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Penalty method for a class of differential nonlinear system arising in contact mechanics

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Abstract

The main goal of this paper is to study a class of differential nonlinear system involving parabolic variational and history-dependent hemivariational inequalities in Banach spaces by using the penalty method. We first construct a penalized problem for such a nonlinear system and then derive the existence and uniqueness of its solution to obtain an approximating sequence for the nonlinear system. Moreover, we prove the strong convergence of the obtained approximating sequence to the solution of the original nonlinear system when the penalty parameter converges to zero. Finally, we apply the obtained convergence result to a long-memory elastic frictional contact problem with wear and damage in mechanics.

First part title: Introduction
Second part title: Preliminaries
Third part title: Convergence result for (1.1)
Fourth part title: An application

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1 Introduction

Let V, X, Y and W be separable, reflexive Banach spaces, V^* and Y^* be the dual spaces of V and Y, respectively, and Y_1 be a separable Hilbert space satisfying $Y \subset Y_1 \subset Y^*$. Moreover, assume that $M : V \hookrightarrow X$ is a compact embedding operator, K_V with $0_V \in K_V$ and K_Y are nonempty, closed, and convex subsets of V and Y, respectively. Let I be the time interval $[0, T]$ with $T > 0$. Very recently, in order to model an elastic frictional contact problem with long memory, damage, and wear, Chen et al. [4] introduced the following differential nonlinear system driven by a differential equation, a history-dependent hemivariational inequality and a parabolic variational inequality: find $u : I \to K_V$, $\zeta : I \to K_Y$ and $w : I \to$

W such that, for all $t \in I = [0, T]$ with $T > 0$,

$$\begin{cases} \dot{w}(t) = F(t, w(t), u(t)), \\ \langle A(t, u(t)) + \int_0^t B(t-s, u(s), \zeta(s)) \, ds, v - u(t) \rangle_{V^* \times V} \\ \quad + j^0(w(t), Mu(t), Mu(t); Mv(t) - Mu(t)) \geq \langle f(t), v - u(t) \rangle_{V^* \times V}, \quad \forall v \in K_V, \quad (1.1) \\ \langle \dot{\zeta}(t), \eta - \zeta(t) \rangle_{Y^* \times Y} + a(\zeta(t), \eta - \zeta(t)) \geq \langle \phi(t, u(t), \zeta(t)), \eta - \zeta(t) \rangle_{Y^* \times Y}, \quad \forall \eta \in K_Y, \\ w(0) = w_0, \qquad \zeta(0) = \zeta_0. \end{cases}$$

Moreover, they gave a unique solvability result for (1.1) by using Banach's fixed-point theorem and applied it to the long-memory elastic frictional contact problem with wear and damage in mechanics.

We would like to mention that (1.1) is an extended model that can be used to describe many real problems such as the long-memory elastic frictional contact problem with wear and damage in mechanics, engineering operation research, network equilibrium problems, and so on [2, 4–8, 23, 30]. Moreover, to choose suitable spaces and maps, many known differential variational inequalities (DVIs) and differential hemivariational inequalities (DHVIs) can be considered as special cases of (1.1) (see, for example, [12–15, 18, 25, 26, 29] and the references therein).

Among the studies on variational inequalities (VIs) and hemivariational inequalities (HVIs), constructing approximating sequences for their solutions and further discussing their convergence analysis are crucially important [10, 11]. It is well known that the penalty method is a kind of efficient approximating method for various problems. It is also constantly used for the study of VIs and HVIs (see, for example, [3, 21, 28, 31]). Due to the close relationship with VIs and HVIs, differential variational inequalities (DVIs) and differential hemivariational inequalities (DHVIs) are studied by employing the penalty method, such as Liu and Zeng [16, 17] and Weng et al. [27]. As the generalization of DVIs and DHVIs, the differential variational-hemivariational inequalities (DVHVIs) have drawn the attention of researchers in operations research and contact mechanics. With the penalty method, Tang et al. [25], Liu et al. [16], and Lu et al. [19] recently studied different DVHVIs, obtained their convergence results, and gave the corresponding applications in contact mechanics. However, to the best of our knowledge, there are no results in the literature concerning the penalty method for (1.1). The motivation of the present work is to make an attempt in this direction.

The main goal of this paper is to obtain a convergence result for (1.1) by employing the penalty method. The main contributions of this paper are twofold. First, we construct a penalized problem for (1.1) and show a convergence result, i.e., the solution of (1.1) can be approached as the penalty parameter converges to zero. Secondly, we apply the obtained convergence result to the long-memory elastic frictional contact problem with wear and damage in mechanics.

The rest of the paper is structured as follows. In Sect. 2, we introduce some preliminary materials that will be used in the following sections. In Sect. 3, we construct approximating sequences of solutions to (1.1) by the penalty method and derive its convergence. Finally, in Sect. 4, we apply the obtained convergence result to a long-memory elastic frictional contact problem with wear and damage in mechanics.

2 Preliminaries

Let $(X, \| \cdot \|_X)$ be a real Banach space with its dual X^* and $\langle \cdot, \cdot \rangle_{X^* \times X}$ denote the duality pairing between X^* and X. In this section, we recall some known definitions and lemmas that will be used subsequently (see [20, 22] for more details). Moreover, the symbols "$\rightarrow$" and "$\rightharpoonup$" represent the strong and weak convergence in various spaces, respectively.

Definition 2.1 A functional $j : X \rightarrow \mathbb{R}$ is lower semicontinuous if and only if for any convergence sequence $\{u_n\}_{n=1}^{\infty} \subset X$ satisfying $u_n \rightarrow u \in X$, one has $\liminf_{n \to \infty} j(u_n) \geq j(u)$.

Definition 2.2 A functional $j : X \rightarrow \mathbb{R} \cup \{\infty\}$ is called proper if $j(v) > -\infty$ for all $v \in X$ and there exists a point $u \in X$ such that $j(u) < +\infty$.

Definition 2.3 Let $j : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Define the convex subdifferential of j at u by

$$\partial_C j(u) = \left\{ u^* \in X^* \mid j(v) - j(u) \geq \langle u^*, v - u \rangle_{X^* \times X} \text{ for all } v \in X \right\}.$$

Definition 2.4 Let $j : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of j at x in the direction $v \in X$ is given by

$$j^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The Clarke subdifferential of j at x is a subset of the dual space X^* defined by

$$\partial j(x) = \left\{ \xi \in X^* \mid j^0(x; v) \geq \langle \xi, v \rangle_{X^* \times X} \text{ for all } v \in X \right\}.$$

For a set-valued operator $A : X \rightarrow 2^{X^*}$, the graph of A is denoted by $G(A)$, i.e.,

$$G(A) := \left\{ (u, u^*) \in X \times X^* \mid u^* \in A(u) \right\}.$$

Definition 2.5 A set-valued operator $A : X \rightarrow 2^{X^*}$ is called monotone if

$$\langle u^* - v^*, u - v \rangle_{X^* \times X} \geq 0, \quad \forall (u, u^*), (v, v^*) \in G(A).$$

Moreover, a monotone operator A is called maximal monotone if for any $(u, u^*) \in X \times X^*$ satisfying

$$\langle u^* - v^*, u - v \rangle_{X^* \times X} \geq 0, \quad \forall (v, v^*) \in G(A),$$

one has $(u, u^*) \in G(A)$.

For a proper, convex and lower semicontinuous functional $j : X \rightarrow \mathbb{R} \cup \{\infty\}$, it is well known that $\partial_C j : X \rightarrow 2^{X^*}$ is maximal monotone.

Definition 2.6 A single-valued operator $A : X \rightarrow X^*$ is said to be

(1) strongly monotone, if there exists $m_A > 0$ such that

$$\langle Au - Av, u - v \rangle_{X^* \times X} \geq m_A \|u - v\|_X^2 \quad \text{for all } v \in X;$$

(2) bounded, if A maps bounded sets of X into bounded sets of X^*;
(3) pseudomonotone, if it is bounded and $u_n \rightharpoonup u$ in X with

$$\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle_{X^* \times X} \leq 0,$$

which implies that $\liminf_{n \to \infty} \langle Au_n, u_n - v \rangle_{X^* \times X} \geq \langle Au, u - v \rangle_{X^* \times X}$ for all $v \in X$;
(4) demicontinuous, if $u_n \to u$ in X implies that $Au_n \rightharpoonup Au$ in X^*;
(5) hemicontinuous at u, if for each $u, v, w \in X$, $F(t) := \langle A(u + tv), w \rangle_{X^* \times X}$ is continuous on $[0,1]$.

Definition 2.7 An operator $P : X \to X^*$ is said to be a penalty operator of the set $K \subset X$ if P is bounded, demicontinuous, monotone, and $K = \{x \in X \mid Px = 0_{X^*}\}$, where 0_{X^*} represents the zero element of X^*.

Lemma 2.1 ([20, Proposition 3.23]) *If the operator $A : X \to X^*$ is bounded, demicontinuous, and monotone, then A is pseudomonotone.*

Lemma 2.2 ([20, Proposition 3.74]) *If $j : X \to \mathbb{R}$ is a locally Lipschitz function, then for every $v \in X$, one has*

$$j^0(x; v) = \max\{\langle \xi, v \rangle_{X^* \times X} : \xi \in \partial j(x)\}.$$

3 Convergence result for (1.1)

In this section, we first use the penalty method to construct a penalized problem of (1.1) and show that the penalized problem has a unique solution by employing Theorem 3.1 of Chen at al. [4]. Then, we show a convergence result that the solution of (1.1) can be approximated by the penalized problem as the penalty parameter converges to 0.

We assume that (V, H, V^*) and (Y, Y_1, Y^*) are two Gelfand triplets of Banach spaces that have continuous, compact, and dense embeddings, M is the embedding operator of $V \hookrightarrow H$, M^* is the adjoint operator of M, and $\|M\|$ and $\|M^*\|$ are the norms of M and M^*, respectively. K_V is a convex subset of V. Let $P : V \to V^*$ be a penalty operator of K_V. In order to develop the approximation procedure of (1.1), we need to construct the penalized problem of (1.1). For any given $\rho > 0$, the penalized problem of (1.1) can be constructed as follows.

