap C)$ , which means that the following best approximation of Fan type (see [34, 35]) holds:

$$0 < d_p(F(x_n), \overline{U} \cap C) = P_U(F(x_n) - x_n) = d_p(F(x_n), I_U^p(x_n) \cap C).$$

Now, by the continuity of  $P_U$ , it follows that the following best approximation of Fan type is also true:

$$0 < P_U(F(x_n) - x_n) = d_p(F(x_n), \overline{U} \cap C) = d_p(F(x_n), I_U^p(x_n) \cap C) = d_p(F(x_n), \overline{I_U^p(x_n)} \cap C);$$

and we have the conclusion below due to that  $\lim_{n \rightarrow \infty} x_n = x_0$  and the continuity of  $F$  (actually  $x_0 \neq F(x_0)$ ):

$$\begin{aligned} P_U(F(x_0) - x_0) &= d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), I_U^p(x_0) \cap C) \\ &= d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(F(x_0)) - 1)^p > 0. \end{aligned}$$

This completes the proof.  $\square$

**Remark 5.2** We note that Theorem 5.2 also improves the corresponding best approximation for 1-set contractive mappings given by Li et al. [67], Liu [70], Xu [129], Xu et al. [130], and the results from the references therein; and 3): When  $p = 1$ , we have the similar best approximation result for the mapping  $F$  in the locally convex spaces with the outward set boundary condition below (see Theorem 3 of Park [86] and related discussion in the references therein).

Although the main focus of this paper studies best approximation, fixed point theorems for single-valued mappings, when a  $p$ -vector space  $E$  (for  $p = 1$ ) is a locally convex space (LCS), we can also have the following best approximation for upper semicontinuous set-valued mappings by applying Theorem 4.6 with arguments used by Theorem 5.1 and Theorem 5.2 (see also the discussion given by Yuan [134] and the references therein).



**Theorem 5.3** (Best approximation for USC set-valued mappings in LCS) *Let  $U$  be a bounded open convex subset of a locally convex space  $E$  (i.e.,  $p = 1$ ) with zero  $0 \in \text{int } U = U$  (the interior  $\text{int } U = U$  as  $U$  is open), and let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ . Assume that  $F : \overline{U} \cap C \rightarrow 2^C$  is a 1-set-contractive upper semicontinuous mapping satisfying condition (H) or (H1). Then there exist  $x_0 \in \overline{U} \cap C$  and  $y_0 \in F(x_0)$  such that  $P_U(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C)$ , where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . More precisely, we have that either (I) or (II) holds:*

- (I)  *$F$  has a fixed point  $x_0 \in U \cap C$ , i.e.,  $x_0 \in F(x_0)$  (so that  $P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) = 0$ ), or*
- (II) *There exist  $x_0 \in \partial_C(U)$  and  $y_0 \in F(x_0)$  with  $y_0 \notin \overline{U}$  with*

$$P_U(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C) > 0.$$

*Proof* Following the proof used in Theorem 5.1 and Theorem 5.2, then applying Theorem 4.6 for  $p = 1$ , the conclusion follows. This completes the proof.  $\square$

Now, by the application of Theorem 5.2 with Remark 5.2 and the argument used in Theorem 5.2, we have the following general principle for the existence of solutions for Birkhoff–Kellogg problems in  $p$ -seminorm spaces for locally  $p$ -convex spaces, where  $0 < p \leq 1$ .

**Theorem 5.4** (Principle of Birkhoff–Kellogg alternative) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  ( $0 \leq p \leq 1$ ) with zero  $0 \in \text{int } U = U$  (the interior  $\text{int } U$  as  $U$  is open), and let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ . Assume that  $F : \overline{U} \cap C \rightarrow C$  is a single-valued 1-set-contractive continuous mapping satisfying the (H) or (H1) condition. Then  $F$  has at least one of the following two properties:*

- (I)  *$F$  has a fixed point  $x_0 \in U \cap C$  such that  $x_0 = F(x_0)$ ,*
- (II) *There exist  $x_0 \in \partial_C(U)$ ,  $F(x_0) \notin \overline{U}$ , and  $\lambda = \frac{1}{(P_U(F(x_0)))^{\frac{1}{p}}} \in (0, 1)$  such that  $x_0 = \lambda F(x_0)$ .*

*In addition, if for each  $x \in \partial_C U$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ), then the best approximation between  $\{x_0\}$  and  $F(x_0)$  is given by*

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(F(x_0)) - 1)^p > 0.$$

*Proof* If (I) is not the case, then (II) is proved by Remark 5.2 and by following the proof in Theorem 5.2 for case ii):  $F(x_0) \notin \overline{U}$  with  $F(x_0) = f(x_0)$ , where  $f$  is the restriction of the continuous retraction  $r$  with respect to the set  $U$  in  $E$  defined in the proof of Theorem 5.2. Indeed, as  $F(x_0) \notin \overline{U}$ , it follows that  $P_U(F(x_0)) > 1$  and  $x_0 = f(F(x_0)) = F(x_0) \frac{1}{(P_U(F(x_0)))^{\frac{1}{p}}}$ . Now, let  $\lambda = \frac{1}{(P_U(F(x_0)))^{\frac{1}{p}}}$ , we have  $\lambda < 1$  and  $x_0 = \lambda F(x_0)$ . Finally, the additionally assumption in (II) allows us to have the best approximation between  $x_0$  and  $F(x_0)$  obtained by following the proof of Theorem 5.2 as  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) > 0$ . This completes the proof.  $\square$

As an application of Theorem 5.3 for the nonself upper semicontinuous set-valued mappings discussed in Theorem 5.4, we have the following general principle of Birkhoff–Kellogg alternative in locally convex spaces.

**Theorem 5.5** (Principle of Birkhoff–Kellogg alternative in LCS) *Let  $U$  be a bounded open  $p$ -convex subset of an LCS  $E$  with the zero  $0 \in U$  and  $C$  be a closed convex subset of  $E$  with also zero  $0 \in C$ . Assume that  $F : \overline{U} \cap C \rightarrow 2^C$  is a set-valued 1-set contractive and upper semicontinuous mapping satisfying the (H) or (H1) condition. Then it has at least one of the following two properties:*

- (I)  *$F$  has a fixed point  $x_0 \in U \cap C$  such that  $x_0 \in F(x_0)$ ,*
- (II) *There exist  $x_0 \in \partial_C(U)$  and  $y_0 \in F(x_0)$  with  $y_0 \notin \overline{U}$  and  $\lambda \in (0, 1)$  such that  $x_0 = \lambda y_0$ , and the best approximation between  $\{x_0\}$  and  $F(x_0)$  is given by*  

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_P(y_0, \overline{I_U^p(x_0)} \cap C) > 0.$$

On the other hand, by the proof of Theorem 5.2, we note that for case (II) of Theorem 5.2, the assumption “each  $x \in \partial_C U$  with  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$ ” is only used to guarantee the best approximation “ $P_U(F(x_0) - x_0) = d_P(F(x_0), \overline{U} \cap C) = d_P(F(x_0), \overline{I_U^p(x_0)} \cap C) > 0$ ”, thus we have the following Leray–Schauder alternative in  $p$ -vector spaces, which, of course, includes the corresponding results in locally convex spaces as special cases.

**Theorem 5.6** (Leray–Schauder nonlinear alternative) *Let  $C$  be a closed  $p$ -convex subset of  $p$ -seminorm space  $E$  with  $0 \leq p \leq 1$  and the zero  $0 \in C$ . Assume that  $F : C \rightarrow C$  is a single-valued 1-set contractive and continuous mapping satisfying the (H) or (H1) condition above. Let  $\varepsilon(F) := \{x \in C : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}$ . Then either  $F$  has a fixed point in  $C$  or the set  $\varepsilon(F)$  is unbounded.*

*Proof* We prove the conclusion by assuming that  $F$  has no fixed point, then we claim that the set  $\varepsilon(F)$  is unbounded. Otherwise, assume that the set  $\varepsilon(F)$  is bounded and that  $P$  is the continuous  $p$ -seminorm for  $E$ , then there exists  $r > 0$  such that the set  $B(0, r) := \{x \in E : P(x) < r\}$ , which contains the set  $\varepsilon(F)$ , i.e.,  $\varepsilon(F) \subset B(0, r)$ , which means, for any  $x \in \varepsilon(F)$ ,  $P(x) < r$ . Then  $B(0, r)$  is an open  $p$ -convex subset of  $E$  and the zero  $0 \in B(0, r)$  by Lemma 2.2 and Remark 2.4. Now, let  $U := B(0, r)$  in Theorem 5.4. It follows that the mapping  $F : B(0, r) \cap C \rightarrow C$  satisfies all general conditions of Theorem 5.4, and we have that any  $x_0 \in \partial_C B(0, r)$ , no any  $\lambda \in (0, 1)$  such that  $x_0 = \lambda F(x_0)$ . Indeed, for any  $x \in \varepsilon(F)$ , it follows that  $P(x) < r$  as  $\varepsilon(F) \subset B(0, r)$ , but for any  $x_0 \in \partial_C B(0, r)$ , we have  $P(x_0) = r$ , thus conclusion (II) of Theorem 5.4 does not hold. By Theorem 5.4 again,  $F$  must have a fixed point, but this contradicts our assumption that  $F$  is fixed point free. This completes the proof.  $\square$

Now, assume a given  $p$ -vector space  $E$  equipped with the  $P$ -seminorm (by assuming it is continuous at zero) for  $0 < p \leq 1$ , then we know that  $P : E \rightarrow \mathbb{R}^+$ ,  $P^{-1}(0) = 0$ ,  $P(\lambda x) = |\lambda|^p P(x)$  for any  $x \in E$  and  $\lambda \in \mathbb{R}$ . Then we have the following useful result for fixed points due to Rothe and Altman types in  $p$ -vector spaces, in particular, for locally  $p$ -convex spaces, which plays an important role in optimization problems, variational inequalities, and complementarity problems (see Isac [51] or Yuan [133] and the references therein for related study in detail).

**Corollary 5.1** *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  and zero  $0 \in U$ , plus  $C$  is a closed  $p$ -convex subset of  $E$  with  $U \subset C$ , where  $0 < p \leq 1$ . Assume that  $F : \overline{U} \rightarrow C$  is a single-valued 1-set contractive continuous mapping satisfying the (H) or (H1) condition. If one of the following is satisfied:*

- (1) (Rothe type condition):  $P_U(F(x)) \leq P_U(x)$  for  $x \in \partial U$ ;  
 (2) (Petryshyn type condition):  $P_U(F(x)) \leq P_U(F(x) - x)$  for  $x \in \partial U$ ;  
 (3) (Altman type condition):  $|P_U(F(x))|^{\frac{2}{p}} \leq [P_U(F(x)) - x]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$  for  $x \in \partial U$ ;  
 then  $F$  has at least one fixed point.

*Proof* By conditions (1), (2), and (3), it follows that the conclusion of (II) in Theorem 5.4 “there exist  $x_0 \in \partial_C(U)$  and  $\lambda \in (0, 1)$  such that  $x_0 \neq \lambda F(x_0)$ ” does not hold, thus by the alternative of Theorem 5.4,  $F$  has a fixed point. This completes the proof.  $\square$

By the fact that for  $p = 1$ , when a  $p$ -vector space is a locally convex space, we have the following classical Fan’s best approximation (see [34]), which is a powerful tool for nonlinear functional analysis in supporting the study in optimization, mathematical programming, games theory, and mathematical economics, and other related topics in applied mathematics.

**Corollary 5.2** (Fan’s best approximation in LCS) *Let  $U$  be a bounded open convex subset of a locally convex space  $E$  with the zero  $0 \in U$  and  $C$  be a closed convex subset of  $E$  with also zero  $0 \in C$ , and assume that  $F : \overline{U} \cap C \rightarrow C$  is a set-valued 1-set contractive and continuous mapping satisfying the (H) or (H1) condition. Assume  $P_U$  to be the Minkowski  $p$ -functional of  $U$  in  $E$ . Then there exists  $x_0 \in \overline{U} \cap C$  such that  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C)$ . More precisely, we have that either (I) or (II) holds:*

- (I)  $F$  has a fixed point  $x_0 \in U \cap C$ , i.e.,  $x_0 = F(x_0)$  (so that  
 $0 = P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C)$ );  
 (II) There exist  $x_0 \in \partial_C(U)$  and  $F(x_0) \notin \overline{U}$  with

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1 > 0.$$

*Proof* When  $p = 1$ , it automatically satisfies the inequality  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$ . Now if  $F$  has no fixed points, by Theorem 5.4, indeed we have that for  $x_0 \in \partial_C(U)$ ,  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1$ . The conclusions are given by Theorem 5.2 (or Theorem 5.3). The proof is complete.  $\square$

We would like to point out that similar results on the Rothe and Leray–Schauder alternative have been developed by Isac [51], Park [85], Potter [97], Shahzad [109, 110], Xiao and Zhu [124], Yuan [134], and the related references therein as tools of nonlinear analysis in  $p$ -vector spaces.

## 6 Nonlinear alternatives principle for the class of single-valued 1-set class contractive mappings

As applications of results in Sect. 5, we now establish general results for the existence of solutions for the Birkhoff–Kellogg problem and the principle of Leray–Schauder alternatives in locally  $p$ -convex spaces for  $0 < p \leq 1$ .

**Theorem 6.1** (Birkhoff–Kellogg alternative in locally  $p$ -convex spaces) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  (where  $0 \leq p \leq 1$ ) with the zero  $0 \in U$ , let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ , and assume that*

$F : \overline{U} \cap C \rightarrow C$  is a single-valued 1-set contractive and continuous mapping satisfying condition (H) or (H1). In addition, for each  $x \in \partial_C(U)$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ), where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . Then we have that either (I) or (II) holds:

- (I) There exists  $x_0 \in \overline{U} \cap C$ ; or
- (II) There exists  $x_0 \in \partial_C(U)$  with  $F(x_0) \notin \overline{U}$  and  $\lambda > 1$  such that  $\lambda x_0 = F(x_0)$ , i.e.,  $F(x_0) \in \{\lambda x_0 : \lambda > 1\} \neq \emptyset$ .

*Proof* By following the argument and symbols used in the proof of Theorem 5.2, we have that either

- (1)  $F$  has a fixed point  $x_0 \in U \cap C$ ; or
- (2) There exist  $x_0 \in \partial_C(U)$  and  $x_0 = f(F(x_0))$  such that

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1 > 0,$$

where  $\partial_C(U)$  denotes the boundary of  $U$  relative to  $C$  in  $E$ , and  $f$  is the restriction of the continuous retraction  $r$  with respect to the set  $U$  in  $E$  defined in the proof of Theorem 5.2.

If  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . As given by the proof of Theorem 5.2, we have that  $F(x_0) \notin \overline{U}$ , thus  $P_U(F(x_0)) > 1$  and  $x_0 = f(F(x_0)) = \frac{F(x_0)}{(P_U(F(x_0)))^{\frac{1}{p}}}$ , which means  $F(x_0) = (P_U(F(x_0)))^{\frac{1}{p}} x_0$ . Let  $\lambda = (P_U(F(x_0)))^{\frac{1}{p}}$ , then  $\lambda > 1$ , and we have  $\lambda x_0 = F(x_0)$ . This completes the proof.  $\square$

**Theorem 6.2** (Birkhoff–Kellogg alternative in LCS) *Let  $U$  be a bounded open convex subset of a locally convex space  $E$  with the zero  $0 \in U$ , let  $C$  be a closed convex subset of  $E$  with also zero  $0 \in C$ , and assume that  $F : \overline{U} \cap C \rightarrow C$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). Then we have that either (I) or (II) holds:*

- (I) There exists  $x_0 \in \overline{U} \cap C$  such that  $x_0 = F(x_0)$ ; or
- (II) There exists  $x_0 \in \partial_C(U)$  with  $F(x_0) \notin \overline{U}$  and  $\lambda > 1$  such that  $\lambda x_0 = F(x_0)$ , i.e.,  $F(x_0) \in \{\lambda x_0 : \lambda > 1\} \neq \emptyset$ .

