Fixed Point Theory and Algorithms
for Sciences and Engineering

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Analysis of a debonding model of two elastic 2D-bars

Meir Shillor[1][*] and Kenneth L. Kuttler[2]

[*]Correspondence:
shillor@oakland.edu
[1]Department of Mathematics and
Statistics, Oakland University,
Rochester, MI 48309, USA
Full list of author information is
available at the end of the article

Abstract

This work establishes the existence of a weak solution to a new model for the process of debonding of two elastic 2D-bars caused by humidity and vibrations. A version of the model was first presented in the PCM-CMM-2019 conference in Krakow, Poland, and was published in (Shillor in J. Theor. Appl. Mech. 58(2): 295–305 2020). The existence of a weak solution is proved by regularizing the problem and then setting it in an abstract form that allows the use of tools for pseudo-differential operators and a fixed point theorem. Questions of further analysis of the solutions, effective numerical methods and simulations, as well as possible controls, are unresolved, yet.

MSC: 35K86; 35L86; 74K10; 74M99

Keywords: 2D bar; Dynamic contact; Debonding; Humidity diffusion

1 Introduction

This work establishes the existence of a weak solution to a system of partial differentia equations that models the process of debonding, caused by humidity and vibrations, of two elastic 2D-bars.

In many industrial settings, in transportation and everyday life, metallic plates are usually joined by welding. However, when the plates are made of different or nonmetallic materials, they have to be joined with screws or adhesively, or both. This is the case in planes, cars, and other vehicles, where increasingly more parts are made of plastics. One of the issues with car plates or other parts that are glued together is the process of debonding, that is, the process by which the adhesive is loosing its strength over time. This is noticeable in places where there is high humidity and considerable vibrations (driving off the road) and also large variations in temperature. The debonding of plates that are adhesively bonded too often has negative or even catastrophic consequences.

There is a very large mechanical engineering literature on debonding, for obvious reasons, see, e.g., [12, 15, 17] and the references therein. There is, also, a growing mathematical literature dealing with models of adhesive contact that include the processes of debonding, because of their relations to contact processes. There exist many publications in the Mathematical Theory of Contact Mechanics (MTCM) [6–8, 11, 16, 20, 22] that deal with contact processes with adhesion, and the host of references therein, and note

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that these references are only a tip of the iceberg. More specific references that deal with adhesive bonding are [2–4, 8, 22], and the references therein.