Problem 3.1 Find $u_\rho : I \to V$, $\zeta_\rho : I \to K_Y$ and $w_\rho : I \to W$ such that, for all $t \in I$,

$$\dot{w}_\rho(t) = F(t, w_\rho(t), u_\rho(t)), \tag{3.1}$$

$$\left\langle A(t, u_\rho(t)) + \int_0^t B(t - s, u_\rho(s), \zeta_\rho(s))\, ds, v - u_\rho(t) \right\rangle_{V^* \times V}$$

$$+ \frac{1}{\rho} \langle Pu_\rho(t), v - u_\rho(t) \rangle_{V^* \times V} + j^0(w_\rho(t), Mu_\rho(t), Mu_\rho(t); Mv(t) - Mu_\rho(t))$$

$$\geq \langle f(t), v - u_\rho(t) \rangle_{V^* \times V}, \quad \forall v \in V, \tag{3.2}$$

$$\langle \dot{\zeta}_\rho(t), \eta - \zeta_\rho(t) \rangle_{Y^* \times Y} + a\left(\zeta_\rho(t), \eta - \zeta_\rho(t) \right)$$

$$\geq \left\langle \phi\left(t, u_\rho(t), \zeta_\rho(t) \right), \eta - \zeta_\rho(t) \right\rangle_{Y^* \times Y}, \quad \forall \eta \in K_Y, \tag{3.3}$$

$$w_\rho(0) = w_0, \qquad \zeta_\rho(0) = \zeta_0. \tag{3.4}$$

Remark 3.1 We note that in Problem 3.1, we can consider the penalty operators for both K_V and K_Y. Since our main interest is to provide tools in analyzing Problem 3.1, we restrict ourselves to study penalty operators for K_V. The case in which K_Y is considered can be solved likewise.

In order to study Problem 3.1, we need the following assumptions on the data.

H(A): The operator $A : I \times V \to V^*$ satisfies

    (a) $A(\cdot, v)$ is continuous on I for any given $v \in V$;

    (b) For any given $t \in I$, $A(t, \cdot)$ is hemicontinuous, pseudomonotone, and strongly monotone with $m_A > 0$ on V, i.e.,

$$\left\langle A(t, u_1) - A(t, u_2), u_1 - u_2 \right\rangle_{V^* \times V} \geq m_A \| u_1 - u_2 \|_V^2, \quad \forall (t, u_1, u_2) \in I \times V \times V;$$

    (c) $A(t, 0_V) = 0_{V^*}$ for all $t \in I$.

H(B): The operator $B : I \times V \times Y \to V^*$ satisfies

    (a) $B(\cdot, v, \zeta)$ is continuous on I for any given $v \in V$ and $\zeta \in Y$;

    (b) $B(t, \cdot, \cdot)$ is Lipschitz continuous with $L_B > 0$ on $V \times Y$ for any given $t \in I$, i.e.,

$$\left\| B(t, u_1, \zeta_1) - B(t, u_2, \zeta_2) \right\|_{V^*} \leq L_B \left(\| u_1 - u_2 \|_V + \| \zeta_1 - \zeta_2 \|_Y \right),$$

$$\forall t \in I, \forall u_1, u_2 \in V, \forall \zeta_1, \zeta_2 \in Y;$$

    (c) There exists $\varrho \in L^2(I; \mathbb{R}^+)$ such that

$$\left\| B(t, u, \zeta) \right\|_{V^*} \leq \varrho(t) \left(\| \zeta \|_Y + \| u \|_V \right), \quad \forall (t, u, \zeta) \in I \times V \times Y.$$

H(j): The functional $j : W \times X \times X \to \mathbb{R}$ satisfies

    (a) $j(w, u, \cdot)$ is locally Lipschitz on X for any given $(w, u) \in W \times X$;

    (b) There exist two constants $c_0, c_1 > 0$ such that

$$\left\| \partial j(w, x, y) \right\|_{X^*} \leq c_1 \left(1 + \| w \|_W + \| x \|_X \right) + c_0 \| y \|_X, \quad \forall (x, y, w) \in X \times X \times W;$$

    (c) There exist $\alpha_0 > 0$ and $\alpha_1 > 0$ such that

$$j^0(w_1, Mu_1, Mv_1; Mv_2 - Mv_1) + j^0(w_2, Mu_2, Mv_2; Mv_1 - Mv_2)$$

$$\leq \alpha_0 \| w_1 - w_2 \|_W \| v_1 - v_2 \|_V + \alpha_1 \| u_1 - u_2 \|_V \| v_1 - v_2 \|_V,$$

$$\forall w_1, w_2 \in W, \forall u_1, u_2, v_1, v_2 \in V.$$

H(F): The operator $F : I \times W \times V \to W$ satisfies

    (a) $F(\cdot, w, v)$ is continuous on I for any given $(w, v) \in W \times V$;

(b) $F(t, \cdot, \cdot)$ is Lipschitz continuous with $L_F > 0$ on $V \times Y$ for any given $t \in I$, i.e.,

$$\left\| F(t, w_1, u_1) - F(t, w_2, u_2) \right\|_W \leq L_F \big(\|u_1 - u_2\|_V + \|w_1 - w_2\|_W \big),$$

$$\forall t \in I, \forall w_1, w_2 \in W, \forall u_1, u_2 \in V.$$

H(ϕ): The operator $\phi : I \times V \times Y \to Y_1$ satisfies
    (a) $\phi(t, \cdot, \cdot)$ is Lipschitz continuous with $L_\phi > 0$ on $V \times Y$ for any given $t \in I$, i.e.,

$$\left\| \phi(t, u, \zeta) - \phi(t, v, \eta) \right\|_{Y_1} \leq L_\phi \big(\|u - v\|_V + \|\zeta - \eta\|_{Y_1} \big), \quad \forall t \in I, \forall u, v \in V, \forall \zeta, \eta \in Y;$$

    (b) $\phi(\cdot, 0_V, 0_Y) \in L^2(I; Y_1)$.

H(a): The functional $a : Y \times Y \to \mathbb{R}$ is a continuous bilinear symmetric coercive functional and there exist $a_1 \in \mathbb{R}$ and $a_2 > 0$ such that

$$a(\eta, \eta) + a_1 \|\eta\|_{Y_1}^2 \geq a_2 \|\eta\|_Y^2, \quad \forall \eta \in Y.$$

Remark 3.2 By Theorem 3.1 in [4], we know that (1.1) has a unique solution $(\zeta^*, u^*, w^*) \in (H^1(I; Y_1) \cap L^2(I; Y)) \times C(I; K_Y) \times C(I; W)$, providing H(A)–H(a) hold with $m_A > \max\{c_0 \|M\|_{L(V;X)}^2, \alpha_1\}$.

Remark 3.3 H(j)(c) is equivalent to the following condition

$$\langle \xi_1 - \xi_2, Mv_1 - Mv_2 \rangle_X \geq -\alpha_0 \|w_1 - w_2\|_W \|v_1 - v_2\|_V - \alpha_1 \|u_1 - u_2\|_V \|v_1 - v_2\|_V$$

for all $w_1, w_2 \in W$ and all $u_1, u_2, v_1, v_2 \in V$ with $\xi_i \in \partial j(w_i, Mu_i, Mv_i)$, $i = 1, 2$.

First, to solve the history-dependent hemivariational inequality in Problem 3.1, we consider the following auxiliary problem.

Problem 3.2 For any given $\rho > 0$, $\zeta \in H^1(I; Y_1) \cap L^2(I; Y)$, $w \in C(I; W)$ and $f \in C(I; V^*)$, find $u_{\rho w \zeta} : I \to V$ such that, for all $t \in I$,

$$\left\langle A\big(t, u_{\rho w \zeta}(t)\big) + \int_0^t B\big(t - s, u_{\rho w \zeta}(s), \zeta(s)\big) ds, v - u_{\rho w \zeta}(t) \right\rangle_{V^* \times V}$$

$$+ j^0 \big(w(t), M u_{\rho w \zeta}(t), M u_{\rho w \zeta}(t); Mv(t) - M u_{\rho w \zeta}(t) \big)$$

$$+ \frac{1}{\rho} \big\langle P u_{\rho w \zeta}(t), v - u_{\rho w \zeta}(t) \big\rangle_{V^* \times V} \geq \big\langle f(t), v - u_{\rho w \zeta}(t) \big\rangle_{V^* \times V}, \quad \forall v \in V. \qquad (3.5)$$

Lemma 3.1 *Assume that H(A), H(B), and H(j) hold. If $m_A > \max\{c_0 \|M\|_{L(V;X)}^2, \alpha_1\}$, then one has the following conclusions:*
    (i) *for any given $\rho > 0$, $\zeta \in H^1(I; Y_1) \cap L^2(I; Y)$, $w \in C(I; W)$ and $f \in C(I; V^*)$, Problem 3.2 has a unique solution $u_{\rho w \zeta} \in C(I; V)$;*
    (ii) *$u_{\rho w \zeta}$ converges strongly to $u_{w \zeta}$ as $\rho \to 0$, where $u_{w \zeta} \in C(I; K_V)$ is the unique solution of the following problem: for any given $\zeta \in (H^1(I; Y_1) \cap L^2(I; Y))$, $w \in C(I; W)$ and*

$f \in C(I; V^*)$, find $u_{w\zeta} : I \to K_V$ such that, for all $t \in I$,

$$
\left\langle A\big(t, u_{w\zeta}(t)\big) + \int_0^t B\big(t - s, u_{w\zeta}(s), \zeta(s)\big) \, ds, v - u_{w\zeta}(t) \right\rangle_{V^* \times V}
$$
$$
+ j^0\big(w(t), Mu_{w\zeta}(t), Mu_{w\zeta}(t); Mv(t) - Mu_{w\zeta}(t)\big)
$$
$$
\geq \big\langle f(t), v - u_{w\zeta}(t) \big\rangle_{V^* \times V}, \quad \forall v \in K_V. \tag{3.6}
$$

Proof (i) Consider a function $A_\rho : I \times V \to V^*$ defined by

$$
A_\rho(t, u) = A(t, u) + \frac{1}{\rho} Pu, \quad \forall(t, u) \in I \times V.
$$

Since P is bounded, demicontinuous, monotone, and $K_V = \{u \in V \mid Pu = 0_{V^*}\}$, it follows from H(A) that $A_\rho(\cdot, v)$ is continuous for any given $v \in V$ and $A_\rho(t, \cdot)$ is hemicontinuous, pseudomonotone, and strong monotone with $A_\rho(t, 0_V) = 0_{V^*}$ for all $t \in I$. This shows that Problem 3.2 satisfies all the hypotheses of Lemma 3.2 in [4] and so Problem 3.2 has a unique solution $u_{\rho w\zeta} \in C(I; V)$.