*Proof* When  $p = 1$ , it automatically satisfies the inequality  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x_0) - x)$ , and indeed we have that for  $x_0 \in \partial_C(U)$ , we have  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{W_U(x_0)} \cap C) = P_U(F(x_0)) - 1$ . The conclusions are given by 5.4. The proof is complete.  $\square$

Indeed, we have the following fixed points for nonself mappings in  $p$ -vector spaces for  $0 < p \leq 1$  under different boundary conditions in locally  $p$ -convex spaces.

**Theorem 6.3** (Fixed points of nonself mappings in a locally  $p$ -convex space) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  (where  $0 \leq p \leq 1$ ) with the zero  $0 \in U$ , let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ , and assume that  $F : \overline{U} \cap C \rightarrow C$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). In addition, for each  $x \in \partial_C(U)$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ), where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . If  $F$  satisfies any one of the following conditions for any  $x \in \partial_C(U) \setminus F(x)$ :*

- (i)  $P_U(F(x) - z) < P_U(F(x) - x)$  for some  $z \in \overline{I_U(x)} \cap C$ ;

- (ii) There exists  $\lambda$  with  $|\lambda| < 1$  such that  $\lambda x + (1 - \lambda)F(x) \in \overline{I_U(x)} \cap C$ ;
- (iii)  $F(x) \in \overline{I_U(x)} \cap C$ ;
- (iv)  $F(x) \in \{\lambda x : \lambda > 1\} = \emptyset$ ;
- (v)  $F(\partial U) \subset \overline{U} \cap C$ ;
- (vi)  $P_U(F(x) - x) \neq ((P_U(F(x)))^{\frac{1}{p}} - 1)^p$ ;

then  $F$  must have a fixed point.

*Proof* By following the argument and symbols used in the proof of Theorem 5.2 (see also Theorem 5.4), we have that either

- (1)  $F$  has a fixed point  $x_0 \in U \cap C$ ; or
- (2) There exists  $x_0 \in \partial_C(U)$  with  $x_0 = f(F(x_0))$  such that

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1 > 0,$$

where  $\partial_C(U)$  denotes the boundary of  $U$  relative to  $C$  in  $E$ , and  $f$  is the restriction of the continuous retraction  $r$  with respect to the set  $U$  in  $E$ .

First, suppose that  $F$  satisfies condition (i). If  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . Then, by condition (i), it follows that  $P_U(F(x_0) - z) < P_U(F(x_0) - x_0)$  for some  $z \in \overline{I_U(x)} \cap C$ , this contradicts with the best approximation equations given by (2), thus  $F$  must have a fixed point.

Second, suppose that  $F$  satisfies condition (ii). If  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . Then by condition (ii), there exists  $\lambda > 1$  such that  $\lambda x_0 + (1 - \lambda)F(x_0) \in \overline{I_U(x)} \cap C$ . It follows that

$$\begin{aligned} P_U(F(x_0) - x_0) &\leq P_U(F(x_0) - (\lambda x_0 + (1 - \lambda)F(x_0))) = P_U(\lambda(F(x_0) - x_0)) \\ &= |\lambda|^p P_U(F(x_0) - x_0) < P_U(F(x_0) - x_0) \end{aligned}$$

this is impossible, and thus  $F$  must have a fixed point in  $\overline{U} \cap C$ .

Third, suppose that  $F$  satisfies condition (iii), i.e.,  $F(x) \in \overline{I_U(x)} \cap C$ ; then by (2) we have that  $P_U(F(x_0) - x_0) = 0$ , and thus  $x_0 = F(x_0)$ , which means  $F$  has a fixed point.

Fourth, suppose that  $F$  satisfies condition (iv). If  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . As given by the proof of Theorem 5.2, we have that  $F(x_0) \notin \overline{U}$ , thus  $P_U(F(x_0)) > 1$  and  $x_0 = f(F(x_0)) = \frac{F(x_0)}{(P_U(F(x_0)))^{\frac{1}{p}}}$ , which means  $F(x_0) = (P_U(F(x_0)))^{\frac{1}{p}} x_0$ , where  $(P_U(F(x_0)))^{\frac{1}{p}} > 1$ , this contradicts assumption (iv), thus  $F$  must have a fixed point in  $\overline{U} \cap C$ .

Fifth, suppose that  $F$  satisfies condition (v), then  $x_0 \neq F(x_0)$ . As  $x_0 \in \partial_C U$ , now by condition (v), we have that  $F(\partial U) \subset \overline{U} \cap C$ . It follows that for  $F(x_0)$  we have  $F(x_0) \in \overline{U} \cap C$ , thus  $F(x_0) \notin \overline{U} \setminus \cap C$ , which implies that  $0 < P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = 0$ , this is impossible, thus  $F$  must have a fixed point. Here, like pointed out by Remark 5.2, we know that based on condition (v), the mapping  $F$  has a fixed point by applying  $F(\partial U) \subset \overline{U} \cap C$  is enough, not needing the general hypothesis: “for each  $x \in \partial_C(U)$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$ ”.

Finally, suppose that  $F$  satisfies condition (vi). If  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . Then condition (v) implies that  $P_U(F(x_0) - x_0) \neq ((P_U(F(x_0)))^{\frac{1}{p}} - 1)^p$ , but our proof in Theorem 5.2 shows that  $P_U(F(x_0) - x_0) = ((P_U(F(x_0)))^{\frac{1}{p}} - 1)^p$ , this is impossible, thus  $F$  must have a fixed point. Then the proof is complete.  $\square$

Now, by taking the set  $C$  in Theorem 6.1 as the whole locally  $p$ -convex space  $E$  itself, we have the following general results for nonself continuous mappings, which include the results of Rothe, Petryshyn, Altman, and Leray–Schauder type fixed points as special cases in locally convex spaces.

Taking  $p = 1$  and  $C = E$  in Theorem 6.3, we have the following fixed points for nonself single-valued mappings in locally convex spaces (LCS), and the corresponding results for upper semicontinuous set-valued mappings are discussed by Yuan [134] and related references therein.

**Theorem 6.4** (Fixed points of nonself mappings with boundary conditions) *Let  $U$  be a bounded open convex subset of the LCS  $E$  with the zero  $0 \in U$ , and assume that  $F : \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). If  $F$  satisfies any one of the following conditions for any  $x \in \partial(U) \setminus F(x)$ :*

- (i)  $P_U(F(x) - z) < P_U(F(x) - x)$  for some  $z \in \overline{I_U(x)}$ ;
- (ii) *There exists  $\lambda$  with  $|\lambda| < 1$  such that  $\lambda x + (1 - \lambda)F(x) \in \overline{I_U(x)}$ ;*
- (iii)  $F(x) \in \overline{I_U(x)}$ ;
- (iv)  $F(x) \in \{\lambda x : \lambda > 1\} = \emptyset$ ;
- (v)  $F(\partial(U)) \subset \overline{U}$ ;
- (vi)  $P_U(F(x) - x) \neq P_U(F(x)) - 1$ ;

*then  $F$  must have a fixed point.*

In what follows, based on the best approximation theorem in  $p$ -seminorm space, we will also give some fixed point theorems for nonself mappings with various boundary conditions which are related to the study for the existence of solutions for PDE and differential equations with boundary problems (see Browder [15], Petryshyn [93, 94], Reich [99]), which would play roles in nonlinear analysis for a  $p$ -seminorm space as shown below.

First, as discussed by Remark 5.2, the proof of Theorem 5.2 with the strongly boundary condition “ $F(\partial(U)) \subset \overline{U} \cap C$ ” only, we can prove that  $F$  has a fixed point, thus we have the following fixed point theorem of Rothe type in  $p$ -vector spaces.

**Theorem 6.5** (Rothe type) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  (where  $0 \leq p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1) and such that  $F(\partial(U)) \subset \overline{U}$ , then  $F$  must have a fixed point.*

Now, as applications of Theorem 6.5, we give the following Leray–Schauder alternative in locally  $p$ -convex spaces for nonself mappings associated with the boundary condition which often appear in the applications (see Isac [51] and the references therein for the study of complementary problems and related topics in optimization).

**Theorem 6.6** (Leray–Schauder alternative in locally  $p$ -convex spaces) *Let  $E$  be a locally  $p$ -convex space  $E$ , where  $0 < p \leq 1$ ,  $B \subset E$  is a bounded closed  $p$ -convex such that  $0 \in \text{int } B$ . Let  $F : [0, 1] \times B \rightarrow E$  be 1-set contractive and continuous, satisfying condition (H) or (H1), and such that the set  $F([0, 1] \times B)$  is relatively compact in  $E$ . If the following assumptions are satisfied:*

- (1)  $x \neq F(t, x)$  for all  $x \notin \partial B$  and  $t \in [0, 1]$ ,
- (2)  $F(\{0\} \times \partial B) \subset B$ ,



then there is an element  $x^* \in B$  such that  $x^* = F(1, x^*)$ .

*Proof* For  $n \in \mathbb{N}$ , we consider the mapping

$$F_n(x) = \begin{cases} F(\frac{1-P_B(x)}{\epsilon_n}, \frac{x}{P_B(x)}), & \text{if } 1 - \epsilon_n \leq P_B(x) \leq 1, \\ F(1, \frac{x}{1-\epsilon_n}), & \text{if } P_B(x) < 1 - \epsilon_n, \end{cases} \quad (3)$$

where  $P_B$  is the Minkowski  $p$ -functional of  $B$  and  $\{\epsilon_n\}_{n \in \mathbb{N}}$  is a sequence of real numbers such that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  and  $0 < \epsilon_n < \frac{1}{2}$  for any  $n \in \mathbb{N}$ , and we also observe that the mapping  $F_n$  is 1-set contractive continuous with nonempty closed  $p$ -convex values on  $B$ . From assumption (2), we have that  $F_n(\partial B) \subset B$ , and the assumptions of Theorem 6.5 are satisfied, then for each  $n \in \mathbb{N}$ , there exists an element  $u_n \in B$  such that  $u_n = F_n(u_n)$ .

We first prove the following statement: “It is impossible to have an infinite number of the elements  $u_n$  satisfying the following inequality:  $1 - \epsilon_n \leq P_B(u_n) \leq 1$ .”

If not, we assume to have an infinite number of the elements  $u_n$  satisfying the following inequality:

$$1 - \epsilon_n \leq P_B(u_n) \leq 1.$$

As  $F_n(B)$  is relatively compact and by the definition of mappings  $F_n$ , we have that  $\{u_n\}_{n \in \mathbb{N}}$  is contained in a compact set in  $E$ . Without loss of generality (indeed, each compact set is also countably compact), we define the sequence  $\{t_n\}_{n \in \mathbb{N}}$  by  $t_n := \frac{1-P_B(u_n)}{\epsilon_n}$  for each  $n \in \mathbb{N}$ . Then we have that  $\{t_n\}_{n \in \mathbb{N}} \subset [0, 1]$ , and we may assume that  $\lim_{n \rightarrow \infty} t_n = t \in [0, 1]$ . The corresponding subsequence of  $\{u_n\}_{n \in \mathbb{N}}$  is denoted again by  $\{u_n\}_{n \in \mathbb{N}}$ , and it also satisfies the inequality  $1 - \epsilon_n \leq P_B(u_n) \leq 1$ , which implies that  $\lim_{n \rightarrow \infty} P_B(u_n) = 1$ .

Now let  $u^*$  be an accumulation point of  $\{u_n\}_{n \in \mathbb{N}}$ , thus we have  $\lim_{n \rightarrow \infty} (t_n, \frac{u_n}{P_B(u_n)}, u_n) = (t, u^*, u^*)$ . By the fact that  $F$  is compact, we assume that  $u_n = F(t_n, \frac{u_n}{P_B(u_n)})$  for each  $n \in \mathbb{N}$ . It follows that  $u^* = F(t, u^*)$ , this contradicts with assumption (1) as we have  $\lim_{n \rightarrow \infty} P_B(u_n) = 1$  (which means that  $u^* \in \partial B$ , this is impossible).

Thus it is impossible “to have an infinite number of elements  $u_n$  satisfy the inequality  $1 - \epsilon_n \leq P_B(u_n) \leq 1$ ”, which means that there is only a finite number of elements of sequence  $\{u_n\}_{n \in \mathbb{N}}$  satisfying the inequality  $1 - \epsilon_n \leq P_B(u_n) \leq 1$ . Now, without loss of generality, for  $n \in \mathbb{N}$ , we have the following inequality:

$$P_B(u_n) < 1 - \epsilon_n.$$

By the fact that  $\lim_{n \rightarrow \infty} (1 - \epsilon_n) = 1$ ,  $u_n \in F(1, \frac{u_n}{1-\epsilon_n})$  for all  $n \in \mathbb{N}$  and assume that  $\lim_{n \rightarrow \infty} u_n = u^*$ , then the continuity of  $F$  with nonempty closed values implies that by  $u_n = F(1, \frac{u_n}{1-\epsilon_n})$  for each  $n \in \mathbb{N}$ ,  $u^* = F(1, u^*)$ . This completes the proof.  $\square$

As a special case of Theorem 6.6, we have the following principle for the implicit form of Leray–Schauder type alternative in locally  $p$ -convex spaces for  $0 < p \leq 1$ .

**Corollary 6.1** (Implicit Leray–Schauder alternative) *Let  $E$  be a locally  $p$ -convex space  $E$ , where  $0 < p \leq 1$ ,  $B \subset E$  be a bounded closed  $p$ -convex such that  $0 \in \text{int} B$ . Let  $F : [0, 1] \times B \rightarrow E$  be 1-set contractive and continuous, satisfying condition (H) or (H1), and let the set  $F([0, 1] \times B)$  be relatively compact in  $E$ . If the following assumptions are satisfied:*

- (1)  $F(\{0\} \times \partial B) \subset B$ ,
- (2)  $x \neq F(0, x)$  for all  $x \in \partial B$ ,

then at least one of the following properties is satisfied:

- (i) There exists  $x^* \in B$  such that  $x^* = F(1, x^*)$ ; or
- (ii) There exists  $(\lambda^*, x^*) \in (0, 1) \times \partial B$  such that  $x^* = F(\lambda^*, x^*)$ .

*Proof* The result is an immediate consequence of Theorem 6.6, this completes the proof.  $\square$

We would like to point out that similar results on Rothe and Leray–Schauder alternative have been developed by Furi and Pera [37], Granas and Dugundji [46], Górniewicz [44], Górniewicz et al. [45], Isac [51], Li et al. [67], Liu [70], Park [85], Potter [97], Shahzad [109, 110], Xu [129], Xu et al. [130], and related references therein as tools of nonlinear analysis in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces, and some Birkhoff–Kellogg type theorems for general class mappings in topological vector spaces are also established by Agarwal et al. [1], Agarwal and O'Regan [2, 3], Park [87] (see the references therein for more details); and in particular, recently O'Regan [80] used the Leray–Schauder type coincidence theory to establish some Birkhoff–Kellogg problem, Furi–Pera type results for a general class of 1-set contractive mappings.

Before closing this section, we would like to share with readers that as the application of the best approximation result for 1-set contractive mappings, we just establish some fixed point theorems and the general principle of Leray–Schauder alternative for nonself mappings, which seem to play important roles in the nonlinear analysis under the framework of locally  $p$ -convex (seminorm) spaces, as the achievement of nonlinear analysis under the framework for underlying locally topological vector spaces, normed spaces, or in Banach spaces.