This work is based, in part, on the plenary lecture presented by MS in the PCM-CMM-2019 conference held in Krakow, Poland, 8–12 September 2019, parts of which were published in [19]. It is the continuation of the work [13], where the process of debonding of two linear beams was modeled, analyzed, and simulated. Additional information and references can be found in [13, 18].
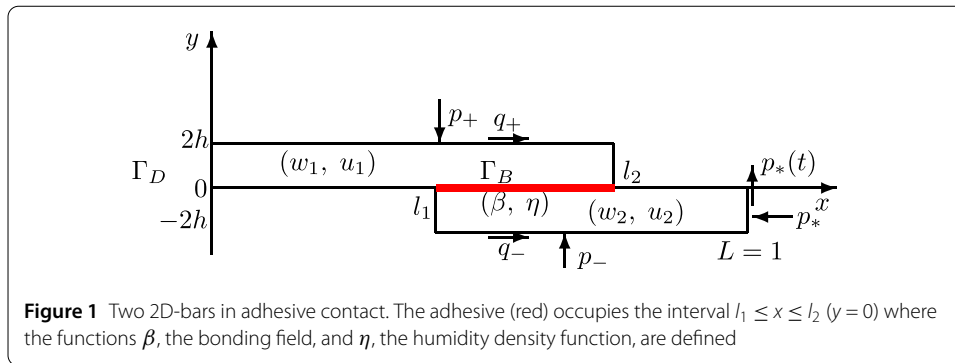
Here, we extend the model in [13] to the setting where two 2D bars are adhesively joined. The model for the 2D bar was introduced in [9, 10] and was rigorously derived in [23], where the existence of the weak solution to the problem of quasistatic contact was established. Numerical simulations of quasistatic contact between a 2D bar and a reactive foundation can be found in [1], and additional information in [1, 19, 23]. Starting with a 3D thermoelastic system and using symmetry and the smallness of the thickness, relative to the other dimensions, a cross-section of a thermoelastic plate is obtained. The interest in the new bar system lies in the fact that while the horizontal displacement field depends on x and y, the vertical displacement depends only on x, which simplifies the mathematical structure, making it 1.5D system, leading to more transparent contact conditions. It also makes it more interesting, since it forms a rather unusual mathematical problem. Indeed, it consists of two 2D partial differential equations (PDEs) for the horizontal displacements, two for the temperatures, two one-dimensional PDEs for the motion of the central axes, and two diffusion equations on the contact boundary. Since the bonding field β has to satisfy $0 \leq \beta \leq 1$, its evolution is modeled with a subdifferential set-inclusion on the segment where the bars are bonded. Thus, the model is highly nonlinear and nonsmooth, has an unusual structure, and one cannot apply directly any of the usual theoretical tools to analyze it. As a first step in its analysis, we establish the existence of a weak solution for a version of the model that excludes thermal effects, thus, the two heat equations for the temperature are omitted. The model is regularized, truncated and, once the necessary a priori estimates are obtained, we pass to the regularization limit and obtain a weak solution. The uniqueness of the solution is unlikely, in view of the mathematical complexity of the model.

The paper is structured as follows. The basic dynamic system, consisting of the motion of the bonded two 2D-bars, is described in Sect. 2, following [19, 23], however, thermal effects are excluded. This is an extension of the model that was introduced, analyzed, and simulated in [13]. The proof of the existence of weak solutions for the model is provided in Sect. 3. It is based on approximations, a priori estimates and a fixed point argument. The further analysis of the model is of interest for future study.

2 The dynamics of two bonded 2D-bars

This section presents a highly nonlinear and nonsmooth system that models the dynamics of two elastic 2D-bars that are bonded. However, here we take into account the effects of mechanical vibrations and humidity on the debonding process. The setting is depicted in Fig. 1.

We let $u(x, y, t)$ denote the horizontal displacement of the 2D-bar; and using symmetry and a simple assumption, the vertical displacement of the central horizontal axis $w(x, t)$ depends only on x. Thus, this is actually a "1.5D" system, since w depends only on x. Below, we use the subscripts x, y, t to denote partial derivatives.

Figure 1 Two 2D-bars in adhesive contact. The adhesive (red) occupies the interval $l_1 \leq x \leq l_2$ ($y = 0$) where the functions β, the bonding field, and η, the humidity density function, are defined

The model we study describes the debonding process of two such 2D-bars, caused by mechanical vibrations and humidity diffusion. Our main interest lies in the dynamics of the bonding field $\beta = \beta(x, t)$ that is defined on the contact surface, Γ_B in Fig. 1(red).

The setting is depicted in Fig. 1. The top 2D-bar occupies the domain $\Omega_1 = \{0 < x < l_2, 0 < y < 2h\}$ and the bottom 2D-bar occupies $\Omega_2 = \{l_1 < x < L, -2h < y < 0\}$, where $0 < l_1 < l_2 < 1$, and, for the sake of simplicity, we scaled all the distances so that the length is $L = 1$. The quantities with index 1 refer to the top bar and those with 2 to the bottom bar, thus for $i = 1, 2$, we refer to bar Ω_i. The horizontal displacements $u_i = u_i(x, y, t)$ are defined on Ω_i. The vertical displacement of the central line $w_1 = w_1(x, t)$ is defined on $\Omega_1 \cap \{y = h\}$, and $w_2 = w_2(x, t)$ on $\Omega_2 \cap \{y = -h\}$. We denote by ρ_i the 2D material density, E_i the Young modulus, and G_i the shear modulus of bar $i = 1, 2$. These are positive mechanical, found from measurements.