(ii) For fixed $\eta \in C(I; K_V)$, we consider the auxiliary problem for (3.5) as follows: find a map $u_{\rho w\zeta\eta} : (0, T) \to V$ such that, for any $t \in I$ and any $v \in V$,

$$
\big\langle A\big(t, u_{\rho w\zeta\eta}(t)\big), v - u_{\rho w\zeta\eta}(t) \big\rangle_{V^* \times V} + \frac{1}{\rho} \big\langle Pu_{\rho w\zeta\eta}(t), v - u_{\rho w\zeta\eta}(t) \big\rangle_{V^* \times V}
$$
$$
+ j^0\big(w(t), Mu_{\rho w\zeta\eta}(t), Mu_{\rho w\zeta\eta}(t); Mv - Mu_{\rho w\zeta\eta}(t)\big)
$$
$$
\geq \big\langle f_\eta(t), v - u_{\rho w\zeta\eta}(t) \big\rangle_{V^* \times V}, \tag{3.7}
$$

where f_η is defined by

$$
f_\eta(t) = f(t) - \int_0^t B\big(t - s, \eta(s), \zeta(s)\big) \, ds.
$$

Let $u_0 \in K_V$ be fixed. Inserting $v = u_0$ into (3.7), we have

$$
\big\langle A(t, u_{\rho w\zeta}(t), u_0 - u_{\rho w\zeta}(t) \big\rangle_{V^* \times V} + \frac{1}{\rho} \big\langle Pu_{\rho w\zeta}(t), u_0 - u_{\rho w\zeta}(t) \big\rangle_{V^* \times V}
$$
$$
+ j^0\big(w(t), Mu_{\rho w\zeta}(t), Mu_{\rho w\zeta}(t); Mu_0 - Mu_{\rho w\zeta}(t)\big) \geq \big\langle f_\eta(t), u_0 - u_{\rho w\zeta}(t) \big\rangle_{V^* \times V}.
$$

It follows from the strong monotonicity of A that

$$
m_A \big\| u_0 - u_{\rho w\zeta}(t) \big\|_V^2 \leq \big\langle A\big(t, u_{\rho w\zeta}(t)\big) - A(t, u_0), u_{\rho w\zeta}(t) - u_0 \big\rangle_{V^* \times V}
$$
$$
\leq \big\langle A(t, u_0), u_0 - u_{\rho w\zeta}(t) \big\rangle_{V^* \times V} + \frac{1}{\rho} \big\langle Pu_{\rho w\zeta}(t), u_0 - u_{\rho w\zeta}(t) \big\rangle_{V^* \times V}
$$
$$
+ j^0\big(w(t), Mu_{\rho w\zeta}(t), Mu_{\rho w\zeta}(t); Mu_0 - Mu_{\rho w\zeta}(t)\big)
$$
$$
- \big\langle f_\eta(t), u_0 - u_{\rho w\zeta}(t) \big\rangle_{V^* \times V}
$$

for all $t \in [0, T]$. As P is monotone, $Pv = 0$ for all $v \in K_V$ and $u_0 \in K_V$, one has

$$m_A \big\| u_0 - u_{\rho w \zeta}(t) \big\|_V^2$$

$$\leq \big\langle A(t, u_0), u_0 - u_{\rho w \zeta}(t) \big\rangle_{V^* \times V} + \frac{1}{\rho} \big\langle P u_{\rho w \zeta}(t) - P u_0, u_0 - u_{\rho w \zeta}(t) \big\rangle_{V^* \times V}$$

$$+ j^0 \big(w(t), M u_{\rho w \zeta}(t), M u_{\rho w \zeta}(t); M u_0 - M u_{\rho w \zeta}(t) \big) - \big\langle f_\eta(t), u_0 - u_{\rho w \zeta}(t) \big\rangle_{V^* \times V}$$

$$\leq \big\langle A(t, u_0), u_0 - u_{\rho w \zeta}(t) \big\rangle_{V^* \times V} - \big\langle f_\eta(t), u_0 - u_{\rho w \zeta}(t) \big\rangle_{V^* \times V}$$

$$+ j^0 \big(w(t), M u_{\rho w \zeta}(t), M u_{\rho w \zeta}(t); M u_0 - M u_{\rho w \zeta}(t) \big) \tag{3.8}$$

for all $t \in [0, T]$. Thus, Remark 3.3 implies that

$$j^0 \big(w(t), M u_0, M u_0; M u_0 - M u_{\rho w \zeta}(t) \big) + j^0 \big(w(t), M u_{\rho w \zeta}(t), M u_{\rho w \zeta}(t); M u_{\rho w \zeta}(t) - M u_0 \big)$$

$$\leq \alpha_1 \big\| u_0 - u_{\rho w \zeta}(t) \big\|_V^2. \tag{3.9}$$

From (3.8), (3.9), and H(j)(b), we have

$$(m_A - \alpha_1) \big\| u_0 - u_{\rho w \zeta}(t) \big\|_V^2$$

$$\leq m_A \big\| u_0 - u_{\rho w \zeta}(t) \big\|_V^2 + j^0 \big(w(t), M u_0, M u_0; M u_0 - M u_{\rho w \zeta}(t) \big)$$

$$+ j^0 \big(w(t), M u_{\rho w \zeta}(t), M u_{\rho w \zeta}(t); M u_{\rho w \zeta}(t) - M u_0 \big)$$

$$\leq \big(\big\| A(t, u_0) \big\|_{V^*} + \big\| f_\eta(t) \big\|_{V^*} \big) \big\| u_0 - u_{\rho w \zeta}(t) \big\|_V$$

$$+ \|M\|^* \big(c_1 \big(1 + \|w\|_{C(I;W)} + \|M\| \|u_0\|_V \big) + c_0 \|M\| \|u_0\|_V \big) \big\| u_0 - u_{\rho w \zeta}(t) \big\|_V$$

and so

$$(m_A - \alpha_1) \big\| u_0 - u_{\rho w \zeta}(t) \big\|_V$$

$$\leq \big(\big\| A(t, u_0) \big\|_{V^*} + \big\| f_\eta(t) \big\|_{V^*} \big)$$

$$+ \|M\|^* \big(c_1 \big(1 + \|w\|_{C(I;W)} + \|M\| \|u_0\|_V \big) + c_0 \|M\| \|u_0\|_V \big). \tag{3.10}$$

Moreover, A is pseudomonotone, so A is bounded, then there exists a constant N, such that

$$\big\| A(t, u_0) \big\|_{V^*} \leq N \tag{3.11}$$

and H(B)(c) implies that

$$\big\| f_\eta(t) \big\|_{V^*} \leq \big\| f(t) \big\|_{V^*} + \int_0^t \big\| B \big(t - s, \eta(s), \zeta(s) \big) \big\| \, ds$$

$$\leq \|f\|_{C(I,V^*)} + \sqrt{T} \|\eta\|_{C(I;V)} \bigg(\int_0^t \big| \varrho(t-s) \big|^2 \, ds \bigg)^{\frac{1}{2}}$$

$$+ \bigg(\int_0^t \big| \varrho(t-s) \big|^2 \, ds \bigg)^{\frac{1}{2}} \bigg(\int_0^t \big\| \zeta(s) \big\|^2 \, ds \bigg)^{\frac{1}{2}}$$

$$\leq \|f\|_{C(I,V^*)} + \sqrt{T}\|\eta\|_{C(I;V)}\|\varrho\|_{L^2(I;\mathbb{R}^+)}$$
$$+ \|\varrho\|_{L^2(I;\mathbb{R}^+)}\|\zeta\|_{L^2(I;Y)}. \tag{3.12}$$

Combining (3.10), (3.11), and (3.12), one has

$$(m_A - \alpha_1)\|u_0 - u_{\rho w \zeta}(t)\|_V$$
$$\leq N + \|f\|_{C(I,V^*)} + \sqrt{T}\|\eta\|_{C(I;V)}\|\varrho\|_{L^2(I;\mathbb{R}^+)} + \|\varrho\|_{L^2(I;\mathbb{R}^+)}\|\zeta\|_{L^2(I;Y)}$$
$$+ \|M\|^*\big(c_1\big(1 + \|w\|_{C(I;W)} + \|M\|\|u_0\|_V\big) + c_0\|M\|\|u_0\|_V\big),$$

which implies that the sequence $\{u_{\rho w \zeta}(t)\}_{\rho > 0}$ is uniformly bounded. Therefore, for any given $t \in [0, T]$, there exists a subsequence $\{u_\rho(t)\}_{\rho > 0}$ such that $u_\rho(t) \rightharpoonup \tilde{u}(t)$ in V as $\rho \to 0$ for some $\tilde{u}(t) \in V$.