## 7 Fixed points for the class of 1-set contractive mappings

In this section, based on the best approximation Theorem 5.2 for classes of 1-set contractive mappings developed in Sect. 5, we show how it can be used as a useful tool to establish fixed point theorems for nonself upper semicontinuous mappings in locally  $p$ -convex (seminorm) spaces for  $p \in (0, 1]$ , which include norm spaces, uniformly convex Banach spaces as special classes.

By following Browder [15], Li [66], Goebel and Kirk [41], Petryshyn [93, 94], Tan and Yuan [117], Xu [129] and the references therein, we recall some definitions for  $p$ -seminorm spaces, where  $p \in (0, 1]$ .

**Definition 7.1** Let  $D$  be a nonempty (bounded) closed subset of locally  $p$ -convex spaces  $(E, \|\cdot\|_p)$ , where  $p \in (0, 1]$ . Suppose that  $f : D \rightarrow X$  is a (single-valued) mapping, then: (1)  $f$  is said to be nonexpansive if for each  $x, y \in D$ , we have  $\|f(x) - f(y)\|_p \leq \|x - y\|_p$ ; (2)  $f$  (actually,  $(I - f)$ ) is said to be demiclosed (see Browder [15]) at  $y \in X$  if for any sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $D$ , the conditions  $x_n \rightarrow x_0 \in D$  weakly and  $(I - f)(x_n) \rightarrow y_0$  strongly imply that  $(I - f)(x_0) = y_0$ , where  $I$  is the identity mapping; (3)  $f$  is said to be hemicompact (see p. 379 of Tan and Yuan [117]) if each sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $D$  has a convergent subsequence with the limit  $x_0$  such that  $x_0 = f(x_0)$ , whenever  $\lim_{n \rightarrow \infty} d_p(x_n, f(x_n)) = 0$ , here  $d_p(x_n, f(x_n)) :=$

$\inf\{P_U(x_n - z) : z \in f(x_n)\}$ , and  $P_U$  is the Minkowski  $p$ -functional for any  $U \in \mathcal{U}$ , which is the family of all nonempty open  $p$ -convex subsets containing the zero in  $E$ ; (4)  $f$  is said to be demicompact (by Petryshyn [93]) if each sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $D$  has a convergent subsequence whenever  $\{x_n - f(x_n)\}_{n \in \mathbb{N}}$  is a convergent sequence in  $X$ ; (5)  $f$  is said to be a semiclosed 1-set contractive mapping if  $f$  is 1-set contractive mapping, and  $(I - f)$  is closed, where  $I$  is the identity mapping (by Li [66]); and (6)  $f$  is said to be semicontractive (see Petryshyn [94] and Browder [15]) if there exists a mapping  $V : D \times D \rightarrow 2^X$  such that  $f(x) = V(x, x)$  for each  $x \in D$ , with (a) for each fixed  $x \in D$ ,  $V(\cdot, x)$  is nonexpansive from  $D$  to  $X$ ; and (b) for each fixed  $x \in D$ ,  $V(x, \cdot)$  is completely continuous from  $D$  to  $X$ , uniformly for  $u$  in a bounded subset of  $D$  (which means if  $v_j$  converges weakly to  $v$  in  $D$  and  $u_j$  is a bounded sequence in  $D$ , then  $V(u_j, v_j) - V(u_j, v) \rightarrow 0$ , strongly in  $D$ ).

From the definition above, we first observe that definitions (1) to (6) for set-valued mappings can be given in a similar way with the Hausdorff metric  $H$  (we omit their detailed definitions here to save space). Secondly, if  $f$  is a continuous demicompact mapping, then  $(I - f)$  is closed, where  $I$  is the identity mapping on  $X$ . It is also clear from the definitions that every demicompact map is hemicompact in seminorm spaces, but the converse is not true by the example in p. 380 by Tan and Yuan [117]. It is evident that if  $f$  is demicompact, then  $I - f$  is demiclosed. It is known that for each condensing mapping  $f$ , when  $D$  or  $f(D)$  is bounded, then  $f$  is hemicompact; and also  $f$  is demicompact in metric spaces by Lemma 2.1 and Lemma 2.2 of Tan and Yuan [117], respectively. In addition, it is known that every nonexpansive map is a 1-set-contractive mapping; and also if  $f$  is a hemicompact 1-set-contractive mapping, then  $f$  is a 1-set-contractive mapping satisfying the following (H1) condition (which is the same as “condition (H1)” in Sect. 5, but slightly different from condition (H) used there in Sect. 5):

(H1) condition: Let  $D$  be a nonempty bounded subset of a space  $E$  and assume  $F : \overline{D} \rightarrow 2^E$  to be a set-valued mapping. If  $\{x_n\}_{n \in \mathbb{N}}$  is any sequence in  $D$  such that for each  $x_n$  there exists  $y_n \in F(x_n)$  with  $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ , then there exists a point  $x \in \overline{D}$  such that  $x \in F(x)$ .

We first note that the “(H1) condition” above is actually the same as the “condition (C)” used by Theorem 1 of Petryshyn [94]. Secondly, it was shown by Browder [15] that indeed the nonexpansive mapping in a uniformly convex Banach space  $X$  enjoys condition (H1) as shown below.

**Lemma 7.1** *Let  $D$  be a nonempty bounded convex subset of a uniformly convex Banach space  $E$ . Assume that  $F : \overline{D} \rightarrow E$  is a nonexpansive (single-valued) mapping, then the mapping  $P := I - F$  defined by  $P(x) := (x - F(x))$  for each  $x \in \overline{D}$  is demiclosed, and in particular, the “(H1) condition” holds.*

*Proof* By following the argument given in p. 329 (see the proof of Theorem 2.2 and Corollary 2.1) by Petryshyn [94], the mapping  $F$  is demiclosed (which actually is called Browder’s demiclosedness principle), which says that by the assumption of (H1) condition, if  $\{x_n\}_{n \in \mathbb{N}}$  is any sequence in  $D$  such that for each  $x_n$  there exists  $y_n \in F(x_n)$  with  $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ , then we have  $0 \in (I - F)(\overline{D})$ , which means that there exists  $x_0 \in \overline{D}$  with  $0 \in (I - F)(x_0)$ , this implies that  $x_0 \in F(x_0)$ . The proof is complete.  $\square$

**Remark 7.1** When a  $p$ -vector space  $E$  is with a  $p$ -norm, then the “(H) condition” satisfies the “(H1) condition”. The (H1) condition is mainly supported by the so-called demiclosedness principle after the work by Browder [15].

By applying Theorem 5.2, we have the following result for nonself mappings in  $p$ -seminorm spaces for  $p \in (0, 1]$ .

**Theorem 7.1** *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex (or seminorm) space  $E$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1) above. In addition, for any  $x \in \partial \overline{U}$ , we have  $\lambda x \neq F(x)$  for any  $\lambda > 1$  (i.e., the “Leray–Schauder boundary condition”), then  $F$  has at least one fixed point.*

*Proof* By Theorem 5.2 with  $C = E$ , it follows that we have that either (I) or (II) holds:

(I)  $F$  has a fixed point  $x_0 \in U$ , i.e.,  $P_U(F(x_0) - x_0) = 0$ .

(II) There exists  $x_0 \in \partial(U)$  with  $P_U(F(x_0) - x_0) = (P_U^{\frac{1}{p}}(F(x_0)) - 1)^p > 0$ .

If  $F$  has no fixed point, then (II) holds and  $x_0 \neq F(x_0)$ . By the proof of Theorem 5.2, we have that  $x_0 = f(F(x_0))$  and  $F(x_0) \notin \overline{U}$ . Thus  $P_U(F(x_0)) > 1$  and  $x_0 = f(F(x_0)) = \frac{F(x_0)}{(P_U(F(x_0)))^{\frac{1}{p}}}$ ,

which means  $F(x_0) = (P_U(F(x_0)))^{\frac{1}{p}} x_0$ , where  $(P_U(F(x_0)))^{\frac{1}{p}} > 1$ , this contradicts the assumption. Thus  $F$  must have a fixed point. The proof is complete.  $\square$

By following the idea used and developed by Browder [15], Li [66], Li et al. [67], Goebel and Kirk [41], Petryshyn [93, 94], Tan and Yuan [117], Xu [129], Xu et al. [130] and the references therein, we have a number of existence theorems for the principle of Leray–Schauder type alternatives in locally  $p$ -convex spaces or  $p$ -seminorm spaces  $(E, \|\cdot\|_p)$  for  $p \in (0, 1]$  as follows.

**Theorem 7.2** *Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$  such that, for each  $x \in \partial \overline{U}$ , we have that for any  $y \in F(x)$ ,  $\|y - x\|_p^{\alpha/p} \geq \|y\|_p^{(\alpha+\beta)/p} \|x\|_p^{-\beta/p} - \|x\|_p^{\alpha/p}$ . Then  $F$  has at least one fixed point.*

*Proof* We prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume  $F$  has no fixed point, by the boundary condition of Theorem 7.1, there exist  $x_0 \in \partial \overline{U}$ ,  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t-1)^\alpha - t^{\alpha+\beta} + 1$  for  $t \geq 1$ . We observe that  $f$  is a strictly decreasing function for  $t \in [1, \infty)$  as the derivative of  $f'(t) = \alpha(t-1)^{\alpha-1} - (\alpha+\beta)t^{\alpha+\beta-1} < 0$  by the differentiation, thus we have  $t^{\alpha+\beta} - 1 > (t-1)^\alpha$  for  $t \in (1, \infty)$ . By combining the boundary condition, we have that  $\|F(x_0) - x_0\|_p^{\alpha/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{\alpha/p} < (\lambda_0^{\alpha+\beta} - 1) \|x_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} = \|F(x_0)\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} - \|x_0\|_p^{\alpha/p}$ , which contradicts the boundary condition given by Theorem 7.2. Thus, the conclusion follows and the proof is complete.  $\square$

**Theorem 7.3** *Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a 1-set contractive and continuous*

mapping satisfying condition (H) or (H1). In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$  such that, for each  $x \in \partial \overline{U}$ , we have that  $\|F(x) + x\|_p^{(\alpha+\beta)/p} \leq \|F(x)\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|x\|_p^{(\alpha+\beta)/p}$ . Then  $F$  has at least one fixed point.

*Proof* We prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume  $F$  has no fixed point, by the boundary condition of Theorem 7.1, there exist  $x_0 \in \partial \overline{U}$  and  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t+1)^{\alpha+\beta} - t^\alpha - 1$  for  $t \geq 1$ . We then can show that  $f$  is a strictly increasing function for  $t \in [1, \infty)$ , thus we have  $t^\alpha + 1 < (t+1)^{\alpha+\beta}$  for  $t \in (1, \infty)$ . By the boundary condition given in Theorem 7.3, we have that

$$\begin{aligned} \|F(x_0) + x_0\|_p^{(\alpha+\beta)/p} &= (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > (\lambda_0^\alpha + 1) \|x_0\|_p^{(\alpha+\beta)/p} \\ &= \|F(x_0)\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|x_0\|_p^{\alpha/p}, \end{aligned}$$

which contradicts the boundary condition given by Theorem 7.3. Thus, the conclusion follows and the proof is complete.  $\square$

**Theorem 7.4** Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F: \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$  (or, alternatively,  $\alpha > 1$ ,  $\beta \geq 0$ ) such that, for each  $x \in \partial \overline{U}$ , we have that  $\|F(x) - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} \geq \|F(x)\|_p^{\alpha/p} \|F(x) + x\|_p^{\beta/p} - \|x\|_p^{(\alpha+\beta)/p}$ . Then  $F$  has at least one fixed point.

*Proof* The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume  $F$  has no fixed point, by the boundary condition of Theorem 7.1, there exist  $x_0 \in \partial \overline{U}$  and  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t-1)^\alpha - t^\alpha(t-1)^\beta + 1$  for  $t \geq 1$ . We then can show that  $f$  is a strictly decreasing function for  $t \in [1, \infty)$ , thus we have  $(t-1)^\alpha < t^\alpha(t+1)^\beta - 1$  for  $t \in (1, \infty)$ .

By the boundary condition given in Theorem 7.4, we have that

$$\begin{aligned} \|F(x_0) - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} &= (\lambda_0 - 1)^\alpha \|x_0\|_p^{(\alpha+\beta)/p} < (\lambda_0^\alpha (\lambda_0 + 1)^\beta - 1) \|x_0\|_p^{(\alpha+\beta)/p} \\ &= \|F(x_0)\|_p^{\alpha/p} \|F(x_0) + x_0\|_p^{\beta/p} - \|x_0\|_p^{(\alpha+\beta)/p}, \end{aligned}$$

which contradicts the boundary condition given by Theorem 7.4. Thus, the conclusion follows and the proof is complete.  $\square$

**Theorem 7.5** Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F: \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$ , we have that  $\|F(x) + x\|_p^{(\alpha+\beta)/p} \leq \|F(x) - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|F(x)\|_p^{\beta/p} \|x\|_p^{\alpha/p}$ . Then  $F$  has at least one fixed point.

*Proof* The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume  $F$  has no fixed point, by

the boundary condition of Theorem 7.1, there exist  $x_0 \in \partial \overline{U}$  and  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t+1)^{\alpha+\beta} - (t-1)^\alpha - t^\beta$  for  $t \geq 1$ . We then can show that  $f$  is a strictly increasing function for  $t \in [1, \infty)$ , thus we have  $(t+1)^{\alpha+\beta} > (t-1)^\alpha + t^\beta$  for  $t \in (1, \infty)$ .

By the boundary condition given in Theorem 7.5, we have that  $\|F(x_0) + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > ((\lambda_0 - 1)^\alpha + \lambda_0^\beta) \|x_0\|_p^{(\alpha+\beta)/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|\lambda_0 x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} = \|F(x_0) - x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} + \|F(x_0)\|_p^{\beta/p} \|x_0\|_p^{\alpha/p}$ , which implies that

$$\|F(x_0) + x_0\|_p^{(\alpha+\beta)/p} > \|F(x_0) - x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} + \|F(x_0)\|_p^{\beta/p} \|x_0\|_p^{\alpha/p},$$

this contradicts the boundary condition given by Theorem 7.5. Thus, the conclusion follows and the proof is complete.  $\square$

As an application of Theorem 7.1 by testing the Leray–Schauder boundary condition, we have the following conclusion for each special case, and thus we omit their detailed proofs here.

**Corollary 7.1** *Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). Then  $F$  has at least one fixed point if one of the following conditions holds for  $x \in \partial \overline{U}$ :*

- (i)  $\|F(x)\|_p \leq \|x\|_p$ ,
- (ii)  $\|F(x)\|_p \leq \|F(x) - x\|_p$ ,
- (iii)  $\|F(x) + x\|_p \leq \|F(x)\|_p$ ,
- (iv)  $\|F(x) + x\|_p \leq \|x\|_p$ ,
- (v)  $\|F(x) + x\|_p \leq \|F(x) - x\|_p$ ,
- (vi)  $\|F(x)\|_p \cdot \|F(x) + x\|_p \leq \|x\|_p^2$ ,
- (vii)  $\|F(x)\|_p \cdot \|F(x) + x\|_p \leq \|F(x) - x\|_p \cdot \|x\|_p$ .

If the  $p$ -seminorm space  $E$  is a uniformly convex Banach space  $(E, \|\cdot\|)$  (for  $p$ -norm space with  $p = 1$ ), then we have the following general existence result (which actually is true for nonexpansive set-valued mappings).