The debonding process takes place on the surface $\Gamma_B = \{l_1 < x < l_2, y = 0\}$ that is occupied by the glue (red segment in Fig. 1), where the bonding field $\beta = \beta(x, t)$ and the humidity field $\eta = \eta(x, t)$ are defined. We assume that there are no body forces acting in the system; the vertical tractions $p_+ = p_+(x, t)$ act on the top boundary of Ω_1 while it is clamped on Γ_D, and free at $x = l_2$ and on the bottom ($0 < x < l_1$). Similarly, the vertical tractions $p_- = p_-(x, t)$ act on the bottom boundary of Ω_2, it is free on $x = l_1$, and on the top for $l_2 \leq x < 1$, and, finally a traction $p_*(t)$ acts at $x = 1$.

The bonding of the two bars is modeled by the *bonding field* $\beta = \beta(x, t)$, which is defined on Γ_B. It represents the fraction of active surface bonds and has the character of a damage variable (see, e.g., [2, 3, 8, 20]), namely, $0 \leq \beta \leq 1$; indeed, when $\beta = 1$, all the adhesive bonds at the point are active; when $\beta = 0$, all the bonds are severed; and when $0 < \beta < 1$, the fraction β are active bonds, which describes the decrease in the load carrying capacity of the adhesive.

The debonding process on Γ_B is controlled by the *debonding source function* Φ, which depends on the bonding field, humidity, and the mechanical tractions. Mathematically, we write it as

$$\Phi(x, t) = \Phi\big(|u_2 - u_1|, (w_1 - w_2), \beta, \eta\big),$$

on $\overline{\Gamma}_B$. This function contains the information about the processes and must be determined in conjunction with experimental data, as was noted in [13], but in a simpler setting. We assume that the debonding process also depends on diffusion as the field at a point is affected by what happens in neighboring points.

To take into account the constraints $0 \le \beta \le 1$ and $w_2(x,t) + h \le w_1(x,t) - h$ on Γ_B (so there is no interpenetration there), we proceed as follows. We let $I_{[0,1]}(\beta)$ and $I_{[0,\infty)}(r)$ be the indicator functions of the intervals $[0, 1]$ and $[0, \infty)$, with subdifferential $\partial I_{[0,1]}(\beta)$ and $\partial I_{[0,\infty)}(r)$, respectively.

To prevent the top bar from penetrating the lower bar, we add the term

$$(-1)^i \partial I_{[0,\infty)}\big(w_1(x,t) - w_2(x,t) - 2h\big)$$

to the right-hand side of the equation of motion for w_i, for $i = 1, 2$. This guarantees that when $w_1(x,t) - w_2(x,t) - 2h > 0$, so there is a gap between the bars, this term vanishes. When the top bar tries to interpenetrate the bottom bar, $w_1(x,t) - w_2(x,t) - 2h = 0$, the term generates enough resistance force to prevent the penetration. More precisely, there exists a fictitious force $\xi \in \partial I_{[0,\infty)}(w_1(x,t) - w_2(x,t) - 2h)$ such that $\xi = 0$ when $w_1(x,t) - w_2(x,t) - 2h > 0$, and when $w_1(x,t) - w_2(x,t) - 2h = 0$ then $-\xi > 0$ is pushing the top bar up, and $\xi < 0$ pushing the bottom bar down, preventing penetration. We note that ξ need not be a function, only a distribution, which complicates the mathematical analysis considerably.

The evolution equation of the bonding field, see, e.g., [8, 20], is assumed to have the form of a constrained diffusion equation, or set-inclusion,

$$\beta_t - k_\beta \beta_{xx} + \Phi\big(|u_2 - u_1|, (w_1 - w_2), \beta, \eta\big) \in -\partial I_{[0,1]}(\beta), \tag{2.1}$$

$$\beta(x, 0) = \beta_0(x) \in [0, 1],$$

where k_β is the bonding diffusion coefficient, and the subdifferential on the right-hand side of (2.1) guarantees that $0 \le \beta \le 1$.