Next, we show that $\tilde{u} \in C(I; K_V)$. In fact, according to the monotonicity of A, we have

$$\frac{1}{\rho}\langle Pu_\rho(t), u_\rho(t) - v\rangle_{V^* \times V}$$
$$\leq \langle A\big(t, u_\rho(t)\big), v - u_\rho(t)\rangle_{V^* \times V} + \langle f_\eta(t), u_\rho(t) - v\rangle_{V^* \times V}$$
$$+ j^0\big(w(t), Mu_\rho(t), Mu_\rho(t); Mv - Mu_\rho(t)\big)$$
$$\leq \langle A(t, v), v - u_\rho(t)\rangle_{V^* \times V} + \langle f_\eta(t), u_\rho(t) - v\rangle_{V^* \times V}$$
$$+ j^0\big(w(t), Mu_\rho(t), Mu_\rho(t); Mv - Mu_\rho(t)\big) \tag{3.13}$$

for all $v \in V$. Taking $v = \tilde{u}(t)$ into (3.13), one has

$$\frac{1}{\rho}\langle Pu_\rho(t), u_\rho(t) - \tilde{u}(t)\rangle_{V^* \times V}$$
$$\leq \langle A\big(t, \tilde{u}(t)\big) - f_\eta(t), \tilde{u}(t) - u_\rho(t)\rangle_{V^* \times V} + j^0\big(w(t), Mu_\rho(t), Mu_\rho(t); M\tilde{u}(t) - Mu_\rho(t)\big).$$

Combining H(j), the continuity of f_η and the compactness of M, we have

$$\limsup_{\rho \to 0}\langle Pu_\rho(t), u_\rho(t) - \tilde{u}(t)\rangle_{V^* \times V} \leq 0. \tag{3.14}$$

Because of Lemma 2.1, P is pseudomonotone, it follows from (3.13) and (3.14) that

$$\langle P\tilde{u}(t), \tilde{u}(t) - v\rangle_{V^* \times V} \leq \liminf_{\rho \to 0}\langle Pu_\rho(t), u_\rho(t) - v\rangle_{V^* \times V}$$
$$\leq \limsup_{\rho \to 0}\langle Pu_\rho(t), u_\rho(t) - v\rangle_{V^* \times V} \leq 0$$

for all $v \in V$. Since $v \in V$ is arbitrary, we know that $P\tilde{u}(t) = 0$ and so $\tilde{u}(t) \in K_V$. Moreover, according to (3.7) and $Pv = 0$ for all $v \in K_V$, one has

$$\langle A\big(t, u_\rho(t)\big), u_\rho(t) - v\rangle_{V^* \times V}$$
$$\leq -\frac{1}{\rho}\langle Pv - Pu_\rho(t), u_0 - u_\rho(t)\rangle_{V^* \times V} + \langle f_\eta(t), u_\rho(t) - v\rangle_{V^* \times V}$$

$$+ j^0\big(w(t), Mu_\rho(t), Mu_\rho(t); Mv - Mu_\rho(t)\big)$$

$$\leq j^0\big(w(t), Mu_\rho(t), Mu_\rho(t); Mv - Mu_\rho(t)\big) + \big\langle f_\eta(t), u_\rho(t) - v\big\rangle_{V^* \times V} \tag{3.15}$$

for all $v \in K_V$. Taking $v = \tilde{u}(t)$ in (3.15) and passing to the upper limit as $\rho \to 0$, we have

$$\limsup_{\rho \to 0} \big\langle A\big(t, u_\rho(t)\big), u_\rho(t) - \tilde{u}(t)\big\rangle_{V^* \times V} \leq 0.$$

Moreover, the pseudomonotonicity of A implies that

$$\big\langle A\big(t, \tilde{u}(t)\big), \tilde{u}(t) - v\big\rangle_{V^* \times V} \leq \liminf_{\rho \to 0} \big\langle A\big(t, u_\rho(t)\big), u_\rho(t) - v\big\rangle_{V^* \times V}. \tag{3.16}$$

Passing to the upper limit as $\rho \to 0$ in (3.15), we obtain

$$\limsup_{\rho \to 0} \big\langle A\big(t, u_\rho(t)\big), u_\rho(t) - v\big\rangle_{V^* \times V}$$

$$\leq j^0\big(w(t), M\tilde{u}(t), M\tilde{u}(t); Mv - M\tilde{u}(t)\big) + \big\langle f_\eta(t), \tilde{u}(t) - v\big\rangle_{V^* \times V}. \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\big\langle A\big(t, \tilde{u}(t)\big), v - \tilde{u}(t)\big\rangle_{V^* \times V} + j^0\big(w(t), M\tilde{u}(t), M\tilde{u}(t); Mv - M\tilde{u}(t)\big) + \big\langle f_\eta(t), \tilde{u}(t) - v\big\rangle_{V^* \times V} \geq 0.$$

Since (3.6) has a unique solution, we know that $\tilde{u}(t) = u_{w\zeta}(t)$ and so $\tilde{u} \in C(I, K_V)$.

Finally, we show the strong convergence of $\{u_{\rho w\zeta}(t)\}$. Indeed, because $\{u_{\rho w\zeta}(t)\}$ is bounded and for any weakly convergent subsequence of $\{u_{\rho w\zeta}(t)\}$ converges weakly to the same limit $u_{w\zeta}(t)$, by Theorem 1.20 in [24], we know that the whole sequence $\{u_{\rho w\zeta}(t)\}$ converges weakly to $u_{w\zeta}(t)$ for any $t \in I$. On the other hand, using the monotonicity of P, one has

$$\big\langle A(t, v), u_{\rho w\zeta}(t) - v\big\rangle_{V^* \times V} \leq \big\langle A\big(t, u_{\rho w\zeta}(t)\big), u_{\rho w\zeta}(t) - v\big\rangle_{V^* \times V}. \tag{3.18}$$

Similar to the proof of (3.15), we have

$$\big\langle A\big(t, u_{\rho w\zeta}(t)\big), u_{\rho w\zeta}(t) - v\big\rangle_{V^* \times V}$$

$$\leq j^0\big(w(t), Mu_{\rho w\zeta}(t), Mu_{\rho w\zeta}(t); Mv - Mu_{\rho w\zeta}(t)\big)$$

$$+ \big\langle f_\eta(t), u_{\rho w\zeta}(t) - v\big\rangle_{V^* \times V}. \tag{3.19}$$

Taking $v = u_{w\zeta}(t)$ in (3.18) and (3.19), and then passing to the limit as $\rho \to 0$, one has

$$\lim_{\rho \to 0} \big\langle A\big(t, u_{\rho w\zeta}(t)\big), u_{\rho w\zeta}(t) - u_{w\zeta}(t)\big\rangle_{V^* \times V} = 0.$$

Using $u_{\rho w\zeta}(t) \rightharpoonup u_{w\zeta}(t)$ in V as $\rho \to 0$, it follows from the strong monotonicity of A that

$$\lim_{\rho \to 0} m_A \big\| u_{\rho w\zeta}(t) - u_{w\zeta}(t)\big\|_V^2 \leq \lim_{\rho \to 0} \big\langle A\big(t, u_{w\zeta}(t)\big) - A\big(t, u_{\rho w\zeta}(t)\big), u_{w\zeta}(t) - u_{\rho w\zeta}(t)\big\rangle_{V^* \times V}$$

$$= 0$$

for all $t \in I$. Consequently, we conclude for each $t \in I$, $u_{\rho w\zeta}(t) \to u_{w\zeta}(t)$ in V as $\rho \to 0$. $\square$

Next, we consider the following auxiliary problem that includes a history-dependent hemivariational inequality and a differential equation in Problem 3.1.

Problem 3.3 For any given $\rho > 0$, $\zeta \in H^1(I; Y_1) \cap L^2(I; Y)$ and $f \in C(I; V^*)$, consider the following problem: find $u_{\rho\zeta} : I \to V$ and $w_{\rho\zeta} : I \to W$ such that, for any $t \in I$,

$$
\begin{cases}
\dot{w}_{\rho\zeta}(t) = F(t, w_{\rho\zeta}(t), u_{\rho\zeta}(t)), \\
\langle A(t, u_{\rho\zeta}(t)) + \int_0^t B(t-s, u_{\rho\zeta}(s), \zeta(s))\, ds, v - u_{\rho\zeta}(t)\rangle_{V^* \times V} \\
\quad + \frac{1}{\rho}\langle Pu_{\rho\zeta}(t), v - u_{\rho\zeta}(t)\rangle_{V^* \times V} + j^0(w_{\rho\zeta}(t), Mu_{\rho\zeta}(t), Mu_{\rho\zeta}(t); Mv(t) - Mu_{\rho\zeta}(t)) \\
\quad \geq \langle f(t), v - u_{\rho\zeta}(t)\rangle_{V^* \times V}, \quad \forall v \in V, \\
w_{\rho\zeta}(0) = w_0.
\end{cases}
$$

Lemma 3.2 *Assume that H(A), H(B), H(j), and H(F) hold. If $m_A > \max\{c_0\|M\|^2_{L(V;X)}, \alpha_1\}$, then one has the following conclusions:*

(i) *for any given $\rho > 0$, $\zeta \in H^1(I; Y_1) \cap L^2(I; Y)$ and $f \in C(I; V^*)$, Problem 3.3 has a unique solution $(u_{\rho\zeta}, w_{\rho\zeta}) \in C(I; V) \times C^1(I; W)$;*

(ii) *$(u_{\rho\zeta}, w_{\rho\zeta})$ converges strongly to (u_ζ, w_ζ) as $\rho \to 0$, where (u_ζ, w_ζ) is the unique solution of the following problem: for any given $\zeta \in H^1(I; Y_1) \cap L^2(I; Y)$ and $f \in C(I; V^*)$, find $u_\zeta : I \to K_V$ and $w_\zeta : I \to W$ such that, for any $t \in I$,*

$$
\begin{cases}
\dot{w}_\zeta(t) = F(t, w_\zeta(t), u_\zeta(t)), \\
\langle A(t, u_\zeta(t)) + \int_0^t B(t-s, u_\zeta(s), \zeta(s))\, ds, v - u_\zeta(t)\rangle_{V^* \times V} \\
\quad + j^0(w_\zeta(t), Mu_\zeta(t), Mu_\zeta(t); Mv(t) - Mu_\zeta(t)) \geq \langle f(t), v - u_\zeta(t)\rangle_{V^* \times V}, \quad \forall v \in K_V, \\
w_\zeta(0) = w_0.
\end{cases}
$$

Proof (i) Define an operator $S : C^1(I; W) \to C(I; V)$ by setting $S(w_\rho)(t) = u_{\rho w\zeta}(t)$. From Lemma 3.3 in [4], for each given $\rho > 0$, $\zeta \in H^1(I; Y_1) \cap L^2(I; Y)$,

$$
\dot{w}_{\rho\zeta}(t) = F\big(t, w_{\rho\zeta}(t), S(w_{\rho\zeta})(t)\big)
$$

has a unique solution $w_{\rho\zeta} \in C^1(I; W)$ and $(S(w_\rho), w_{\rho\zeta}) \in C(I; V) \times C^1(I; W)$ is the unique solution of Problem 3.3.