**Theorem 7.6** *Let  $U$  be a bounded open convex subset of a uniformly convex Banach space  $(E, \|\cdot\|)$  (with  $p = 1$ ) with zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a semicontractive and continuous single-valued mapping with nonempty values. In addition, for any  $x \in \partial \overline{U}$ , we have  $\lambda x \neq F(x)$  for any  $\lambda > 1$  (i.e., the “Leray–Schauder boundary condition”). Then  $F$  has at least one fixed point.*

*Proof* By the assumption that  $F$  is a semicontractive and continuous single-valued mapping with nonempty values, it follows by Lemma 3.2 in p. 338 of Petryshyn [94] that  $f$  is a 1-set contractive single-valued mapping. Moreover, by the assumption that  $E$  is a uniformly convex Banach space, indeed  $(I - F)$  is closed at zero, i.e.,  $F$  is semiclosed (see Browder [15] or Goebel and Kirk [41]). Thus all assumptions of Theorem 7.1 are satisfied with the (H1) condition. The conclusion follows by Theorem 7.1, and the proof is complete.  $\square$



Lemma 7.1 shows that a single-valued nonexpansive mapping defined in a uniformly convex Banach space (see also Theorem 7.6) satisfies the (H1) condition. Actually, the nonexpansive set-valued mappings defined on a special class of Banach spaces with the so-called “Opial condition” do not only satisfy condition (H1), but also belong to the classes of semiclosed 1-set contractive mappings as shown below.

Now let  $K(X)$  denote the family of all nonempty compact convex subsets of a topological vector space  $X$ . The notion of the so-called “Opial condition” first given by Opial [79] says that a Banach space  $X$  is said to satisfy the Opial condition if  $\liminf_{n \rightarrow \infty} \|w_n - w\| < \liminf_{n \rightarrow \infty} \|w_n - p\|$  whenever  $(w_n)$  is a sequence in  $X$  weakly convergent to  $w$  and  $p \neq w$ , we know that the Opial condition plays an important role in the fixed point theory, e.g., see Lami Dozo [64], Goebel and Kirk [42], Xu [127], and the references therein. The following result shows that there exist nonexpansive set-valued mappings in Banach spaces with the Opial condition (see Lami Dozo [64] satisfying condition (H1)).

**Lemma 7.2** *Let  $C$  be a convex weakly compact subset of a Banach space  $X$  which satisfies the Opial condition. Let  $T : C \rightarrow K(C)$  be a nonexpansive set-valued mapping with nonempty compact values. Then the graph of  $(I - T)$  is closed in  $(X, \sigma(X, X^*) \times (X, \|\cdot\|))$ , thus  $T$  satisfies the “(H1) condition”, where  $I$  denotes the identity on  $X$ ,  $\sigma(X, X^*)$  the weak topology, and  $\|\cdot\|$  the norm (or strong) topology.*

*Proof* By following Theorem 3.1 of Lami Dozo [64], it follows that the mapping  $T$  is demiclosed, thus  $T$  satisfies the “(H1) condition”. The proof is complete.  $\square$

As an application of Lemma 7.2, we have the following results for nonexpansive mappings.

**Theorem 7.7** *Let  $C$  be a nonempty convex weakly compact subset of a Banach space  $X$  which satisfies the Opial condition and  $0 \in \text{int } C$ . Let  $T : C \rightarrow K(X)$  be a nonexpansive set-valued mapping with nonempty compact convex values. In addition, for any  $x \in \partial \overline{C}$ , we have  $\lambda x \neq F(x)$  for any  $\lambda > 1$  (i.e., the “Leray–Schauder boundary condition”). Then  $F$  has at least one fixed point.*

*Proof* As  $T$  is nonexpansive, it is 1-set contractive. By Lemma 7.1, it is then semicontractive and continuous. Then the (H1) condition of Theorem 7.1 is satisfied. The conclusion follows by Theorem 7.1, and the proof is complete.  $\square$

Before the end of this section, by considering the  $p$ -seminorm space  $(E, \|\cdot\|)$  is a seminorm space with  $p = 1$ , the following result is a special case of the corresponding results from Theorem 7.2 to Theorem 7.5, and thus we omit its proof.

**Corollary 7.2** *Let  $U$  be a bounded open convex subset of a normed space  $(E, \|\cdot\|)$ . Assume that  $F : \overline{U} \rightarrow E$  is a 1-set contractive and continuous mapping satisfying condition (H) or (H1). Then  $F$  has at least one fixed point if there exist  $\alpha > 1$ ,  $\beta \geq 0$  such that any one of the following conditions is satisfied:*

- (i) For each  $x \in \partial \overline{U}$ ,  $\|F(x) - x\|^\alpha \geq \|F(x)\|^{(\alpha+\beta)} \|x\|^{-\beta} - \|x\|^\alpha$ ,
- (ii) For each  $x \in \partial \overline{U}$ ,  $\|F(x) + x\|^{(\alpha+\beta)} \leq \|F(x)\|^\alpha \|x\|^\beta + \|x\|^{(\alpha+\beta)}$ ,
- (iii) For each  $x \in \partial \overline{U}$ ,  $\|F(x) - x\|^\alpha \|x\|^\beta \geq \|F(x)\|^\alpha \|F(x) + x\|^\beta - \|x\|^{(\alpha+\beta)}$ ,

(iv) For each  $x \in \partial \overline{U}$ ,  $\|F(x) + x\|^{(\alpha+\beta)} \leq \|F(x) - x\|^\alpha \|x\|^\beta + \|F(x)\|^\beta \|x\|^\alpha$ .

**Remark 7.2** As discussed by Lemma 7.1 and the proof of Theorem 7.6, when the  $p$ -vector space is a uniformly convex Banach space, the semicontractive or nonexpansive mappings automatically satisfy condition (H) or (H1). Moreover, our results from Theorem 7.1 to Theorem 7.6, Corollary 7.1 and Corollary 7.2 also improve or unify the corresponding results given by Browder [15], Li [66], Li et al. [67], Goebel and Kirk [41], Petryshyn [93, 94], Reich [99], Tan and Yuan [117], Xu [126], Xu [129], Xu et al. [130], and results from the reference therein by extending the nonself mappings to the classes of 1-set contractive set-valued mappings in  $p$ -seminorm spaces with  $p \in (0, 1]$  (including the normed space or Banach space when  $p = 1$ , and for  $p$ -seminorm spaces).

## 8 Fixed points for the class of semiclosed 1-set contractive mappings in $p$ -seminorm spaces

In order to study the fixed point theory for a class of semiclosed 1-set contractive mappings in  $p$ -seminorm spaces, we first introduce the following definition which is a set-valued generalization of single-value semiclosed 1-set mappings first discussed by Li [66], Xu [129] (see also Li et al. [67], Xu et al. [130], and the references therein).

**Definition 8.1** Let  $D$  be a nonempty (bounded) closed subset of  $p$ -vector space  $(E, \|\cdot\|_p)$  with  $p$ -seminorm for  $p$ -vector spaces, where  $p \in (0, 1]$  (which include norm spaces or Banach spaces as special classes), and suppose that  $T : D \rightarrow X$  is a set-valued mapping. Then  $T$  is said to be a semiclosed 1-set contraction mapping if  $T$  is 1-set contraction and  $(I - T)$  is closed, which means that for a given net  $\{x_n\}_{n \in I}$ , for each  $i \in I$ , there exists  $y_i \in T(x_i)$  with  $\lim_{i \in I} (x_i - y_i) = 0$ , then  $0 \in (I - T)(\overline{D})$ , i.e., there exists  $x_0 \in \overline{D}$  such that  $x_0 \in T(x_0)$ .

**Remark 8.1** By Lemmas 7.1 and 7.2, it follows that each nonexpansive (single-valued) mapping defined on a subset of uniformly convex Banach spaces and nonexpansive set-valued mappings defined on a subset of Banach spaces satisfying the Opial condition are semiclosed 1-set contractive mappings (see also Goebel [40], Goebel and Kirk [41], Petrusel et al. [92], Xu [127], Yangai [131] for related discussion and the references therein). In particular, under the setting of metric spaces or Banach spaces with certain property, it is clear that each semiclosed 1-set contractive mapping satisfies condition (H1).

Although we know that compared to the single-valued case, based on the study in the literature about the approximation of fixed points for multi-valued mappings, a well-known counterexample due to Pietramala [95] (see also Muglia and Marino [75]) proved in 1991 that Browder approximation Theorem 1 given by Browder [13] cannot be extended to the genuine multivalued case even on a finite dimensional space  $\mathbb{R}^2$ . Moreover, if a Banach space  $X$  satisfies the Opial property (see Opial [79]) that is, if  $x_n$  weakly converges to  $x$ , then we have that  $\limsup \|x_n - x\| < \limsup \|x_n - y\|$  for all  $x \in X$  and  $y \neq x$ , then  $I - f$  is demiclosed at 0 (see Lami Dozo [64], Yanagi [131], and related references therein) provided  $f : C \rightarrow K(C)$  is nonexpansive (here  $K(C)$  denotes the family of nonempty compact subsets of  $C$ ). We know that all Hilbert spaces and  $L^p$  spaces  $p \in (1, \infty)$  have the Opial property, but it seems that whether  $I - f$  is demiclosed at zero 0 if  $f$  is a nonexpansive set-valued mapping defined on the space  $X$  which is uniformly convex (e.g.,  $L[0, 1]$ ,  $1 < p < \infty$ ,

$\neq 2$ ) and  $f : C \rightarrow K(C)$  is nonexpansive. Here we remark that for a single-valued nonexpansive mapping  $f$  is yes, which is the famous theorem of Browder [12]. A remarkable fixed point theorem for multi-valued mappings is Lim's result in [69], which says that: If  $C$  is a nonempty closed bounded convex subset of a uniformly convex Banach space  $X$  and  $f : C \rightarrow K(C)$  is nonexpansive, then  $f$  has a fixed point.

Now, based on the concept for the semiclosed 1-set contractive mappings, we give the existence results for their best approximation, fixed points, and related nonlinear alternative under the framework of  $p$ -seminorm spaces for  $p \in (0, 1]$ .

**Theorem 8.1** (Schauder fixed point theorem for semiclosed 1-set contractive mappings) *Let  $U$  be a nonempty bounded open subset of a (Hausdorff) locally  $p$ -convex space  $E$  and its zero  $0 \in U$ , and  $C \subset E$  be a closed  $p$ -convex subset of  $E$  such that  $0 \in C$  with  $0 < p \leq 1$ . If  $F : C \cap \overline{U} \rightarrow C \cap \overline{U}$  is continuous and semiclosed 1-set contractive, then  $T$  has at least one fixed point in  $C \cap \overline{U}$ .*

*Proof* As the mapping  $T$  is 1-set contractive, take an increasing sequence  $\{\lambda_n\}$  such that  $0 < \lambda_n < 1$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ , where  $n \in \mathbb{N}$ . Now we define a mapping  $F_n : C \rightarrow C$  by  $F_n(x) := \lambda_n F(x)$  for each  $x \in C$  and  $n \in \mathbb{N}$ . Then it follows that  $F_n$  is a  $\lambda_n$ -set-contractive mapping with  $0 < \lambda_n < 1$ . By Theorem 4.5 on the condensing mapping  $F_n$  in a  $p$ -vector space with  $p$ -seminorm  $P_U$  for each  $n \in \mathbb{N}$ , there exists  $x_n \in C$  such that  $x_n \in F_n(x_n) = \lambda_n F(x_n)$ . Thus we have  $x_n = \lambda_n F(x_n)$ . Let  $P_U$  be the Minkowski  $p$ -functional of  $U$  in  $E$ . It follows that  $P_U$  is continuous as  $0 \in \text{int}(U) = U$ . Note that for each  $n \in \mathbb{N}$ ,  $\lambda_n x_n \in \overline{U} \cap C$ , which imply that  $x_n = r(\lambda_n F(x_n)) = \lambda_n F(x_n)$ , thus  $P_U(\lambda_n F(x_n)) \leq 1$  by Lemma 2.2. Note that

$$\begin{aligned} P_U(F(x_n) - x_n) &= P_U(F(x_n) - \lambda_n F(x_n)) = P_U\left(\frac{(1 - \lambda_n)\lambda_n F(x_n)}{\lambda_n}\right) \\ &\leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n F(x_n)) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that  $\lim_{n \rightarrow \infty} P_U(F(x_n) - x_n) = 0$ . Now, by the assumption that  $F$  is semiclosed, which means that  $(I - F)$  is closed at zero, there exists one point  $x_0 \in \overline{C}$  such that  $0 \in (I - F)(\overline{C})$ , thus we have that  $x_0 = F(x_0)$ .

Indeed, without loss of generality, we assume that  $\lim_{n \rightarrow \infty} x_n = x_0$  with  $x_n = \lambda_n F(x_n)$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ , which implies that  $x_0 = \lim_{n \rightarrow \infty} (\lambda_n F(x_n))$ , which means  $F(x_0) := \lim_{n \rightarrow \infty} F(x_n) = x_0$ , thus  $x_0 = F(x_0)$ . We complete the proof.  $\square$

**Theorem 8.2** (Best approximation for semiclosed 1-set contractive mappings) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  ( $0 \leq p \leq 1$ ) with the zero  $0 \in U$ , and let  $C$  be a (bounded) closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ . Assume  $F : \overline{U} \cap C \rightarrow C$  is a semiclosed 1-set contractive and continuous mapping, and for each  $x \in \partial_C U$  with  $F(x) \notin \overline{U}$ ,  $(P_U^{\frac{1}{p}}(F(x)) - 1)^p \leq P_U(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ). Then we have that there exist  $x_0 \in C \cap \overline{U}$  and  $F(x_0)$  such that  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C)$ , where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . More precisely, we have that either (I) or (II) holds:*

- (I)  $F$  has a fixed point  $x_0 \in U \cap C$ , i.e.,  $x_0 = F(x_0)$  (so that  $0 = P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C)$ );

(II) There exist  $x_0 \in \partial_C(U)$  and  $F(x_0) \notin \overline{U}$  with

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(F(x_0)) - 1)^p > 0.$$

*Proof* Let  $r : E \rightarrow U$  be a retraction mapping defined by  $r(x) := \frac{x}{\max\{1, (P_U(x))^{\frac{1}{p}}\}}$  for each  $x \in E$ , where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . Since the space  $E$ 's zero  $0 \in U$  ( $= \text{int } U$  as  $U$  is open), it follows that  $r$  is continuous by Lemma 2.2. As the mapping  $F$  is 1-set contractive, taking an increasing sequence  $\{\lambda_n\}$  such that  $0 < \lambda_n < 1$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ , where  $n \in \mathbb{N}$ . Now we define a mapping  $F_n : C \cap \overline{U} \rightarrow C$  by  $F_n(x) := \lambda_n F \circ r(x)$  for each  $x \in C \cap \overline{U}$  and  $n \in \mathbb{N}$ . Then it follows that  $F_n$  is a  $\lambda_n$ -set-contractive mapping with  $0 < \lambda_n < 1$  for each  $n \in \mathbb{N}$ . As  $C$  and  $\overline{U}$  are  $p$ -convex, we have  $r(C) \subset C$  and  $r(\overline{U}) \subset \overline{U}$ , so  $r(C \cap \overline{U}) \subset C \cap \overline{U}$ . Thus  $F_n$  is a self-mapping defined on  $C \cap \overline{U}$ . By Theorem 4.5 for condensing mapping  $F_n$ , for each  $n \in \mathbb{N}$ , there exists  $z_n \in C \cap \overline{U}$  such that  $z_n \in F_n(z_n) = \lambda_n F \circ r(z_n)$ . Let  $x_n = r(z_n)$ , then we have  $x_n \in C \cap \overline{U}$  with  $x_n = r(\lambda_n F(x_n))$  such that (1) or (2) holds for each  $n \in \mathbb{N}$ :

(1):  $\lambda_n F(x_n) \in C \cap \overline{U}$ ; or (2):  $\lambda_n F(x_n) \in C \setminus \overline{U}$ .