Next, we assume that the humidity function evolves by nonlinear diffusion from the edges $(x = l_1, y = 0)$ and $(x = l_2, y = 0)$, where the adhesive layer is exposed to the environment. The coefficient of diffusion D is assumed to depend on the bonding field, stress, and humidity density in the adhesive layer. Thus,

$$\eta_t - \big(D\big(|u_2 - u_1|, (w_1 - w_2), \beta, \eta\big)\eta_x\big)_x = 0, \quad \eta(x, 0) = \eta_0(x). \tag{2.2}$$

The boundary conditions are

$$\eta(l_1, t) = \eta_L, \qquad \eta(l_2, t) = \eta_R, \tag{2.3}$$

were $\eta_L(t)$ and $\eta_R(t)$ are given humidity densities at $x = l_1$ and $x = l_2$, respectively. In most applications it is reasonable to assume that $\eta_L = \eta_R = \eta_{amb}$, where η_{amb} is the ambient humidity density. However, these may be different and both varying with time. Also, $0 \le \eta_0(x)$ is the initial humidity density in the adhesive, which in practice is likely to be unknown.

The model for the *debonding of two adhesively joined 2D-bars caused by vibrations and humidity* is the following.

Problem 1 Find the functions u_i, defined on Ω_i, $i = 1, 2$; the functions w_1 defined on $\Omega_1 \cap \{y = h\}$ and w_2 defined on $\Omega_2 \cap \{y = -h\}$; and the functions β, η defined on Γ_B, such that the following equations and conditions hold for $0 < t \le T$:

$$\rho_i u_{itt} - E_i u_{ixx} - G_i u_{iyy} = 0, \tag{2.4}$$

$$2\rho_i w_{itt} - G_i w_{ixx} - (E_i - G_i) u_{ixy}$$
$$\in (-1)^i \partial I_{[0,\infty)} \big(w_1(x,t) - w_2(x,t) - 2h\big), \tag{2.5}$$

where $(x,y) \in \Omega_i, t \in [0,T]$;

$$\beta_t - k_\beta \beta_{xx} + \Phi\big(|u_2 - u_1|, (w_1 - w_2), \beta, \eta\big) \in -\partial I_{[0,1]}(\beta), \tag{2.6}$$

$$\eta_t - \big(D(|u_2 - u_1|, (w_1 - w_2), \beta, \eta)\eta_x\big)_x = 0, \quad \eta(x,0) = \eta_0(x). \tag{2.7}$$

The boundary conditions for β and η are

$$\beta_x(l_1, t) = \beta_x(l_2, t) = 0, \qquad \eta(l_1, t) = \eta_L, \qquad \eta(l_2, t) = \eta_R, \tag{2.8}$$

and the initial conditions

$$\beta(x,0) = \beta_0(x) \in [0,1], \qquad \eta(x,0) = \eta_0(x). \tag{2.9}$$

The balance of vertical and horizontal tractions on Γ_B, due to the adhesive, $\sigma_{\beta v}$ and $\sigma_{\beta h}$, is assumed to be

$$\sigma_{\beta v} = \beta K_{\beta v}(\eta)\big(w_1(x,t) - w_2(x,t) - 2h\big), \tag{2.10}$$

$$\sigma_{\beta h} = \beta K_{\beta h}(\eta)\big|u_1(x,0,t) - u_2(x,0,t)\big|. \tag{2.11}$$

Here, $K_{\beta v}$ and $K_{\beta h}$ are the adhesive stiffnesses, which depend on the humidity.