(ii) Consider an operator $\Lambda : C(I; W) \to C^1(I; K_V)$ defined as follows:

$$
\Lambda w_\zeta(t) = \int_0^t F\big(s, w_\zeta(s), S(w_\zeta)(s)\big)\, ds + w_0, \quad \forall t \in [0, T].
$$

Then, by the proof of Lemma 3.3 in [4], we know that Λ has a unique fixed point $w_\zeta(t)$. It follows from H(F) that

$$
\begin{aligned}
\big\|w_{\rho\zeta}(t) - w_\zeta(t)\big\| &= \big\|\Lambda w_{\rho\zeta}(t) - \Lambda w_\zeta(t)\big\| \\
&= \left\|\int_0^t F\big(s, w_{\rho\zeta}(s), S(w_{\rho\zeta})(s)\big)\, ds - \int_0^t F\big(s, w_\zeta(s), S(w_\zeta)(s)\big)\, ds\right\| \\
&\leq L_F \int_0^t \big\|w_{\rho\zeta}(s) - w_\zeta(s)\big\|\, ds + L_F \int_0^t \big\|u_{\rho w\zeta}(s) - u_{w\zeta}(s)\big\|\, ds. \tag{3.20}
\end{aligned}
$$

Now, Gronwall's inequality yields

$$
\begin{aligned}
&\big\| w_{\rho\zeta}(t) - w_{\zeta}(t) \big\| \\
&\le L_F \int_0^t \big\| u_{\rho w\zeta}(s) - u_{w\zeta}(s) \big\| \, ds + L_F^2 e^{L_F T} \int_0^t \int_0^t \big\| u_{\rho w\zeta}(s) - u_{w\zeta}(s) \big\| \, ds \, dl \\
&\le \big(L_F + T L_F^2 e^{L_F T} \big) \int_0^t \big\| u_{\rho w\zeta}(s) - u_{w\zeta}(s) \big\| \, ds.
\end{aligned}
\tag{3.21}
$$

Since for each $s \in I$, $u_{\rho w\zeta}(s) \to u_{w\zeta}(s)$ in V as $\rho \to 0$ and $u_{\rho w\zeta}, u_{w\zeta} \in C(I; V)$, one has

$$
w_{\rho\zeta}(t) \to w_{\zeta}(t) \quad \text{in } W \text{ as } \rho \to 0 \text{ for each } t \in I.
$$

Letting $u_{w\zeta}(t) = u_{\zeta}(t)$ and $u_{\rho w\zeta}(t) = u_{\rho\zeta}(t)$, we can conclude that

$$
\big(u_{\rho\zeta}(t), w_{\rho\zeta}(t) \big) \to \big(u_{\zeta}(t), w_{\zeta}(t) \big) \quad \text{as } \rho \to 0
$$

for each $t \in I$. $\qquad\square$

Finally, we only need to solve the following parabolic variational inequality.

Problem 3.4 For any given $\rho > 0$, consider the following problem: find $\zeta_\rho : I \to K_Y$ such that, for all $t \in I$,

$$
\big\langle \dot{\zeta}_\rho(t), \eta - \zeta_\rho(t) \big\rangle_{Y^* \times Y} + a\big(\zeta_\rho(t), \eta - \zeta_\rho(t) \big) \ge \big\langle \phi\big(t, u_\rho(t), \zeta_\rho(t)\big), \eta - \zeta_\rho(t) \big\rangle_{Y^* \times Y},
$$
$$
\forall \eta \in K_Y
\tag{3.22}
$$

with $\zeta_\rho(0) = \zeta_0$.

Lemma 3.3 ([1, 9]) *Suppose that condition H(a) holds. Then, for any given $\lambda \in L^2(I; Y_1)$, there exists a unique $\zeta \in H^1(I; Y_1) \cap L^2(I; Y)$ such that*

$$
\big\langle \dot{\zeta}(t), \eta - \zeta(t) \big\rangle_{Y_1} + a\big(\zeta(t), \eta - \zeta(t) \big) \ge \big\langle \lambda(t), \eta - \zeta(t) \big\rangle_{Y_1}, \quad \forall \eta \in K_Y
\tag{3.23}
$$

with $\zeta(0) = \zeta_0 \in K_Y$. Moreover, if ζ_i is the unique solution to problem (3.23) for $\lambda_i \in L^2(I; Y_1)$ with $i = 1, 2$, then

$$
\big\| \zeta_1(t) - \zeta_2(t) \big\|_{Y_1}^2 \le d_1 \int_0^t \big\| \lambda_1(s) - \lambda_2(s) \big\|_{Y_1}^2 \, ds \quad \text{for a.e. } t \in (0, T)
\tag{3.24}
$$

with $d_1 > 0$.

Lemma 3.4 *Assume that $H(A)$, $H(B)$, $H(j)$, $H(F)$, $H(\phi)$, and $H(a)$ hold. If $m_A > \max\{c_0 \times \|M\|_{L(V;X)}^2, \alpha_1\}$, then one has the following conclusions:*
   (i) *for any given $\rho > 0$, Problem 3.4 has a unique solution $\zeta_\rho \in H^1(I; Y_1) \cap L^2(I; Y)$;*

(ii) ζ_ρ *converges strongly to ζ as $\rho \to 0$, where ζ is the the unique solution of the following problem: find $\zeta : I \to K_Y$ such that, for all $t \in I$,*

$$\langle \dot{\zeta}(t), \eta - \zeta(t) \rangle_{Y^* \times Y} + a\big(\zeta(t), \eta - \zeta(t)\big) \geq \langle \phi\big(t, u_\zeta(t), \zeta(t)\big), \eta - \zeta(t) \rangle_{Y^* \times Y},$$

$$\forall \eta \in K_Y \tag{3.25}$$

with $\zeta(0) = \zeta_0$.

Proof (i) Let $u_\rho(t) = u_{\rho\zeta}(t)$ in Lemma 3.2. Then, it follows from Lemma 3.5 in [4] that Problem 3.4 has a unique solution $\zeta_\rho \in H^1(I; Y_1) \cap L^2(I; Y)$.

(ii) Let $\phi_\zeta(t) := \phi(t, u_\zeta(t), \zeta(t))$. Then, by taking $\lambda = \phi_\zeta$ in Lemma 3.3 and using H(ϕ), we have

$$\big\|\zeta(t) - \zeta_\rho(t)\big\|_{Y_1}^2 \leq d_1 \int_0^t \big\|\phi\big(s, u_{\rho\zeta}(s), \zeta_\rho(s)\big) - \phi\big(s, u_\zeta(s), \zeta(s)\big)\big\|_{Y_1}^2 ds$$

$$\leq 2d_1 L_\phi^2 \int_0^t \big\|u_{\rho\zeta}(s) - u_\zeta(s)\big\|_V^2 + \big\|\zeta(s) - \zeta_\rho(s)\big\|_{Y_1}^2 ds. \tag{3.26}$$

Now, Gronwall's inequality yields

$$\big\|\zeta(t) - \zeta_\rho(t)\big\|_{Y_1}^2$$

$$\leq 2d_1 L_\phi^2 \int_0^t \big\|u_{\rho\zeta}(s) - u_\zeta(s)\big\|_V^2 ds + 4d_1^2 L_\phi^4 e^{2d_1 L_\phi^2 T} \int_0^t \int_0^t \big\|u_{\rho\zeta}(s) - u_\zeta(s)\big\|_V^2 ds\, dl$$

$$\leq \big(2d_1 L_\phi^2 + 4d_1^2 L_\phi^4 T e^{2d_1 L_\phi^2 T}\big) \int_0^t \big\|u_{\rho\zeta}(s) - u_\zeta(s)\big\|_V^2 ds. \tag{3.27}$$

Since for each $s \in I$, $u_{\rho\zeta}(s) \to u_\zeta(s)$ in V as $\rho \to 0$ and $u_{\rho\zeta} \in C(I; V)$, $u_{w\zeta} \in C(I; K_V)$, one has

$$\zeta_\rho(t) \to \zeta(t) \quad \text{as } \rho \to 0$$

for each $t \in I$. □

Theorem 3.1 *Suppose that the assumptions H(A), H(B), H(j), H(F), H(ϕ), and H(a) hold and $m_A > \max\{c_0 \|M\|_{L(V;X)}^2, \alpha_1\}$. Then, one has the following conclusions:*

(i) *for any given $\rho > 0$, Problem 3.1 has a unique solution*
$(\zeta_\rho, u_{\rho\zeta}, w_{\rho\zeta}) \in (H^1(I; Y_1) \cap L^2(I; Y)) \times C(I; V) \times C(I; W)$;

(ii) *$(\zeta_\rho, u_{\rho\zeta}, w_{\rho\zeta})$ converges strongly to (ζ^*, u^*, w^*) as $\rho \to 0$, where (ζ^*, u^*, w^*) is the unique solution of (1.1).*

Proof Let $(u_{\rho\zeta}, w_{\rho\zeta})$ be the same as in Lemma 3.2, ζ_ρ be the same as in Lemma 3.4, and $(\zeta_\rho, u_{\rho\zeta}, w_{\rho\zeta}) = (\zeta^*, u^*, w^*)$ be the same as in Remark 3.2. Then, it is easy to see that the conclusions (i) and (ii) are true. This finishes the proof. □

4 An application

In this section, we use the abstract results obtained in Sect. 3 to study the long-memory elastic frictional contact problem with wear and damage. To this end, we first recall some notations.

Let $\mathbb{S}^d$ denote the second-order symmetric tensors on $\mathbb{R}^d$. For any given $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$, define

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij} := \sum_{i,j=1}^{d} \sigma_{ij}\tau_{ij}$$

and $\|\boldsymbol{\tau}\| = \sqrt{\boldsymbol{\tau} \cdot \boldsymbol{\tau}}$. We use notations $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})) = (\frac{1}{2}(u_{i,j} + u_{j,i}))$, $i, j = 1, 2, \ldots, d$ to denote the displacement vector, the stress tensor and the linearized strain tensor, respectively, where $u_{i,j} := \frac{\partial u_i}{\partial x_j}$. Here and below, the spatial derivative is defined in the sense of distribution. Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with Lipschitz continuous boundary $\Gamma := \partial\Omega$. Let $\boldsymbol{v}$ denote the unit outward normal vector defined *a.e.* on Γ. The normal and tangential components of stress field $\boldsymbol{\sigma}$ and displacement field $\boldsymbol{u}$ on Γ are denoted by $\sigma_v = (\boldsymbol{\sigma}\boldsymbol{v}) \cdot \boldsymbol{v}$, $u_v = \boldsymbol{u} \cdot \boldsymbol{v}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{v} - \sigma_v\boldsymbol{v}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_v\boldsymbol{v}$, respectively.