Now we prove the conclusion by considering the following two cases:

Case (I): For each  $n \in \mathbb{N}$ ,  $\lambda_n F(x_n) \in C \cap \overline{U}$ ; or

Case (II): There exists a positive integer  $n$  such that  $\lambda_n F(x_n) \in C \setminus \overline{U}$ .

First, by case (I), for each  $n \in \mathbb{N}$ ,  $\lambda_n F(x_n) \in \overline{U} \cap C$ , which implies that  $x_n = r(\lambda_n F(x_n)) = \lambda_n F(x_n)$ , thus  $P_U(\lambda_n F(x_n)) \leq 1$  by Lemma 2.2. Note that

$$\begin{aligned} P_U(F(x_n) - x_n) &= P_U(F(x_n) - \lambda_n F(x_n)) = P_U\left(\frac{(1 - \lambda_n)\lambda_n F(x_n)}{\lambda_n}\right) \\ &\leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n F(x_n)) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that  $\lim_{n \rightarrow \infty} P_U(F(x_n) - x_n) = 0$ . Now, by the fact that  $F$  is semiclosed, it implies that there exists a point  $x_0 \in \overline{U}$  (i.e., the consequence  $\{x_n\}_{n \in \mathbb{N}}$  has a convergent subsequence with the limit  $x_0$ ) such that  $x_0 = F(x_0)$ . Indeed, without loss of generality, we assume that  $\lim_{n \rightarrow \infty} x_n = x_0$  with  $x_n = \lambda_n F(x_n)$  and  $\lim_{n \rightarrow \infty} \lambda_n = 1$ , and as  $x_0 = \lim_{n \rightarrow \infty} (\lambda_n F(x_n))$ , it implies that  $F(x_0) = \lim_{n \rightarrow \infty} F(x_n) = x_0$ . Thus there exists  $F(x_0) = x_0$ , we have  $0 = d_p(x_0, F(x_0)) = d(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C)$  as indeed  $x_0 = F(x_0) \in \overline{U} \cap C \subset \overline{I_U^p(x_0)} \cap C$ .

Second, by case (II) there exists a positive integer  $n$  such that  $\lambda_n F(x_n) \in C \setminus \overline{U}$ . Then we have that  $P_U(\lambda_n F(x_n)) > 1$ , and also  $P_U(F(x_n)) > 1$  as  $\lambda_n < 1$ . As  $x_n = r(\lambda_n F(x_n)) = \frac{\lambda_n F(x_n)}{(P_U(\lambda_n F(x_n)))^{\frac{1}{p}}}$ , it implies that  $P_U(x_n) = 1$ , thus  $x_n \in \partial_C(U)$ . Note that

$$P_U(F(x_n) - x_n) = P_U\left(\frac{(P_U(F(x_n))^{\frac{1}{p}} - 1)F(x_n)}{P_U(F(x_n))^{\frac{1}{p}}}\right) = (P_U^{\frac{1}{p}}(F(x_n)) - 1)^p.$$

By the assumption, we have  $(P_U^{\frac{1}{p}}(F(x_n)) - 1)^p \leq P_U(F(x_n) - x)$  for  $x \in C \cap \partial \overline{U}$ , it follows that

$$\begin{aligned} P_U(F(x_n)) - 1 &\leq P_U(F(x_n)) - \sup\{P_U(z) : z \in C \cap \overline{U}\} \\ &\leq \inf\{P_U(F(x_n) - z) : z \in C \cap \overline{U}\} = d_p(F(x_n), C \cap \overline{U}). \end{aligned}$$

Thus we have the best approximation:  $P_U(F(x_n) - x_n) = d_p(F(x_n), \overline{U} \cap C) = (P_U^{\frac{1}{p}}(F(x_n)) - 1)^p > 0$ .

Now we want to show that  $P_U(F(x_n) - x_n) = d_p(F(x_n), \overline{U} \cap C) = d_p(F(x_n), \overline{I_U^p(x_0)} \cap C) > 0$ .

By the fact that  $(\overline{U} \cap C) \subset I_U^p(x_n) \cap C$ , let  $z \in I_U^p(x_n) \cap C \setminus (\overline{U} \cap C)$ , we first claim that  $P_U(F(x_n) - x_n) \leq P_U(F(x_n) - z)$ . If not, we have  $P_U(F(x_n) - x_n) > P_U(F(x_n) - z)$ . As  $z \in I_U^p(x_n) \cap C \setminus (\overline{U} \cap C)$ , there exist  $y \in \overline{U}$  and a nonnegative number  $c$  (actually  $c \geq 1$  as shown soon below) with  $z = x_n + c(y - x_n)$ . Since  $z \in C$ , but  $z \notin \overline{U} \cap C$ , it implies that  $z \notin \overline{U}$ . By the fact that  $x_n \in \overline{U}$  and  $y \in \overline{U}$ , we must have the constant  $c \geq 1$ ; otherwise, it implies that  $z = (1 - c)x_n + cy \in \overline{U}$ , this is impossible by our assumption, i.e.,  $z \notin \overline{U}$ . Thus we have that  $c \geq 1$ , which implies that  $y = \frac{1}{c}z + (1 - \frac{1}{c})x_n \in C$  (as both  $x_n \in C$  and  $z \in C$ ). On the other hand, as  $z \in I_U^p(x_n) \cap C \setminus (\overline{U} \cap C)$  and  $c \geq 1$  with  $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$ , combining with our assumption that for each  $x \in \partial_C \overline{U}$  and  $F(x) \notin \overline{U}$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$ , it then follows that

$$\begin{aligned} P_U(F(x_n) - y) &= P_U\left[\frac{1}{c}(F(x_n) - z) + \left(1 - \frac{1}{c}\right)(F(x_n) - x_n)\right] \\ &\leq \left[\left(\frac{1}{c}\right)^p P_U(F(x_n) - z) + \left(1 - \frac{1}{c}\right)^p P_U(F(x_n) - x_n)\right] \\ &< P_U(F(x_n) - x_n), \end{aligned}$$

which contradicts that  $P_U(F(x_n) - x_n) = d_p(F(x_n), \overline{U} \cap C)$  as shown above. We know that  $y \in \overline{U} \cap C$ , we should have  $P_U(F(x_n) - x_n) \leq P_U(F(x_n) - y)$ ! This helps us to complete the claim  $P_U(F(x_n) - x_n) \leq P_U(F(x_n) - z)$  for any  $z \in I_U^p(x_n) \cap C \setminus (\overline{U} \cap C)$ , which means that the following best approximation of Fan type (see [34, 35]) holds:

$$0 < d_p(F(x_n), \overline{U} \cap C) = P_U(F(x_n) - x_n) = d_p(F(x_n), I_U^p(x_n) \cap C).$$

Now, by the continuity of  $P_U$ , it follows that the following best approximation of Fan type is also true:

$$0 < P_U(F(x_n) - x_n) = d_p(F(x_n), \overline{U} \cap C) = d_p(F(x_n), I_U^p(x_n) \cap C) = d_p(F(x_n), \overline{I_U^p(x_0)} \cap C),$$

and we have that

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), I_U^p(x_0) \cap C) > 0.$$

The proof is complete.  $\square$

For a  $p$ -vector space when  $p = 1$ , we have the following best approximation for LCS.

**Theorem 8.3** (Best approximation for LCS) *Let  $U$  be a bounded open convex subset of a locally convex space  $E$  (i.e.,  $p = 1$ ) with zero  $0 \in \text{int } U = U$  (the interior  $\text{int } U = U$  as  $U$  is open), and let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ . Assume that  $F : \overline{U} \cap C \rightarrow C$  is a semiclosed 1-set-contractive continuous mapping. Then there exists  $x_0 \in \overline{U} \cap X$  such that  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C)$ , where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . More precisely, we have that either (I) or (II) holds:*

- (I)  $F$  has a fixed point  $x_0 \in U \cap C$ , i.e.,  $x_0 = F(x_0)$  (so that  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = 0$ );
- (II) There exist  $x_0 \in \partial_C(U)$  and  $F(x_0) \notin \overline{U}$  with

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) > 0.$$

*Proof* By applying Theorem 5.2 and the same argument used by Theorem 8.2, the conclusion follows. This completes the proof.  $\square$

Now, by the application of Theorems 8.2 and 8.3, we have the following general principle for the existence of solutions for Birkhoff–Kellogg problems in  $p$ -seminorm spaces, where  $(0 < p \leq 1)$ .

**Theorem 8.4** (Principle of Birkhoff–Kellogg alternative) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  ( $0 \leq p \leq 1$ ) with zero  $0 \in \text{int } U = (U)$  (the interior  $\text{int } U$  as  $U$  is open), and let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ . Assume that  $F : \overline{U} \cap C \rightarrow C$  is a semiclosed 1-set-contractive continuous mapping. Then  $F$  has at least one of the following two properties:*

- (I)  $F$  has a fixed point  $x_0 \in U \cap C$  such that  $x_0 = F(x_0)$ , or
- (II) There exist  $x_0 \in \partial_C(U)$  and  $F(x_0) \notin \overline{U}$ , and  $\lambda = \frac{1}{(P_U(F(x_0)))^{\frac{1}{p}}} \in (0, 1)$  such that

$x_0 = \lambda F(x_0)$ . In addition, if for each  $x \in \partial_C U$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ), then the best approximation between  $x_0$  and  $F(x_0)$  is given by

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) = (P_U^{\frac{1}{p}}(F(x_0)) - 1)^p > 0.$$

*Proof* If (I) is not the case, then (II) is proved by Remark 5.2 and by following the proof in Theorem 8.2 for case (ii):  $F(x_0) \in C \setminus \overline{U}$  with  $y_0 = f(F(x_0))$ , where  $f$  is the restriction of the continuous mapping  $r$  restriction to the subset  $U$  in  $E$ . Indeed, as  $y_0 \notin \overline{U}$ , it follows that  $P_U(y_0) > 1$  and  $x_0 = f(y_0) = F(x_0) \frac{1}{(P_U(F(x_0)))^{\frac{1}{p}}}$ . Now let  $\lambda = \frac{1}{(P_U(F(x_0)))^{\frac{1}{p}}}$ , we have  $\lambda < 1$  and  $x_0 = \lambda F(x_0)$ . Finally, the additionally assumption in (II) allows us to have the best approximation between  $x_0$  and  $F(x_0)$  obtained by following the proof of Theorem 8.2 as  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) > 0$ . This completes the proof.  $\square$

As an application of Theorem 8.2 for the nonself mappings, we have the following general principle of Birkhoff–Kellogg alternative in TVS.

**Theorem 8.5** (Principle of Birkhoff–Kellogg alternative in LCS) *Let  $U$  be a bounded open  $p$ -convex subset of the LCS  $E$  with the zero  $0 \in U$ , and let  $C$  be a closed convex subset of  $E$  with also zero  $0 \in C$ . Assume that  $F : \overline{U} \cap C \rightarrow C$  is a semiclosed 1-set contractive and continuous mapping. Then it has at least one of the following two properties:*

- (I)  $F$  has a fixed point  $x_0 \in U \cap C$  such that  $x_0 = F(x_0)$ ; or
- (II) There exist  $x_0 \in \partial_C(U)$  and  $F(x_0) \notin \overline{U}$  and  $\lambda \in (0, 1)$  such that  $x_0 = \lambda F(x_0)$ , and the best approximation between  $\{x_0\}$  and  $F(x_0)$  is given by
- $$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U^p(x_0)} \cap C) > 0.$$



On the other hand, by the proof of Theorem 8.2, we note that for case (II) of Theorem 8.2, the assumption “each  $x \in \partial_C U$  with  $y \in F(x)$ ,  $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ ” is only used to guarantee the best approximation “ $P_U(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{P_U^{\frac{1}{p}}(x_0)} \cap C) > 0$ ”, thus we have the following Leray–Schauder alternative in  $p$ -vector spaces, which, of course, includes the corresponding results in locally convex spaces as special cases.

**Theorem 8.6** (Leray–Schauder nonlinear alternative) *Let  $C$  be a closed  $p$ -convex subset of a  $P$ -seminorm space  $E$  with  $0 \leq p \leq 1$  and the zero  $0 \in C$ . Assume that  $F : C \rightarrow C$  is a semi-closed 1-set contractive and continuous mapping. Let  $\varepsilon(F) := \{x \in C : x \in \lambda F(x) \text{ for some } 0 < \lambda < 1\}$ . Then either  $F$  has a fixed point in  $C$  or the set  $\varepsilon(F)$  is unbounded.*

*Proof* By assuming that case (I) is not true, i.e.,  $F$  has no fixed point, we claim that the set  $\varepsilon(F)$  is unbounded. Otherwise, assume that the set  $\varepsilon(F)$  is bounded, and assume that  $P$  is the continuous  $p$ -seminorm for  $E$ , then there exists  $r > 0$  such that the set  $B(0, r) := \{x \in E : P(x) < r\}$ , which contains the set  $\varepsilon(F)$ , i.e.,  $\varepsilon(F) \subset B(0, r)$ , which means for any  $x \in \varepsilon(F)$ ,  $P(x) < r$ . Then  $B(0, r)$  is an open  $p$ -convex subset of  $E$  and the zero  $0 \in B(0, r)$  by Lemma 2.2 and Remark 2.4. Now let  $U := B(0, r)$  in Theorem 8.4, it follows that the mapping  $F : B(0, r) \cap C \rightarrow 2^C$  satisfies all general conditions of Theorem 8.4, and we have that any  $x_0 \in \partial_C B(0, r)$ , no any  $\lambda \in (0, 1)$  such that  $x_0 = \lambda y_0$ , where  $y_0 \in F(x_0)$ . Indeed, for any  $x \in \varepsilon(F)$ , it follows that  $P(x) < r$  as  $\varepsilon(F) \subset B(0, r)$ , but for any  $x_0 \in \partial_C B(0, r)$ , we have  $P(x_0) = r$ , thus conclusion (II) of Theorem 8.4 does not hold. By Theorem 8.4 again,  $F$  must have a fixed point, but this contradicts our assumption that  $F$  is fixed point free. This completes the proof.  $\square$

Now assume a given  $p$ -vector space  $E$  equipped with the  $P$ -seminorm (by assuming it is continuous at zero) for  $0 < p \leq 1$ , then we know that  $P : E \rightarrow \mathbb{R}^+$ ,  $P^{-1}(0) = 0$ ,  $P(\lambda x) = |\lambda|^p P(x)$  for any  $x \in E$  and  $\lambda \in \mathbb{R}$ . Then we have the following useful result for fixed points due to Rothe and Altman types in  $p$ -vector spaces, which plays important roles in optimization problems, variational inequalities, and complementarity problems.

**Corollary 8.1** *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  and zero  $0 \in U$ , plus  $C$  is a closed  $p$ -convex subset of  $E$  with  $U \subset C$ , where  $0 < p \leq 1$ . Assume that  $F : \overline{U} \rightarrow C$  is a semiclosed 1-set contractive continuous mapping. If one of the following is satisfied:*

- (1) (Rothe type condition):  $P_U(F(X)) \leq P_U(x)$  for any  $x \in \partial U$ ;
  - (2) (Petryshyn type condition):  $P_U(F(X)) \leq P_U(F(X) - x)$  for any  $x \in \partial U$ ;
  - (3) (Altman type condition):  $|P_U(F(X))|^{\frac{2}{p}} \leq [P_U(F(X) - x)]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$  for any  $x \in \partial U$ ;
- then  $F$  has at least one fixed point.*

*Proof* By conditions (1), (2), and (3), it follows that the conclusion of (II) in Theorem 8.4 “there exist  $x_0 \in \partial_C(U)$  and  $\lambda \in (0, 1)$  such that  $x_0 \neq \lambda F(x_0)$ ” does not hold, thus by the alternative of Theorem 8.4,  $F$  has a fixed point. This completes the proof.  $\square$

By the fact that when  $p = 1$  in a  $p$ -vector space is an LCS, we have the following classical Fan’s best approximation (see [34]) as a powerful tool for the study in the optimization, mathematical programming, games theory, mathematical economics, and other related topics in applied mathematics.

**Corollary 8.2** (Fan's best approximation) *Let  $U$  be a bounded open convex subset of a locally convex space  $E$  with the zero  $0 \in U$ , let  $C$  be a closed convex subset of  $E$  with also zero  $0 \in C$ , and assume that  $F : \overline{U} \cap C \rightarrow C$  is a semiclosed 1-set contractive and continuous mapping. Then there exists  $x_0 \in \overline{U} \cap C$  such that  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C)$ , where  $P_U$  is the Minkowski  $p$ -functional of  $U$  in  $E$ . More precisely, we have that either (I) or (II) holds, where  $W_{\overline{U}}(x_0)$  is either the inward set  $I_{\overline{U}}(x_0)$  or the outward set  $O_{\overline{U}}(x_0)$ :*

- (I)  $F$  has a fixed point  $x_0 \in U \cap C$ , i.e.,  $x_0 = F(x_0)$ ;
- (II) There exists  $x_0 \in \partial_C(U)$  with  $F(x_0) \notin \overline{U}$  such that

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1 > 0.$$

*Proof* When  $p = 1$ , it automatically satisfies the inequality  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for each  $x \in \overline{U} \cap C$ . Indeed, we have that for  $x_0 \in \partial_C(U)$ , we have  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1$ . The conclusions are given by Theorem 8.2 (or Theorem 8.3). The proof is complete.  $\square$

We would like to point out that similar results on Rothe and Leray–Schauder alternative have been developed by Isac [51], Park [85], Potter [97], Shahzad [109, 110], Xiao and Zhu [124], and related references therein as tools of nonlinear analysis in topological vector spaces. As mentioned above, when  $p = 1$  and  $F$  is a continuous mapping, then we can obtain a version of Leray–Schauder in locally convex spaces, and we omit detailed statements here due to the limit of the space.

## 9 Nonlinear alternative principle for the class of semiclosed 1-set contractive mappings

As applications of results in Sect. 8, we now establish general results for the existence of solutions for the Birkhoff–Kellogg problem and the principle of Leray–Schauder alternatives for semiclosed 1-set contractive mappings for  $p$ -vector spaces being locally  $p$ -convex spaces for  $0 < p \leq 1$ .

**Theorem 9.1** (Birkhoff–Kellogg alternative in locally  $p$ -convex spaces) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  (where  $0 \leq p \leq 1$ ) with the zero  $0 \in U$ , let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ , and assume that  $F : \overline{U} \cap C \rightarrow 2^C$  is a semiclosed 1-set contractive and continuous mapping, and for each  $x \in \partial_C(U)$  with  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ), where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . Then we have that either (I) or (II) holds:*

- (I) There exists  $x_0 \in \overline{U} \cap C$  such that  $x_0 = F(x_0)$ ;
- (II) There exists  $x_0 \in \partial_C(U)$  with  $F(x_0) \notin \overline{U}$  and  $\lambda > 1$  such that  $\lambda x_0 = F(x_0)$ , i.e.,  $F(x_0) \in \{\lambda x_0 : \lambda > 1\}$ .

*Proof* By following the argument and notations used by Theorem 8.2, we have that either

- (1)  $F$  has a fixed point  $x_0 \in U \cap C$ ; or
- (2) There exists  $x_0 \in \partial_C(U)$  with  $x_0 = f(y_0)$  such that

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1 > 0,$$

where  $\partial_C(U)$  denotes the boundary of  $U$  relative to  $C$  in  $E$  and  $f$  is the restriction of the continuous retraction  $r$  with respect to the set  $U$  in  $E$ .

If  $F$  has no fixed point, then above (2) holds and  $x_0 \neq F(x_0)$ . As given by the proof of Theorem 8.2, we have that  $F(x_0) \notin \overline{U}$ , thus  $P_U(F(x_0)) > 1$  and  $x_0 = f(y_0) = \frac{F(x_0)}{(P_U(F(x_0)))^{\frac{1}{p}}}$ , which means  $F(x_0) = (P_U(F(x_0)))^{\frac{1}{p}} x_0$ . Let  $\lambda = (P_U(F(x_0)))^{\frac{1}{p}}$ , then  $\lambda > 1$ , and we have  $\lambda x_0 = F(x_0)$ . This completes the proof.  $\square$

**Theorem 9.2** (Birkhoff–Kellogg alternative in LCS) *Let  $U$  be a bounded open convex subset of a locally  $p$ -convex space  $E$  with the zero  $0 \in U$ , let  $C$  be a closed convex subset of  $E$  with also zero  $0 \in C$ , and assume that  $F : \overline{U} \cap C \rightarrow C$  is a semiclosed 1-set contractive and continuous mapping. Then we have that either (I) or (II) holds:*

- (I) *There exists  $x_0 \in \overline{U} \cap C$  such that  $x_0 = F(x_0)$ ; or*
- (II) *There exists  $x_0 \in \partial_C(U)$  with  $F(x_0) \notin \overline{U}$  and  $\lambda > 1$  such that  $\lambda x_0 = F(x_0)$ , i.e.,  $F(x_0) \in \{\lambda x_0 : \lambda > 1\}$ .*

*Proof* When  $p = 1$ , then it automatically satisfies the inequality  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for all  $x \in \overline{U} \cap C$ . Indeed, we have that for  $x_0 \in \partial_C(U)$ , we have  $P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1$ . The conclusions are given by Theorems 8.3 and 8.4. The proof is complete.  $\square$

Indeed, we have the following fixed points for nonself mappings in  $p$ -vector spaces for  $0 < p \leq 1$  under different boundary conditions.

**Theorem 9.3** (Fixed points of nonself mappings) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  (where  $0 \leq p \leq 1$ ) with the zero  $0 \in U$ , let  $C$  be a closed  $p$ -convex subset of  $E$  with also zero  $0 \in C$ , and assume that  $F : \overline{U} \cap C \rightarrow C$  is a semiclosed 1-set contractive and continuous mapping. In addition, for each  $x \in \partial_C(U)$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$  (this is trivial when  $p = 1$ ), where  $P_U$  is the Minkowski  $p$ -functional of  $U$ . If  $F$  satisfies any one of the following conditions for any  $x \in \partial_C(U) \setminus F(x)$ :*

- (i)  $P_U(F(x) - z) < P_U(F(x) - x)$  for some  $z \in \overline{I_U(x)} \cap C$ ;
- (ii) *There exists  $\lambda$  with  $|\lambda| < 1$  such that  $\lambda x + (1 - \lambda)F(x) \in \overline{I_U(x)} \cap C$ ;*
- (iii)  $F(x) \in \overline{I_U(x)} \cap C$ ;
- (iv)  $F(x) \in \{\lambda x : \lambda > 1\} = \emptyset$ ;
- (v)  $F(\partial U) \subset \overline{U} \cap C$ ;
- (vi)  $P_U(F(x) - x) \neq ((P_U(F(x)))^{\frac{1}{p}} - 1)^p$ ;

*then  $F$  must have a fixed point.*

*Proof* By following the argument and symbols used in the proof of Theorem 8.2 (see also Theorem 8.4), we have that either

- (1)  $F$  has a fixed point  $x_0 \in U \cap C$ ; or
- (2) There exists  $x_0 \in \partial_C(U)$  with  $x_0 = f(F(x_0))$  such that

$$P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = d_p(F(x_0), \overline{I_U(x_0)} \cap C) = P_U(F(x_0)) - 1 > 0,$$

where  $\partial_C(U)$  denotes the boundary of  $U$  relative to  $C$  in  $E$ , and  $f$  is the restriction of the continuous retraction  $r$  with respect to the set  $U$  in  $E$ .

First, suppose that  $F$  satisfies condition (i). If  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . Then, by condition (i), it follows that  $P_U(F(x_0) - z) < P_U(F(x_0) - x_0)$  for some  $z \in \overline{I_U(x)} \cap C$ . This contradicts the best approximation equations given by (2), thus  $F$  must have a fixed point.

Second, suppose that  $F$  satisfies condition (ii). If  $F$  has no fixed point, then above (2) holds and  $x_0 \neq F(x_0)$ . Then, by condition (ii), there exists  $\lambda > 1$  such that  $\lambda x_0 + (1 - \lambda)F(x_0) \in \overline{I_U(x)} \cap C$ . It follows that

$$\begin{aligned} P_U(F(x_0) - x_0) &\leq P_U(F(x_0) - (\lambda x_0 + (1 - \lambda)F(x_0))) = P_U(\lambda(F(x_0) - x_0)) \\ &= |\lambda|^p P_U(F(x_0) - x_0) < P_U(F(x_0) - x_0), \end{aligned}$$

this is impossible, and thus  $F$  must have a fixed point in  $\overline{U} \cap C$ .

Third, suppose that  $F$  satisfies condition (iii), i.e.,  $F(x) \in \overline{I_U(x)} \cap C$ ; then by (2) we have that  $P_U(F(x_0) - x_0) = 1$ , and thus  $x_0 = F(x_0)$ , which means  $F$  has a fixed point.

Fourth, suppose that  $F$  satisfies condition (iv), and if  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . As given by the proof of Theorem 8.2, we have that  $F(x_0) \notin \overline{U}$ , thus  $P_U(F(x_0)) > 1$  and  $x_0 = f(F(x_0)) = \frac{F(x_0)}{(P_U(F(x_0)))^{\frac{1}{p}}}$ , which means  $F(x_0) = (P_U(F(x_0)))^{\frac{1}{p}} x_0$ , where  $(P_U(F(x_0)))^{\frac{1}{p}} > 1$ . This contradicts assumption (iv), thus  $F$  must have a fixed point in  $\overline{U} \cap C$ .

Fifth, suppose that  $F$  satisfies condition (v), then  $x_0 \neq F(x_0)$ . As  $x_0 \in \partial_C U$ , now by condition (v) we have that  $F(\partial U) \subset \overline{U} \cap C$ , it follows that for any  $x \in \partial_C U$  we have  $F(x) \in \overline{U} \cap C$ , thus  $F(x) \notin \overline{U} \setminus \cap C$ , which implies that  $0 < P_U(F(x_0) - x_0) = d_p(F(x_0), \overline{U} \cap C) = 0$ . This is impossible, thus  $F$  must have a fixed point. Here, as pointed out by Remark 5.2, we know that based on condition (v), the mapping  $F$  has a fixed point by applying  $F(\partial U) \subset \overline{U} \cap C$  is enough, we do not need the general hypothesis: “for each  $x \in \partial_C(U)$ ,  $P_U^{\frac{1}{p}}(F(x)) - 1 \leq P_U^{\frac{1}{p}}(F(x) - x)$  for  $0 < p \leq 1$ ”.

Finally, suppose that  $F$  satisfies condition (vi). If  $F$  has no fixed point, then (2) holds and  $x_0 \neq F(x_0)$ . Then condition (v) implies that  $P_U(F(x_0) - x_0) \neq (P_U(F(x_0)) - 1)^p$ , but our proof in Theorem 5.2 shows that  $P_U(y_0 - x_0) = ((P_U(y_0)) - 1)^p$ , which is impossible, thus  $F$  must have a fixed point. Then the proof is complete.  $\square$

Now, by taking the set  $C$  in Theorem 8.1 as the whole locally  $p$ -convex space  $E$  itself, we have the following general results for nonself upper semicontinuous mappings, which include the results of Rothe, Petryshyn, Altman, and Leray–Schauder types’ fixed points as special cases.

Taking  $p = 1$  and  $C = E$  in Theorem 9.3, we have the following fixed points for nonself continuous mappings associated with inward or outward sets for locally convex spaces, which are locally  $p$ -convex spaces for  $p = 1$ .

**Theorem 9.4** (Fixed points of nonself mappings with boundary conditions) *Let  $U$  be a bounded open convex subset of the LCS  $E$  with the zero  $0 \in U$ , and assume that  $F : \overline{U} \rightarrow E$  is a semiclosed 1-set contractive and continuous mapping. If  $F$  satisfies any one of the following conditions for any  $x \in \partial(U) \setminus F(x)$ :*

- (i)  $P_U(F(x) - z) < P_U(F(x) - x)$  for some  $z \in \overline{I_U(x)}$ ;
- (ii) There exists  $\lambda$  with  $|\lambda| < 1$  such that  $\lambda x + (1 - \lambda)F(x) \in \overline{I_U(x)}$ ;
- (iii)  $F(x) \in \overline{I_U(x)}$ ;

- (iv)  $F(x) \in \{\lambda x : \lambda > 1\} = \emptyset$ ;
  - (v)  $F(\partial(U)) \subset \overline{U}$ ;
  - (vi)  $P_U(F(x) - x) \neq P_U(F(x)) - 1$ ;
- then  $F$  must have a fixed point.

In what follows, based on the best approximation theorem in a  $p$ -seminorm space, we will also give some fixed point theorems for nonself continuous mappings with various boundary conditions, which are related to the study of the existence of solutions for PDE and differential equations with boundary problems (see Browder [15], Petryshyn [93, 94], Reich [99]), which would play roles in nonlinear analysis for  $p$ -seminorm space as shown below.

First, as discussed by Remark 5.2, the proof of Theorem 9.2, with the strongly boundary condition " $F(\partial(U)) \subset \overline{U} \cap C$ " only, we can prove that  $F$  has a fixed point, thus we have the following fixed point theorem of Rothe type in locally  $p$ -convex spaces.

**Theorem 9.5** (Rothe type) *Let  $U$  be a bounded open  $p$ -convex subset of a locally  $p$ -convex space  $E$  (where  $0 \leq p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a semi 1-set contractive and continuous mapping and such that  $F(\partial(U)) \subset \overline{U}$ , then  $F$  must have a fixed point.*

Now, as applications of Theorem 9.5, we give the following Leray–Schauder alternative in  $p$ -vector spaces for nonself set-valued mappings associated with the boundary condition which often appear in the applications (see Isac [51] and the references therein for the study of complementary problems and related topics in optimization).

By using the same argument used in the proof of Theorem 6.6, we have the following result.

**Theorem 9.6** (Leray–Schauder alternative in locally  $p$ -convex spaces) *Let  $E$  be a locally  $p$ -convex space  $E$ , where  $0 < p \leq 1$ ,  $B \subset E$  is bounded closed  $p$ -convex such that  $0 \in \text{int } B$ . Let  $F : [0, 1] \times B \rightarrow E$  be a semiclosed 1-set contractive and continuous mapping, and such that the set  $F([0, 1] \times B)$  is relatively compact in  $E$ . If the following assumptions are satisfied:*

- (1)  $x \neq F(t, x)$  for all  $x \notin \partial B$  and  $t \in [0, 1]$ ,
- (2)  $F(\{0\} \times \partial B) \subset B$ ,

then there is an element  $x^* \in B$  such that  $x^* = F(1, x^*)$ .

*Proof* The conclusion is proved by following the argument used in Theorem 6.6. The proof is complete.  $\square$

As a special case of Theorem 9.6, we have the following principle for the implicit form of Leray–Schauder type alternative in locally  $p$ -convex spaces for  $0 < p \leq 1$ .