Consider a viscoelastic body that occupies Ω. The boundary Γ can be divided into three disjoint measurable parts Γ_1, Γ_2, and Γ_3 with meas(Γ_1) > 0. We are interested in the evolution of the body on the time interval $I := [0, T]$ with $T > 0$. We also use the following abbreviations to simplify the notations $\mathcal{Q} = \Omega \times I$, $\Sigma = \Gamma \times I$, $\Sigma_i = \Gamma_i \times I$, $i = 1, 2, 3$. The time partial derivative for a function $f(\boldsymbol{x}, t)$ is denoted by $\dot{f}(\boldsymbol{x}, t)$. For the sake of simplicity, we do not mention the dependence of different functions on variable $\boldsymbol{x}$.

Thus, the long-memory elastic frictional contact problem with wear and damage can be modeled as follows (see [4]).

Problem 4.1 Find a displacement field $\boldsymbol{u} : \mathcal{Q} \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \mathcal{Q} \to \mathbb{S}^d$, a damage field $\zeta : \mathcal{Q} \to [0, 1]$ and a wear function $w : \Sigma_3 \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\big(t, \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big) + \int_0^t \mathcal{B}\big(t - s, \boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big), \zeta(s)\big)\, ds \quad \text{in } \mathcal{Q}, \tag{4.1}$$

$$\dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) \ni \phi\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \zeta\big) \quad \text{in } \mathcal{Q}, \tag{4.2}$$

$$\frac{\partial\zeta}{\partial v} = 0 \quad \text{on } \Sigma, \tag{4.3}$$

$$-\operatorname{Div}\boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \quad \text{in } \mathcal{Q}, \tag{4.4}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Sigma_1, \tag{4.5}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{v} = \boldsymbol{f}_2(t) \quad \text{on } \Sigma_2, \tag{4.6}$$

$$u_v(t) \le g, \qquad \sigma_v(t) + \xi_v(t) \le 0 \quad \text{on } \Sigma_3, \tag{4.7}$$

$$\big(u_v(t) - g\big)\big(\sigma_v(t) + \xi_v(t)\big) = 0 \quad \text{on } \Sigma_3, \tag{4.8}$$

$$\xi_v(t) \in \partial j_v\big(w(t), u_v(t), u_v(t)\big) \quad \text{on } \Sigma_3, \tag{4.9}$$

$$-\boldsymbol{\sigma}_\tau(t) = \in \partial j_\tau\big(w(t), u_v(t), \boldsymbol{u}_\tau(t)\big) \quad \text{on } \Sigma_3, \tag{4.10}$$

$$\dot{w}(t) = \alpha(t)p\big(u_v(t) - w(t)\big) \quad \text{on } \Sigma_3, \tag{4.11}$$

$$w(0) = 0, \zeta(0) = \zeta_0 \in (0, 1) \quad \text{on } \Gamma_3. \tag{4.12}$$

Relations (4.7)–(4.9) show that the body contacts with a rigid foundation covered by a layer of soft material, where $g > 0$ is the thickness of the soft material.

We use the standard Sobolev spaces on Ω and Γ. In particular, let $H^1 := W^{1,2}(\Omega;\mathbb{R}^d)$ and $H = L^2(\Omega;\mathbb{R}^d)$. Let $V = \{\mathbf{v} \in H^1 \mid \mathbf{v} = 0 \text{ a.e. on } \Gamma_1\}$ endowed with the norm

$$\|\mathbf{u}\|_V := \|\mathbf{u}\|_{H^1} = \|\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^d)} + \|\nabla\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{d\times d})},$$

where $\nabla\mathbf{u} = (\frac{\partial u_i}{\partial x_j})$ for $i,j = 1,\ldots,d$ with $\mathbf{u} \in H^1$. Let $\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}) = (\frac{\partial \sigma_{ij}}{\partial x_j})$ with $\boldsymbol{\sigma} \in W^{1,2}(\Omega;\mathbb{S}^d)$. Then, we have the following Green formula

$$\langle \mathrm{Div}\,\boldsymbol{\sigma},\mathbf{v}\rangle_H + \langle\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\mathbf{u})\rangle_{L^2(\Omega;\mathbb{S}^d)} = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\mathbf{v}\,d\Gamma,$$

where

$$\langle\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\mathbf{u})\rangle_{L^2(\Omega;\mathbb{S}^d)} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{u})\,d\Omega.$$

From the assumption of $\mathrm{meas}(\Gamma_1) > 0$, the space V can be endowed with the inner product

$$\langle\mathbf{u},\mathbf{v}\rangle_V = \langle\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})\rangle_{L^2(\Omega;\mathbb{S}^d)},$$

which yields the completeness of V and allows us to use Korn's inequality.

Let $Y = H^1(\Omega;\mathbb{R})$ and $Y_1 = L^2(\Omega;\mathbb{R})$ endowed with the canonical inner products and norms. Denote two convex sets $K_V = \{\mathbf{v} \in V \mid v_\nu \leq g \text{ a.e. on } \Gamma_3\}$ and $K_Y = \{u \in Y \mid 0 \leq u \leq 1 \text{ a.e. in } \Omega\}$. We define $\gamma : V \to L^2(\Gamma_3;\mathbb{R}^d)$ as the trace operator and assume that j_ν and j_τ admit the regular assumption. Let

$$\langle\mathbf{f}(t),\mathbf{v}-\mathbf{u}\rangle_{V^*\times V} = \langle\mathbf{f_2}(t),\gamma\mathbf{v}-\gamma\mathbf{u}(t)\rangle_{L^2(\Gamma_2;\mathbb{R}^d)} + \langle\mathbf{f}_0(t),\mathbf{v}-\mathbf{u}(t)\rangle_{L^2(\Omega;\mathbb{R}^d)},$$

$$j(w,\gamma\mathbf{u},\gamma\mathbf{v}) = \int_{\Gamma_3} j_\nu(w,u_\nu,v_\nu)\,d\Gamma + \int_{\Gamma_3} j_\tau(w,\mathbf{u}_\tau,\mathbf{v}_\tau)\,d\Gamma,$$

$$a(\zeta,\eta) = \kappa\int_\Omega \nabla\zeta\cdot\nabla\eta\,dx \quad \text{for all } \zeta,\eta\in Y.$$

Then, the variational formulation of Problem 4.1 can be described as follows (see [4]).

Problem 4.2 Find $\mathbf{u} : I \to K_V$, $\zeta : I \to K_Y$ and $w : I \to L^2(\Gamma_3;\mathbb{R})$ such that, for all $t \in I$,

$$\sigma(t) = \mathcal{A}\big(t,\boldsymbol{\varepsilon}(\mathbf{u}(t))\big) + \int_0^t \mathcal{B}\big(t-s,\boldsymbol{\varepsilon}(\mathbf{u}(s)),\zeta(s)\big)\,ds \quad \text{in } \Omega, \tag{4.13}$$

$$\langle\sigma(t),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\mathbf{u}(t))\rangle_{L^2(\Omega;\mathbb{S}^d)} + j^0(w,\gamma\mathbf{u},\gamma\mathbf{u};\gamma\mathbf{v}-\gamma\mathbf{u})$$

$$\geq \langle\mathbf{f}(t),\mathbf{v}-\mathbf{u}(t)\rangle_{V^*\times V}, \quad \forall\mathbf{v}\in K_V, \tag{4.14}$$

$$\langle(\dot\zeta(t),\eta-\zeta(t)\rangle_{Y_1} + a(\zeta,\eta-\zeta) \geq \langle\phi\big(t,\boldsymbol{\varepsilon}(\mathbf{u}(t)),\zeta(t)\big),\eta-\zeta(t)\rangle_{Y_1}, \quad \forall\eta\in K_Y, \tag{4.15}$$

$$\dot w(t) = \alpha(t)p\big(u_\nu(t)-w(t)\big) \quad \text{in } \Gamma_3, \tag{4.16}$$

$$w(0) = 0, \qquad \zeta(0) = \zeta_0 \in (0,1). \tag{4.17}$$

Now, we turn to introduce the following penalized problem concerning Problem 4.1.

Problem 4.3 Find a displacement field $\boldsymbol{u}_\rho : \mathcal{Q} \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}_\rho : \mathcal{Q} \to \mathbb{S}^d$, a damage field $\zeta : \mathcal{Q} \to [0,1]$ and a wear function $w_\rho : \Sigma_3 \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{A}\big(t, \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\rho(t)\big)\big) + \int_0^t \mathcal{B}\big(t-s, \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\rho(s)\big), \zeta_\rho(s)\big)\, ds \quad \text{in } \mathcal{Q}, \tag{4.18}$$

$$\dot{\zeta}_\rho - \kappa \Delta \zeta_\rho + \partial I_{[0,1]}(\zeta_\rho) \ni \phi\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}_\rho(t)\big), \zeta_\rho\big) \quad \text{in } \mathcal{Q}, \tag{4.19}$$

$$\frac{\partial \zeta_\rho}{\partial \nu} = 0 \quad \text{on } \Sigma, \tag{4.20}$$

$$-\operatorname{Div} \boldsymbol{\sigma}_\rho(t) = \boldsymbol{f}_0(t) \quad \text{in } \mathcal{Q}, \tag{4.21}$$

$$\boldsymbol{u}_\rho(t) = \boldsymbol{0} \quad \text{on } \Sigma_1, \tag{4.22}$$

$$\boldsymbol{\sigma}_\rho(t)\boldsymbol{v} = \boldsymbol{f}_2(t) \quad \text{on } \Sigma_2, \tag{4.23}$$

$$\sigma_{\rho\nu}(t) + \xi_{\rho\nu}(t) + \frac{1}{\rho}\big(u_{\rho\nu}(t) - g\big)^+ = 0 \quad \text{on } \Sigma_3, \tag{4.24}$$

$$\xi_{\rho\nu}(t) \in \partial j_\nu\big(w_\rho(t), u_{\rho\nu}(t), u_{\rho\nu}(t)\big) \quad \text{on } \Sigma_3, \tag{4.25}$$

$$-\boldsymbol{\sigma}_{\rho\tau}(t) = \in \partial j_\tau\big(w_\rho(t), u_{\rho\nu}(t), \boldsymbol{u}_{\rho\tau}(t)\big) \quad \text{on } \Sigma_3, \tag{4.26}$$

$$\dot{w}_\rho(t) = \alpha(t) p\big(u_{\rho\nu}(t) - w_\rho(t)\big) \quad \text{on } \Sigma_3, \tag{4.27}$$

$$w_\rho(0) = 0, \zeta_\rho(0) = \zeta_0 \in (0,1) \quad \text{on } \Gamma_3, \tag{4.28}$$

where the operator "+" above a function represents the positive part of it.