**Corollary 9.1** (Implicit Leray–Schauder alternative) *Let  $E$  be a locally  $p$ -convex space  $E$ , where  $0 < p \leq 1$ ,  $B \subset E$  be bounded closed  $p$ -convex such that  $0 \in \text{int } B$ . Let  $F : [0, 1] \times B \rightarrow E$  be semiclosed 1-set contractive and continuous, and let the set  $F([0, 1] \times B)$  be relatively compact in  $E$ . If the following assumptions are satisfied:*

- (1)  $F(\{0\} \times \partial B) \subset B$ ,
- (2)  $x \neq F(0, x)$  for all  $x \in \partial B$ ,

then at least one of the following properties is satisfied:

- (i) There exists  $x^* \in B$  such that  $x^* = F(1, x^*)$ ; or
- (ii) There exists  $(\lambda^*, x^*) \in (0, 1) \times \partial B$  such that  $x^* = F(\lambda^*, x^*)$ .

*Proof* The result is an immediate consequence of Theorem 9.6, this completes the proof.  $\square$

We would like to point out that similar results on Rothe and Leray–Schauder alternative have been developed by Furi and Pera [37], Granas and Dugundji [46], Górniewicz [44], Górniewicz et al. [45], Isac [51], Li et al. [67], Liu [70], Park [85], Potter [97], Shahzad [109, 110], Xu [129], Xu et al. [130] (see related references therein) as tools of nonlinear analysis in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces. Some Birkhoff–Kellogg type theorems for general class mappings in topological vector spaces have also been established by Agarwal et al. [1], Agarwal and O'Regan [2, 3], Park [87] (see the references therein for more details); and in particular, recently O'Regan [80] used the Leray–Schauder type coincidence theory to establish some Birkhoff–Kellogg problem, Furi–Pera type results for a general class of mappings.

Before closing this section, we would like to share that as the application of the best approximation result for 1-set contractive mappings, we can establish fixed point theorems and the general principle of Leray–Schauder alternative for nonself mappings, which seem to play important roles in the development of nonlinear analysis for  $p$ -vector spaces for  $0 < p \leq 1$ , as the natural extension and achievement of nonlinear functional analysis in mathematics for the underling locally convex vector spaces, locally convex spaces, normed spaces, or in Banach spaces.

## 10 Fixed points for the class of semiclosed 1-set contractive mappings

In this section, based on the best approximation Theorem 8.2 established for the 1-set contractive mappings in Sect. 8, we show how it is used as a useful tool for us to develop fixed point theorems for semiclosed 1-set contractive nonself upper semicontinuous mappings in  $p$ -seminorm spaces (for  $p \in (0, 1]$ , by including seminorm, norm spaces, and uniformly convex Banach spaces as special cases).

By following Definition 7.1, we first observe that if  $f$  is a continuous demicompact mapping, then  $(I - f)$  is closed, where  $I$  is the identity mapping on  $X$ . It is also clear from definitions that every demicompact map is hemicompact in seminorm spaces, but the converse is not true in general (e.g., see the example in p. 380 by Tan and Yuan [117]). It is evident that if  $f$  is demicompact, then  $I - f$  is demiclosed. It is known that for each condensing mapping  $f$ , when  $D$  or  $f(D)$  is bounded, then  $f$  is hemicompact; and also  $f$  is demicompact in metric spaces by Lemma 2.1 and Lemma 2.2 of Tan and Yuan [117], respectively. In addition, it is known that every nonexpansive map is a 1-set-contractive map; and also if  $f$  is a hemicompact 1-set-contractive mapping, then  $f$  is a 1-set-contractive mapping satisfying the following “Condition (H1)” (the same as (H1) and slightly different from condition (H) used in Sect. 5):

(H1) *condition*: Let  $D$  be a nonempty bounded subset of a space  $E$ , and assume that  $F: \overline{D} \rightarrow 2^E$  is a set-valued mapping. If  $\{x_n\}_{n \in \mathbb{N}}$  is any sequence in  $D$  such that for each  $x_n$  there exists  $y_n \in F(x_n)$  with  $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ , then there exists a point  $x \in \overline{D}$  such that  $x \in F(x)$ .



We first note that the “(H1) condition” above is actually “condition (C)” used by Theorem 1 of Petryshyn [94]. Indeed, by following Goebel and Kirk [42] (see also Xu [127] and the references therein), Browder [15] (see also [16], p. 103) proved that if  $K$  is a closed and convex subset of a uniformly convex Banach space  $X$ , and if  $T : K \rightarrow X$  is nonexpansive, then the mapping  $f := I - T$  is demiclosed on  $X$ . This result, known as Browder’s demiclosedness principle (Browder’s proof, which was inspired by the technique of Göhde in [43]), is one of the fundamental results in the theory of nonexpansive mappings, which satisfies the “(H1) condition”.

The following is Browder’s demiclosedness principle proved by Browder [15] that says that a nonexpansive mapping in a uniformly convex Banach  $X$  enjoys condition (H1) as shown below.

**Lemma 10.1** *Let  $D$  be a nonempty bounded convex subset of a uniformly convex Banach space  $E$ . Assume that  $F : \overline{D} \rightarrow E$  is a nonexpansive single-valued mapping, then the mapping  $P := I - F$  defined by  $P(x) := (x - F(x))$  for each  $x \in \overline{D}$  is demiclosed, and in particular, the “(H1) condition” holds.*

*Proof* By following the argument given in p. 329 (see also the proof of Theorem 2.2 and Corollary 2.1) by Petryshyn [94], by the Browder demiclosedness principle (see Goebel and Kirk [42] or Xu [127]),  $P = (I - F)$  is closed at zero, thus there exists  $x_0 \in \overline{U}$  such that  $0 \in (I - F)(x_0)$ , which means that  $x_0 \in F(x_0)$ . The proof is complete.  $\square$

On the other hand, by following the notion called “Opial condition” given by Opial [79], which says that a Banach space  $X$  is said to satisfy the Opial condition if  $\liminf_{n \rightarrow \infty} \|w_n - w\| < \liminf_{n \rightarrow \infty} \|w_n - p\|$  whenever  $(w_n)$  is a sequence in  $X$  weakly convergent to  $w$  and  $p \neq w$ , we know that the Opial condition plays an important role in the fixed point theory, e.g., see Lami Dozo [64], Goebel and Kirk [42], Xu [127] and the references therein. Actually, the following result shows that there exists a class of nonexpansive set-valued mappings in Banach spaces with the Opial condition (see Lami Dozo [64] satisfying the “(H1) condition”).

**Lemma 10.2** *Let  $C$  be a nonempty convex weakly compact subset of a Banach space  $X$  which satisfies the Opial condition. Let  $T : C \rightarrow K(C)$  be a nonexpansive set-valued mapping with nonempty compact values. Then the graph of  $(I - T)$  is closed  $(X, \sigma(X, X^*) \times (X, \|\cdot\|))$ , thus  $T$  satisfies the “(H1) condition”, where  $I$  denotes the identity on  $X$ ,  $\sigma(X, X^*)$  the weak topology, and  $\|\cdot\|$  the norm (or strong) topology.*

*Proof* By following Theorem 3.1 of Lami Dozo [64], it follows that the mapping  $T$  is demiclosed, thus  $T$  satisfies the “(H1) condition”. The proof is complete.  $\square$

By Theorem 3.1 of Lami Dozo [64], indeed we have the following statement which is another version by using the term of “distance convergence” for Lemma 10.2.

**Lemma 10.3** *Let  $C$  be a nonempty closed convex subset of a Banach space  $(X, d)$  which satisfies the Opial condition. Let  $T : C \rightarrow K(C)$  be a multi-valued nonexpansive mapping (with fixed points). Let  $(y_n)_{n \in \mathbb{N}}$  be a bounded sequence such that  $\lim_{n \rightarrow \infty} d(y, T(y_n)) = 0$ , then the weak cluster points of  $(y_n)$ ,  $n \in \mathbb{N}$  is a fixed point of  $T$ .*

*Proof* It is Theorem 3.1 of Lami Dozo [64] (see also Lemma 3.2 of Xu and Muglia [128]).  $\square$

We note that another class of set-valued mappings, called  $*$ -nonexpansive mappings in Banach spaces (introduced by Husain and Tarafdar [50], see also Husain and Latif [49]), which was proved to hold the demiclosedness principle in reflexive Banach spaces satisfying the Opial condition by Muglia and Marino (i.e., Lemma 3.4 in [75]), thus the demiclosedness principle also holds in reflexive Banach spaces with duality mapping that is weakly sequentially continuous since these satisfy the Opial condition.

More precisely, let  $C$  be a subset of a Banach space  $(X, \|\cdot\|)$  and  $K(C)$  be the family of compact subsets of  $C$ . By following Husain and Latif [49], a mapping  $W : C \rightarrow K(C)$  is said to be  $*$ -nonexpansive if for all  $x, y \in C$  and  $x^W \in W(x)$  such that  $\|x - x^W\| = d(x, W(x))$ , there exists  $y^W \in W(y)$  with  $\|y - y^W\| = d(y, W(y))$  such that  $\|x^W - y^W\| \leq \|x - y\|$ .

As pointed by Muglia and Marino [75], however,  $*$ -nonexpansivity and multivalued nonexpansivity are not so far. By Theorem 3 of López-Acdeo and Xu [71], it is proved that a multivalued mapping  $W : C \rightarrow K(C)$  is  $*$ -nonexpansive if and only if the metric projection  $P_W(x) := \{u_x \in W(x) : \|x - u_x\| = \inf_{y \in W(x)} \|x - y\|\}$  is nonexpansive.

We now have the following result which is the demiclosedness principle for multivalued  $*$ -nonexpansive mappings given by Lemma 3.4 of Muglia and Marino [75].

**Lemma 10.4** *Let  $X$  be a reflexive space satisfying the Opial condition, and let  $W : X \rightarrow K(X)$  be a  $*$ -nonexpansive multivalued mapping with fixed points (existing) (denoted by  $\text{Fix}(W)$ ). Let  $(y_n)_{n \in \mathbb{N}}$  be a bounded sequence such that  $\lim_{n \rightarrow \infty} d(y_n, W(y_n)) \rightarrow 0$ . Then the weak cluster points of  $(y_n)_{n \in \mathbb{N}}$  belong to  $\text{Fix}(W)$ .*

*Proof* It is Lemma 3.4 of Muglia and Marino [75].  $\square$

**Remark 10.1** We would like to point out that, indeed, Xu [126] proved existence results of fixed points for  $*$ -nonexpansive mappings on strictly convex Banach spaces, and López-Acdeo and Xu in [71] obtained the existence result in the setting of Banach space satisfying the Opial condition, so the assumption on the existence of fixed points for the mapping  $W$  in Lemma 10.4 makes sense for the setting under either strictly convex Banach spaces or Banach spaces satisfying the Opial condition.

Let  $E$  denote a Hausdorff locally convex topological vector space, and let  $\mathfrak{F}$  denote the family of continuous seminorms generating the topology of  $E$ . Also  $C(E)$  will denote the family of nonempty compact subsets of  $E$ . For each  $p \in \mathfrak{F}$  and  $A, B \in C(E)$ , we can define  $\delta(A, B) := \sup\{p(a - b) : a \in A, b \in B\}$  and  $D_p(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} P(a - b), \sup_{b \in B} \inf_{a \in A} P(a - b)\}$ . Although the  $P$  is only a seminorm,  $D_p$  is a Hausdorff metric on  $C(E)$  (e.g., see Ko and Tsai [61]).

**Definition 10.1** Let  $K$  be a nonempty subset of  $E$ . A mapping  $T : K \rightarrow C(E)$  is said to be a multi-valued contraction if there exists a constant  $k_p \in (0, 1)$  such that  $D_p(T(x), T(y)) \leq k_p P(x - y)$ .  $T$  is said to be nonexpansive if for any  $x, y \in K$ , we have  $D_p(T(x), T(y)) \leq P(x - y)$ .

By Chen and Singh [26], we now have the following definition of the Opial condition in locally convex spaces.

**Definition 10.2** The locally convex space  $E$  is said to satisfy the Opial condition if for each  $x \in E$  and every net  $(x_\alpha)$  converging weakly to  $x$ , for each  $P \in \mathfrak{F}$ , we have  $\liminf P(x_\alpha - y) > \liminf P(x_\alpha - x)$  for any  $y \neq x$ .

Now we have the following demiclosedness principle for nonexpansive set-valued mappings in (Hausdorff) local convex spaces  $E$ , which is indeed Theorem 1 of Chen and Singh [26].

**Lemma 10.5** Let  $K$  be a nonempty, weakly compact, and convex subset of  $E$ . Let  $T : K \rightarrow C(E)$  be nonexpansive. If  $E$  satisfies the Opial condition, then graph  $(I - G)$  is closed in  $E_w \times E$ , where  $E_w$  is  $E$  with its weak topology and  $I$  is the identity mapping.

*Proof* The conclusion follows by Theorem 1 of Chen and Singh [26].  $\square$

**Remark 10.2** When a  $p$ -vector space  $E$  is with a  $p$ -norm, then both (H1) and (H) conditions for their convergence can be described by the weak and strong convergence, by the weak topology and strong topology induced by  $p$ -norm for  $p \in (0, 1]$ . Secondly, if a given  $p$ -vector space  $E$  has a nonempty open  $p$ -convex subset  $U$  containing zero, then any mapping satisfying the “(H) condition” is a hemicompact mapping (with respect to  $P_U$  for a given bounded open  $p$ -convex subset  $U$  containing zero of  $p$ -vector space  $E$ ), thus satisfying the “(H) condition” used in Theorem 5.1.

By the fact that each semiclosed 1-set mapping satisfies the “(H1) condition”, we have the existence of fixed points for the class of semiclosed 1-set mappings. First, as an application of Theorem 8.2, we have the following result for nonself mappings in  $p$ -seminorm spaces for  $p \in (0, 1]$ .

**Theorem 10.1** Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $E$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a semiclosed 1-set contractive and continuous mapping. In addition, for any  $x \in \partial \overline{U}$ , we have  $\lambda x \neq F(x)$  for any  $\lambda > 1$  (i.e., the “Leray–Schauder boundary condition”). Then  $F$  has at least one fixed point.

*Proof* By the proof of Theorem 8.2 with  $C = E$ , we actually have that either (I) or (II) holds:

(I)  $F$  has a fixed point  $x_0 \in U$ , i.e.,  $P_U(F(x_0) - x_0) = 0$ .

(II) There exists  $x_0 \in \partial(U)$  with  $P_U(F(x_0) - x_0) = (P_U^{\frac{1}{p}}(F(x_0)) - 1)^p > 0$ .

If  $F$  has no fixed point, then (II) holds and  $x_0 \neq F(x_0)$ . By the proof of Theorem 8.2, thus  $P_U(F(x_0)) > 1$  and  $x_0 = f(F(x_0)) = \frac{F(x_0)}{(P_U(F(x_0)))^{\frac{1}{p}}}$ , which means  $F(x_0) = (P_U(F(x_0)))^{\frac{1}{p}} x_0$ , where  $(P_U(F(x_0)))^{\frac{1}{p}} > 1$ , this contradicts the assumption, thus  $F$  must have a fixed point. The proof is complete.  $\square$

By following the idea used and developed by Browder [15], Li [66], Li et al. [67], Goebel and Kirk [41], Petryshyn [93, 94], Tan and Yuan [117], Xu [129], Xu et al. [130], and the references therein, we have the following existence theorems for the principle of Leray–Schauder type alternatives in  $p$ -seminorm spaces  $(E, \|\cdot\|_p)$  for  $p \in (0, 1]$ .

**Theorem 10.2** Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a semiclosed 1-set contractive

and continuous mapping. In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$  such that, for each  $x \in \partial \overline{U}$ , we have  $\|F(x) - x\|_p^{\alpha/p} \geq \|F(x)\|_p^{(\alpha+\beta)/p} \|x\|_p^{-\beta/p} - \|x\|_p^{\alpha/p}$ . Then  $F$  has at least one fixed point.