It is worth noting that, compared with Problem 4.2, the contact conditions (4.7) and (4.8) are replaced by (4.24) with $\rho > 0$.

Now, we define an operator $P : V \to V^*$ by

$$\langle P\boldsymbol{u}, \boldsymbol{v}\rangle_{V^* \times V} = \int_{\Gamma_3} (u_\nu - g)^+ v_\nu \, d\Gamma, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V.$$

Then, it follows from the arguments in [4] that the variational formulation of Problem 4.3 can be stated as follows.

Problem 4.4 Find $\boldsymbol{u}_\rho : I \to V$, $\zeta_\rho : I \to K_Y$ and $w_\rho : I \to L^2(\Gamma_3; \mathbb{R})$ such that, for all $t \in I$,

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{A}\big(t, \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\rho(t)\big)\big) + \int_0^t \mathcal{B}\big(t-s, \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\rho(s)\big), \zeta_\rho(s)\big)\, ds \quad \text{in } \Omega, \tag{4.29}$$

$$\big\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\rho(t)\big)\big\rangle_{L^2(\Omega; \mathbb{S}^d)} + \frac{1}{\rho}\big\langle P\boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t)\big\rangle_{V^* \times V}$$

$$+ j^0(w, \gamma\boldsymbol{u}_\rho; \gamma\boldsymbol{v} - \gamma\boldsymbol{u}_\rho) \geq \big\langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t)\big\rangle_{V^* \times V}, \quad \forall \boldsymbol{v} \in V, \tag{4.30}$$

$$\big\langle (\dot{\zeta}(t), \eta - \zeta(t)\big\rangle_{Y_1} + a(\zeta_\rho, \eta - \zeta_\rho)$$

$$\geq \big\langle \phi\big(t, \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\rho(t)\big), \zeta_\rho(t)\big), \eta - \zeta_\rho(t)\big\rangle_{Y_1}, \quad \forall \eta \in K_Y, \tag{4.31}$$

$$\dot{w}_\rho(t) = \alpha(t)p\big(u_{\rho v}(t) - w_\rho(t)\big) \quad \text{in } \Gamma_3, \tag{4.32}$$

$$w_\rho(0) = 0, \qquad \zeta_\rho(0) = \zeta_0 \in (0,1). \tag{4.33}$$

In order to solve Problem 4.4, we need the following hypotheses.

$H(1)$: The elasticity operator $\mathcal{A} : \Omega \times I \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\ \mathcal{A}(\cdot, t, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \forall (t, \boldsymbol{\varepsilon}) \in I \times \mathbb{S}^d; \\ (b)\ \mathcal{A}(\boldsymbol{x}, \cdot, \cdot) \text{ is continuous on } I \times \mathbb{S}^d \text{ for a.e. } \boldsymbol{x} \in \Omega; \\ (c)\ \mathcal{A}(\boldsymbol{x}, t, \cdot) \text{ is Lipschitz continous with } L_\mathcal{A} > 0 \text{ for all } t \in I, i.e., \\ \quad \|\mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2)\| \le L_\mathcal{A} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega; \\ (d)\ \mathcal{A}(\boldsymbol{x}, t, \cdot) \text{ is strong monotone with } m_\mathcal{A} > 0 \text{ for all } t \in I, i.e., \\ \quad (\mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_\mathcal{A} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega; \\ (e)\ \text{for all } t \in I \text{ and a.e. } \boldsymbol{x} \in \Omega;, \mathcal{A}(\boldsymbol{x}, t, 0_{\mathbb{S}^d}) = 0_{\mathbb{S}^d}. \end{cases} \tag{4.34}$$

$H(2)$: The relaxation operator $\mathcal{B} : \Omega \times I \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\ \mathcal{B}(\cdot, t, \boldsymbol{\varepsilon}, \zeta) \text{ is measurable on } \Omega, \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d, \forall t \in I, \forall \zeta \in \mathbb{R}; \\ (b)\ \mathcal{B}(\boldsymbol{x}, \cdot, \boldsymbol{\varepsilon}, \zeta) \text{ is continuous on } I \text{ for a.e. } \boldsymbol{x} \in \Omega \text{ and all } (\boldsymbol{\varepsilon}, \zeta) \in \mathbb{S}^d \times \mathbb{R}; \\ (c)\ \mathcal{B}(\boldsymbol{x}, t, \cdot, \cdot) \text{ is Lipschitz continuous with } L_\mathcal{B} > 0, \forall t \in I \text{ and a.e. } \boldsymbol{x} \in \Omega, i.e., \\ \quad \|\mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1, \zeta_1) - \mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2, \zeta_2)\| \le L_\mathcal{B}(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|) \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ all } \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega; \\ (d)\ \text{there exists a function } \varrho_\mathcal{B} \in L^2(I; \mathbb{R}^+) \text{ such that } \|\mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}, \zeta)\| \le \varrho_\mathcal{B}(t)(|\zeta| + \|\boldsymbol{\varepsilon}\|) \\ \quad \text{for all } (t, \boldsymbol{\varepsilon}, \zeta) \in I \times \mathbb{S}^d \times \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{4.35}$$

$H(3)$: The normal compliance function $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$ satisfies

$$\begin{cases} (a)\ p(\cdot, r) \text{ is measurable on } \Gamma_3, \forall r \in \mathbb{R}; \\ (b)\ p(\boldsymbol{x}, \cdot) \text{ is Lipschitz continuous with } \tilde{L}_p > 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, i.e., \\ \quad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \le L_p |r_1 - r_2|, \forall r_1, r_2 \in \mathbb{R}. \end{cases} \tag{4.36}$$

$H(4)$: The damage source function $\phi : \Omega \times I \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \phi(\cdot, t, \boldsymbol{\varepsilon}, \zeta) \text{ is measurable on } \Omega, \text{ for all } t \in I, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}; \\ (b)\ \phi(\boldsymbol{x}, t, \cdot, \cdot) \text{ is Lipschitz continuous with } L_\phi > 0 \text{ for all } t \in I \text{ a.e. } \boldsymbol{x} \in \Omega, i.e., \\ \quad \|\phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \zeta_1) - \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \zeta_2)\| \le \tilde{L}_\phi(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|) \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ all } \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega; \\ (c)\ \phi(\cdot, \cdot, \boldsymbol{0}_{\mathbb{S}^d}, 0_\mathbb{R}) \in L^2(I; L^2(\Omega; \mathbb{R})). \end{cases} \tag{4.37}$$

$H(5)$: The normal compliance function $j_\nu : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$
\begin{cases}
(a)\ j_\nu(\cdot, r_1, r_2, r_3) \text{ is measurable on } \Gamma_3 \text{ for all } r_1, r_2, r_3 \in \mathbb{R} \text{ and there exists} \\
\quad e \in L^2(\Gamma_3, \mathbb{R}) \text{ such that } j_\nu(\cdot, r_1, r_2, e(\cdot)) \in L^1(\Gamma_3, \mathbb{R}) \text{ for all } r_1, r_2 \in \mathbb{R}; \\
(b)\ j_\nu(\mathbf{x}, r_1, r_2, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } \mathbf{x} \in \Gamma_3; \\
(c)\ \text{There are two constants } \tilde{c}_0, \tilde{c}_1 > 0 \text{ such that, for a.e. } \mathbf{x} \in \Gamma_3, \\
\quad |\partial j_\nu(\cdot, r_1, r_2, r_3)| \le \tilde{c}_1(1 + |r_1| + |r_2|) + \tilde{c}_0|r_3|, \forall r_1, r_2, r_3 \in \mathbb{R}; \\
(d)\ \text{There are two constants } \tilde{\alpha}_0, \tilde{\alpha}_1 > 0 \text{ such that for a.e. } \mathbf{x} \in \Gamma_3, \\
\quad j_\nu^0(\mathbf{x}, w_1, s_1, r_1; r_2 - r_1) + j_\nu^0(\mathbf{x}, w_2, s_2, r_2; r_1 - r_2) \\
\quad\quad \le |r_1 - r_2|(\tilde{\alpha}_0|w_1 - w_2| + \tilde{\alpha}_1|s_1 - s_2|), \forall r_1, r_2, s_1, s_2, w_1, w_2 \in \mathbb{R}.
\end{cases}
\tag{4.38}
$$

$H(6)$: The normal compliance function $j_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$
\begin{cases}
(a)\ j_\tau(\cdot, r_1, r_2, \mathbf{r}_3) \text{ is measurable on } \Gamma_3 \text{ for all } r_1, r_2 \in \mathbb{R} \text{ and } \mathbf{r}_3 \in \mathbb{R}^d \text{ and there exists} \\
\quad e \in L^2(\Gamma_3, \mathbb{R}^d) \text{ such that } j_\nu(\cdot, r_1, r_2, e(\cdot)) \in L^1(\Gamma_3, \mathbb{R}) \text{ for all } r_1, r_2 \in \mathbb{R}; \\
(b)\ j_\nu(\mathbf{x}, r_1, r_2, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } \mathbf{x} \in \Gamma_3; \\
(c)\ \text{There are two constants } \bar{c}_0, \bar{c}_1 > 0 \text{ such that for a.e. } \mathbf{x} \in \Gamma_3, \\
\quad |\partial j_\nu(\cdot, r_1, r_2, \mathbf{r}_3)| \le \bar{c}_1(1 + |r_1| + |r_2|) + \bar{c}_0\|\mathbf{r}_3\|, \forall r_1, r_2 \in \mathbb{R}, \forall \mathbf{r}_3 \in \mathbb{R}^d; \\
(d)\ \text{There are two constants } \bar{\alpha}_0, \bar{\alpha}_1 > 0 \text{ such that for a.e. } \mathbf{x} \in \Gamma_3, \\
\quad j_\nu^0(\mathbf{x}, w_1, s_1, \mathbf{r}_1; \mathbf{r}_2 - \mathbf{r}_1) + j_\nu^0(\mathbf{x}, w_2, s_2, \mathbf{r}_2; \mathbf{r}_1 - \mathbf{r}_2) \\
\quad\quad \le \|\mathbf{r}_1 - \mathbf{r}_2\|(\bar{\alpha}_0|w_1 - w_2| + \bar{\alpha}_1|s_1 - s_2|), \forall s_1, s_2, w_1, w_2 \in \mathbb{R}, \forall \mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^d.
\end{cases}
\tag{4.39}
$$

Moreover, we use the data $\mathbf{f}_0, \mathbf{f}_2,$ and α that satisfy the following conditions:

$$
\mathbf{f}_0 \in C(I; H), \qquad \mathbf{f}_2 \in C\big(I; L^2\big(\Gamma_2; \mathbb{R}^d\big)\big), \qquad \alpha \in C\big(I; L^\infty(\Gamma_3; \mathbb{R})\big).
\tag{4.40}
$$

Given Banach spaces $V = \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{u} = 0 \text{ on } \Gamma_1\}$ and $Y = H^1(\Omega; \mathbb{R})$, and Hilbert spaces $H = L^2(\Omega; \mathbb{R}^d)$ and $Y_1 = L^2(\Omega; \mathbb{R})$, by the basic embedding theory of Sobolev spaces, we know that (V, H, V^*) and (Y, Y_1, Y^*) form two Gelfand triples. Let $X = L^2(\Gamma_3; \mathbb{R}^d)$ and $W = L^2(\Gamma_3; \mathbb{R})$.

Now, we give the following theorem.

Theorem 4.1 *Suppose that assumptions* (4.34)–(4.40) *hold and*

$$
m_{\mathcal{A}} > \|\gamma\|^2 \max\{\tilde{\alpha}_1 + \bar{\alpha}_1, \tilde{c}_0 + \bar{c}_0\}.
$$

Then, one has the following conclusions:
  (i) *Problem* 4.2 *has a unique solution*
     $(\zeta, \mathbf{u}_\zeta, w_\zeta) \in (H^1(I; Y_1) \cap L^2(I; Y)) \times C(I; K_V) \times C^1(I; W);$ *for any given $\rho > 0$,*
     Problem 4.4 *has a unique solution*
     $(\zeta_\rho, \mathbf{u}_{\rho\zeta}, w_{\rho\zeta}) \in (H^1(I; Y_1) \cap L^2(I; Y)) \times C^1(I; K_V) \times C^1(I; W);$
  (ii) $(\zeta_\rho \mathbf{u}_{\rho\zeta}, w_{\rho\zeta})$ *converges strongly to* $(\zeta, \mathbf{u}_\zeta, w_\zeta)$ *as* $\rho \to 0$.

Proof The unique solvability of Problem 4.2 is the direct conclusion of Theorem 4.1 in [4]. Define operators $A(t, \cdot) : V \to V^*$, $B(t, \cdot, \cdot) : V \times Y \to V^*$, $P(\cdot) : V \to V^*$, $F(t, \cdot, \cdot) : W \times V \to W$, $\phi(t, \cdot, \cdot) : V \times Y \to Y^*$, a functional $j(\cdot, \cdot, \cdot) : W \times X \times X \to \mathbb{R}$ and a symmetric bilinear form $a(\cdot, \cdot) : Y \times Y \to \mathbb{R}$ by setting

$$
\begin{cases}
\langle A(t, \boldsymbol{u}), \boldsymbol{v} \rangle_{V^* \times V} = \int_\Omega \mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, d\Omega, \\
\langle B(t, \boldsymbol{u}, \zeta), \boldsymbol{v} \rangle_{V^* \times V} = \int_\Omega \mathcal{B}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, d\Omega, \\
\langle P\boldsymbol{u}, \boldsymbol{v} \rangle_{V^* \times V} = \int_{\Gamma_3} (u_\nu - g)^+ v_\nu \, d\Gamma, \\
F(t, w, \boldsymbol{u}) = \alpha(t) p(u_\nu - w), \\
\phi(t, \boldsymbol{u}, \zeta) = \lambda_D (\frac{1-\zeta}{\zeta}) - \frac{1}{2} \lambda_E \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2 + \lambda_w, \\
j(w, \gamma\boldsymbol{u}, \gamma\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(w, u_\nu, v_\nu) \, d\Gamma + \int_{\Gamma_3} \boldsymbol{j}_\tau(w, \boldsymbol{u}_\tau, \boldsymbol{v}_\tau) \, d\Gamma, \\
a(\zeta, \eta) = \kappa \int_\Omega \nabla\zeta \cdot \nabla\eta \, d\Omega
\end{cases}
$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $w \in W$, $\zeta, \eta \in Y$ and $t \in I$. Then, Problem 4.2 can be transformed as follows:

$$
\dot{w}_\rho(t) = F\big(t, w_\rho(t), \boldsymbol{u}_\rho(t)\big),
$$

$$
\left\langle A\big(t, \boldsymbol{u}_\rho(t)\big) + \int_0^t B\big(t - s, \boldsymbol{u}_\rho(s), \zeta_\rho(s)\big) \, ds, \boldsymbol{v} - \boldsymbol{u}_\rho(t) \right\rangle_{V^* \times V} + \frac{1}{\rho} \big\langle P\boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t) \big\rangle_{V^* \times V}
$$

$$
+ j^0\big(w_\rho(t), M\boldsymbol{u}_\rho(t), M\boldsymbol{u}_\rho(t); M\boldsymbol{v} - M\boldsymbol{u}_\rho(t)\big) \geq \big\langle f(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t) \big\rangle_{V^* \times V}, \quad \forall \boldsymbol{v} \in V,
$$

$$
\big\langle \dot{\zeta}_\rho(t), \eta - \zeta_\rho(t) \big\rangle_{Y_1} + a\big(\zeta_\rho(t), \eta - \zeta_\rho(t)\big) \geq \big\langle \phi\big(t, \boldsymbol{u}_\rho(t), \zeta_\rho(t)\big), \eta - \zeta_\rho(t) \big\rangle_{Y_1}, \quad \forall \eta \in K_Y,
$$

$$
w_\rho(0) = 0, \qquad \zeta_\rho(0) = \zeta_0 \in (0, 1).
$$

Next, we show that all the conditions of Theorem 3.1 are satisfied. We only need to prove that H(A)(b) is fulfilled and P is a penalty operator because the other assumptions have been testified in [4].

We first show that H(A)(b) is fulfilled. In fact, from (4.34)(c) and Hölder's inequality, one has

$$
\big\langle A(t, \boldsymbol{u}_1) - A(t, \boldsymbol{u}_2), \boldsymbol{v} \big\rangle_{V^* \times V} \leq \left(\int_\Omega \big\| \mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{u}_1)\big) - \mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{u}_2)\big) \big\|^2 \, d\Omega \right)^{\frac{1}{2}} \|\boldsymbol{v}\|_V
$$

$$
\leq L_{\mathcal{A}} \left(\int_\Omega \big\| \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2) \big\|^2 \, d\Omega \right)^{\frac{1}{2}} \|\boldsymbol{v}\|_V
$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in V$ and all $t \in I$ and so

$$
\big\| A(t, \boldsymbol{u}_1) - A(t, \boldsymbol{u}_2) \big\|_{V^*} \leq L_{\mathcal{A}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V.
$$

This shows that $L_A = L_{\mathcal{A}}$ in (4.34)(c), and so condition H(A)(b) holds.

Moreover, we prove that P is a penalty operator of K_V. Now, we show that P is monotone. For any given $r_1, r_2, g \in \mathbb{R}$, through simple algebraic calculations we have

$$
\big((r_1 - g)^+ - (r_2 - g)^+\big)(r_1 - r_2) \geq 0
$$

and so

$$\left((u_\nu - g)^+ - (v_\nu - g)^+\right)(u_\nu - v_\nu) \geq 0 \quad \text{on } \Gamma_3.$$

This shows that P is monotone. Now, the Sobolev trace theorem states that there exists a positive constant C such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma, \mathbb{R}^d)} \leq C\|\boldsymbol{v}\|_V.$$

From Hölder's inequality, one has

$$\langle P\boldsymbol{u}_1 - P\boldsymbol{u}_2, \boldsymbol{v}\rangle_{V^* \times V} \leq \left(\int_{\Gamma_3} \left((u_{1\nu} - g)^+ - (u_{2\nu} - g)^+\right)^2 d\Gamma\right)^{\frac{1}{2}} \left(\int_{\Gamma_3} v_\nu^2 d\Gamma\right)^{\frac{1}{2}}$$

$$\leq C^2\|\boldsymbol{u_1} - \boldsymbol{u_2}\|_V\|\boldsymbol{v}\|_V$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in V$ and so

$$\|P\boldsymbol{u}_1 - P\boldsymbol{u}_2\|_{V^*} \leq C^2\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V.$$

This shows that P is a bounded and continuous operator. Furthermore, we can see that

$$K_V = \{\boldsymbol{u} \in V \mid P\boldsymbol{u} = \boldsymbol{0}_{V^*}\} = \{\boldsymbol{u} \in V \mid u_\nu \leq g \text{ a.e. on } \Gamma_3\},$$

and so P is a penalty operator of K_V. Thus, we can conclude that Problem 4.4 is equivalent to Problem 3.1 and so Theorem 3.1 ends the proof. $\square$

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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