*Proof* By assuming  $F$  has no fixed point, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 10.1 does not hold. If we assume that  $F$  has no fixed point, by the boundary condition of Theorem 10.1, there exist  $x_0 \in \partial \overline{U}$  and  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t-1)^\alpha - t^{\alpha+\beta} + 1$  for  $t \geq 1$ . We observe that  $f$  is a strictly decreasing function for  $t \in [1, \infty)$  as the derivative of  $f'(t) = \alpha(t-1)^{\alpha-1} - (\alpha+\beta)t^{\alpha+\beta-1} < 0$  by the differentiation, thus we have  $t^{\alpha+\beta} - 1 > (t-1)^\alpha$  for  $t \in (1, \infty)$ . By combining the boundary condition, we have that  $\|F(x_0) - x_0\|_p^{\alpha/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{\alpha/p} < (\lambda_0^{\alpha+\beta} - 1) \|x_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} = \|F(x_0)\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} - \|x_0\|_p^{\alpha/p}$ , which contradicts the boundary condition given by Theorem 10.2. Thus, the conclusion follows and the proof is complete.  $\square$

**Theorem 10.3** Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F: \overline{U} \rightarrow 2^E$  is a semiclosed 1-set contractive and continuous mapping. In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$  such that, for each  $x \in \partial \overline{U}$ , we have  $\|F(x) + x\|_p^{(\alpha+\beta)/p} \leq \|F(x)\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|x\|_p^{(\alpha+\beta)/p}$ . Then  $F$  has at least one fixed point.

*Proof* We prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 10.1 does not hold. If we assume  $F$  has no fixed point, by the boundary condition of Theorem 10.1, there exist  $x_0 \in \partial \overline{U}$  and  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t+1)^{\alpha+\beta} - t^\alpha - 1$  for  $t \geq 1$ . We then can show that  $f$  is a strictly increasing function for  $t \in [1, \infty)$ , thus we have  $t^\alpha + 1 < (t+1)^{\alpha+\beta}$  for  $t \in (1, \infty)$ . By the boundary condition given in Theorem 7.3, we have that

$$\begin{aligned} \|F(x_0) + x_0\|_p^{(\alpha+\beta)/p} &= (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > (\lambda_0^\alpha + 1) \|x_0\|_p^{(\alpha+\beta)/p} \\ &= \|F(x_0)\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|x_0\|_p^{\alpha/p}, \end{aligned}$$

which contradicts the boundary condition given by Theorem 10.3. Thus, the conclusion follows and the proof is complete.  $\square$

**Theorem 10.4** Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F: \overline{U} \rightarrow E$  is a semiclosed 1-set contractive and continuous mapping. In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$  (or, alternatively,  $\alpha > 1$ ,  $\beta \geq 0$ ) such that, for each  $x \in \partial \overline{U}$ , we have that  $\|F(x) - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} \geq \|F(x)\|_p^{\alpha/p} \|F(x) + x\|_p^{\beta/p} - \|x\|_p^{(\alpha+\beta)/p}$ . Then  $F$  has at least one fixed point.

*Proof* The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 10.1 does not hold. If we assume  $F$  has no fixed point, by the boundary condition of Theorem 10.1, there exist  $x_0 \in \partial \overline{U}$  and  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t-1)^\alpha - t^\alpha(t-1)^\beta + 1$  for  $t \geq 1$ . We then can show that  $f$  is a strictly decreasing function for  $t \in [1, \infty)$ , thus we have  $(t-1)^\alpha < t^\alpha(t+1)^\beta - 1$  for  $t \in (1, \infty)$ .

By the boundary condition given in Theorem 10.3, we have that

$$\begin{aligned}\|F(x_0) - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} &= (\lambda_0 - 1)^\alpha \|x_0\|_p^{(\alpha+\beta)/p} < (\lambda_0^\alpha (\lambda_0 + 1)^\beta - 1) \|x_0\|_p^{(\alpha+\beta)/p} \\ &= \|F(x_0)\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} - \|x_0\|_p^{(\alpha+\beta)/p},\end{aligned}$$

which contradicts the boundary condition given by Theorem 10.4. Thus, the conclusion follows and the proof is complete.  $\square$

**Theorem 10.5** *Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F: \overline{U} \rightarrow E$  is a semiclosed 1-set contractive and continuous mapping. In addition, there exist  $\alpha > 1$ ,  $\beta \geq 0$ , we have that  $\|F(x) + x\|_p^{(\alpha+\beta)/p} \leq \|F(x) - x\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|F(x)\|_p^{\beta/p} \|x\|_p^{\alpha/p}$ . Then  $F$  has at least one fixed point.*

*Proof* The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume  $F$  has no fixed point, by the boundary condition of Theorem 10.1, there exist  $x_0 \in \partial \overline{U}$  and  $\lambda_0 > 1$  such that  $F(x_0) = \lambda_0 x_0$ .

Now, consider the function  $f$  defined by  $f(t) := (t+1)^{\alpha+\beta} - (t-1)^\alpha - t^\beta$  for  $t \geq 1$ . We then can show that  $f$  is a strictly increasing function for  $t \in [1, \infty)$ , thus we have  $(t+1)^{\alpha+\beta} > (t-1)^\alpha + t^\beta$  for  $t \in (1, \infty)$ .

By the boundary condition given in Theorem 10.3, we have that  $\|F(x_0) + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > ((\lambda_0 - 1)^\alpha + \lambda_0^\beta) \|x_0\|_p^{(\alpha+\beta)/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|\lambda_0 x_0\|_p^{\beta/p} \|x_0\|_p^{\alpha/p} = \|F(x_0) - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|F(x_0)\|_p^{\beta/p} \|x_0\|_p^{\alpha/p}$ , which implies that

$$\|F(x_0) + x_0\|_p^{(\alpha+\beta)/p} > \|F(x_0) - x_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|F(x_0)\|_p^{\beta/p} \|x_0\|_p^{\alpha/p},$$

this contradicts the boundary condition given by Theorem 10.5. Thus, the conclusion follows and the proof is complete.  $\square$

As an application of Theorem 10.1, by testing the Leray–Schauder boundary condition, we have the following conclusion for each special case, and thus we omit their detailed proofs here.

**Corollary 10.1** *Let  $U$  be a bounded open  $p$ -convex subset of a  $p$ -seminorm space  $(E, \|\cdot\|_p)$  ( $0 < p \leq 1$ ) with the zero  $0 \in U$ . Assume that  $F: \overline{U} \rightarrow E$  is a semiclosed 1-set contractive and continuous mapping. Then  $F$  has at least one fixed point if one of the following (strong) conditions holds for  $x \in \partial \overline{U}$ :*

- (i)  $\|F(x)\|_p \leq \|x\|_p$ ;
- (ii)  $\|F(x)\|_p \leq \|F(x) - x\|_p$ ;
- (iii)  $\|F(x) + x\|_p \leq \|F(x)\|_p$ ;
- (iv)  $\|F(x) + x\|_p \leq \|x\|_p$ ;
- (v)  $\|F(x) + x\|_p \leq \|F(x) - x\|_p$ ;
- (vi)  $\|F(x)\|_p \cdot \|F(x) + x\|_p \leq \|x\|_p^2$ ;
- (vii)  $\|F(x)\|_p \cdot \|F(x) + x\|_p \leq \|F(x) - x\|_p \cdot \|x\|_p$ .

If the  $p$ -seminorm space  $E$  is a uniformly convex Banach space  $(E, \|\cdot\|)$  (for  $p$ -norm space with  $p = 1$ ), then we have the following general existence result which can apply to general nonexpansive (single-valued) mappings, too.

**Theorem 10.6** *Let  $U$  be a bounded open convex subset of a uniformly convex Banach space  $(E, \|\cdot\|)$  (with  $p = 1$ ) with zero  $0 \in U$ . Assume that  $F : \overline{U} \rightarrow E$  is a semicontractive and continuous (single-valued) mapping. In addition, for any  $x \in \partial \overline{U}$ , we have  $\lambda x \neq F(x)$  for any  $\lambda > 1$  (i.e., the “Leray–Schauder boundary condition”). Then  $F$  has at least one fixed point.*

*Proof* By Lemma 10.1,  $F$  is a semiclosed 1-set contractive mapping. Moreover, by the assumption that  $E$  is a uniformly convex Banach space, the mapping  $(I - F)$  is closed at zero, and thus  $F$  is semiclosed at zero (see Browder [15] or Goebel and Kirk [41]). Thus all assumptions of Theorem 10.2 are satisfied. The conclusion follows by Theorem 10.2. The proof is complete.  $\square$

Now we can also have the following result for nonexpansive set-valued mappings (instead of single-valued) in a Banach space  $X$  with the Opial condition.

**Theorem 10.7** *Let  $C$  be a nonempty convex weakly compact subset of a local convex space  $X$  which satisfies the Opial condition and  $0 \in \text{int } C$ . Let  $T : C \rightarrow K(X)$  be a nonexpansive set-valued mapping with nonempty compact convex values. In addition, for any  $x \in \partial \overline{C}$ , we have  $\lambda x \neq F(x)$  for any  $\lambda > 1$  (i.e., the “Leray–Schauder boundary condition”). Then  $F$  has at least one fixed point.*

*Proof* As  $T$  is nonexpansive, it is 1-set contractive. By Lemma 10.2, it is then semicontractive and continuous. By following the idea of Theorem 10.1, indeed using the proof of Theorem 8.2 (or a similar argument used by Theorem 5.2) by applying Theorem 5.3 (instead of Theorem 5.2) for the fixed point theorem of upper semicontinuous set-valued mappings in a locally convex space, the conclusion follows. The proof is complete.  $\square$

By using Lemma 10.4, we have the following result in local convex spaces for  $*$ -nonexpansive single-valued mappings.

**Theorem 10.8** *Let  $C$  be a nonempty (bounded) convex closed subset of a Banach space  $X$  which is either strictly convex or satisfying the Opial condition. Let  $T : C \rightarrow X$  be a  $*$ -nonexpansive and continuous mapping. In addition, for any  $x \in \partial \overline{C}$ , we have  $\lambda x \neq F(x)$  for any  $\lambda > 1$  (i.e., the “Leray–Schauder boundary condition”). Then  $F$  has at least one fixed point.*

*Proof* As  $T$  is  $*$ -nonexpansive, and by the demiclosedness principle for  $*$ -nonexpansive mappings given by Lemma 10.4, it follows that  $T$  satisfies the (H1) condition of Theorem 7.1, then all conditions of Theorem 7.1 are satisfied, then the conclusion follows by Theorem 7.1. The proof is complete.  $\square$

By considering the  $p$ -seminorm space  $(E, \|\cdot\|)$  with a seminorm for  $p = 1$ , the following corollary is a special case of the corresponding results from Theorem 10.2 to Theorem 10.5, and thus we omit its proof.



**Corollary 10.2** *Let  $U$  be a bounded open convex subset of a normed space  $(E, \|\cdot\|)$ . Assume that  $F : \overline{U} \rightarrow E$  is a semiclosed 1-set contractive and continuous mapping. Then  $F$  has at least one fixed point if there exist  $\alpha > 1$ ,  $\beta \geq 0$  such that any one of the following conditions is satisfied:*

- (i) *For each  $x \in \partial \overline{U}$ ,  $\|F(x) - x\|^\alpha \geq \|F(x)\|^{(\alpha+\beta)} \|x\|^{-\beta} - \|x\|^\alpha$ ;*
- (ii) *For each  $x \in \partial \overline{U}$ ,  $\|F(x) + x\|^{(\alpha+\beta)} \leq \|F(x)\|^\alpha \|x\|^\beta + \|x\|^{(\alpha+\beta)}$ ;*
- (iii) *For each  $x \in \partial \overline{U}$ ,  $\|F(x) - x\|^\alpha \|x\|^\beta \geq \|F(x)\|^\alpha \|y + x\|^\beta - \|x\|^{(\alpha+\beta)}$ ;*
- (iv) *For each  $x \in \partial \overline{U}$ ,  $\|F(x) + x\|^{(\alpha+\beta)} \leq \|F(x) - x\|^\alpha \|x\|^\beta + \|F(x)\|^\beta \|x\|^\alpha$ .*

**Remark 10.3** As discussed by Lemma 10.1 and the proof of Theorem 10.6, when the  $p$ -vector space is a uniformly convex Banach space, the semicontractive or nonexpansive mappings automatically satisfy the conditions (see (H1)) required by Theorem 10.1, that is, the mappings are indeed semiclosed. Moreover, our results from Theorem 10.1 to Theorem 10.6, Corollary 10.1, and Corollary 10.2 also improve or unify the corresponding results given by Browder [15], Huang et al. [48], Li [66], Li et al. [67], Goebel and Kirk [41], Mańka [73], Petryshyn [93, 94], Reich [99], Shahzad [108], Tan and Yuan [117], Xu [125], Xu [126], Xu [129], Xu et al. [130], Yuan [134], Yuan [135] and the results from the references therein by extending the nonself mappings to the classes of semiclosed 1-set contractive set-valued mappings in  $p$ -seminorm spaces with  $p \in (0, 1]$  (including the norm space or the Banach space when  $p = 1$  for  $p$ -seminorm spaces).

Before ending this paper, we would like to share with readers that the main goal of this paper is to develop some new results and tools in the natural way for the category of nonlinear analysis for three classes of mappings, which are: 1) condensing; 2) 1-set contractive; and 3) semiclosed mappings under the general framework of locally  $p$ -convex spaces (where  $(0 < p \leq 1)$ ) for (single-valued) continuous mappings instead of set-valued mappings without the strong condition with closed  $p$ -convex values! We do also expect that these new results would become very useful tools for the development of nonlinear functional analysis under the general framework of  $p$ -vector spaces, which include the topological vector spaces as a special classes, and also the related applications for nonlinear problems on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and so on.

As we mentioned at the beginning of this paper, we do expect that nonlinear results and principles of the best approximation theorem established in this paper would play a very important role in the nonlinear analysis under the general framework of  $p$ -vector spaces for  $(0 < p \leq 1)$ , as shown by those results given from Sects. 6 and 7 for both condensing and 1-set contractive mappings; and general new results in nonlinear analysis from Sects. 8, 9, and 10 for semiclosed 1-set contractive mappings for the development of fixed point theorems for nonself mappings, the principle of nonlinear alternative, Rothe type, Leray–Schauder alternative, and related topics, which do not only include corresponding results in the existing literature as special cases, but are expected to be important tools for the study of its nonlinear analysis.

Finally, we would like to point out that the work presented by this paper focuses on the development of nonlinear analysis for single-valued (instead of set-valued) mappings for locally  $p$ -convex spaces. It is essentially very important and, indeed, the continuation of the work given recently by Yuan [134]; therein the attention is given to establishing new results

on fixed points, the principle of nonlinear alternative for nonlinear mappings mainly on set-valued mappings developed in locally  $p$ -convex spaces for  $0 < p \leq 1$ . Although some new results for set-valued mappings in locally  $p$ -convex spaces have been developed (see Gholizadeh et al. [39], Park [89], Qiu and Rolewicz [98], Xiao and Zhu [123, 124], Yuan [134], and others), we still would like to emphasize that the results obtained for set-valued mappings for  $p$ -vector spaces may face some challenges in dealing with true nonlinear problems. One example is that the assumption used for “set-valued mappings with closed  $p$ -convex values” seems too strong as it always means that the zero element is a trivial fixed point of the set-valued mappings, and this simple fact was also discussed by Yuan [134] for  $0 < p \leq 1$ .

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##### Author contributions

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##### Author details

<sup>1</sup>Business School, Chengdu University, Chengdu 610601, China. <sup>2</sup>School of Mathematics, Sichuan University, Chengdu 610065, China. <sup>3</sup>Business School, Sun Yat-Sen University, Guangzhou 510275, China. <sup>4</sup>College of Science, Chongqing University of Technology, Chongqing 4000547, China.

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