Next, the tractions acting on the system are

$$p_+ \quad \text{on } 0 \le x \le l_2, y = 2h, \tag{2.12}$$

$$p_- \quad \text{on } l_1 \le x \le 1, y = -2h, \tag{2.13}$$

$$p_* \quad \text{on } x = 1, -2h \le y \le 2h, \tag{2.14}$$

and zero tractions everywhere else, except for Γ_D, where

$$u_1(0, y, t) = 0, \qquad w_1(0, t) = 0. \tag{2.15}$$

Finally, the initial conditions for u_i, w_i are

$$u_i = u_{i0}, \qquad u_{it} = \overline{u}_{i0}, \qquad w_i = w_{i0}, \qquad w_{it} = \overline{w}_{i0}. \tag{2.16}$$

We note that the interesting aspects of the physics of the process enter via the debonding source function Φ and the humidity diffusion coefficient D, both of which must be found from experimental data in conjunction with numerical optimization. This is a major issue in the applications of the model. However, even qualitatively simple forms of Φ and D may provide very useful information in concrete applications.

3 Existence

The system is highly nonlinear and nonsmooth, so it is unlikely to have any classical solutions, so, we reformulate it and look for weak solutions. Moreover, generally, the solutions are unlikely be unique, therefore, we deal only with proving existence. However, we replace the the nonpenetration condition in (2.5), represented by the subdifferential for w_i, which physically is an idealized condition, with a more physically reasonable one. For the sake of simplicity, we rescale the variables so that $\rho_1 = \rho_2 = 1$. The system is rather complex, involving six dependent variables, and so we do not provide all the details in the proof of existence. We just outline the approach, pointing out that existence can be proved in this manner.

We first introduce the spaces and subspaces where we seek the various functions, others being chosen by analogy. Let,

$$V_1 = \left\{ u \in H^1(\Omega_1) : u(0, \cdot) = 0 \right\}, \qquad V_2 = \left\{ u \in H^1(\Omega_2) \right\},$$

$$W_1 = \left\{ w \in H^1\big((0, l_2) : w(0) = 0\big) \right\}, \qquad W_2 = H^1(l_1, 1),$$

$$Z_1 = H^1(\Gamma_B), \qquad Z_2 = \left\{ \phi \in H^1(\Gamma_B) : \phi = \eta_{amb}, x = l_1, l_2) \right\}$$

which are also the spaces of test functions. Here, H^1 denotes the usual Sobolev space over the relevant domain. We let $H_1 = L^2(\Omega_1)$ and $H_2 = L^2(\Omega_2)$.

We seek u_1 and u_2, for $T > 0$, in the respective spaces, $L^2(0, T; V_1)$ and $L^2((0, T); V_2)$; w_1 and w_2 in $L^2((0, T); W_1)$ and $L^2((0, T); W_2)$; β in $L^2((0, T); Z_1)$, and η in $L^2((0, T); Z_2)$, where for the sake of simplicity, we set $\eta_L = \eta_R = \eta_{amb} = const > 0$.

We start with the equation for u_1, and let $\psi \in V_1$. It follows from (2.4), using ψ as a test function and integrations by part, that

$$- \int_{\Omega_1} (E_1 u_{1xx} + G_1 u_{1yy}) \psi \, dx \, dy$$

$$= - \int_{\Omega_1} \nabla \cdot (E_1 u_{1x}, G_1 u_{1y}) \psi \, dx \, dy$$

$$= - \int_{\Omega_1} \nabla \cdot (E_1 \psi u_{1x}, G_1 \psi u_{1y}) \, dx \, dy + \int_{\Omega_1} (E_1 \psi_x u_{1x} + G_1 \psi_y u_{1y}) \, dx \, dy$$

$$= - \int_{\partial \Omega_1} (E_1 u_{1x}, G_1 u_{1y}) \cdot \mathbf{n} \psi \, dS + \int_{\Omega_1} (E_1 \psi_x u_{1x} + G_1 \psi_y u_{1y}) \, dx \, dy$$

$$= \int_0^{l_2} p_+ \psi(x, 2h) \, dx + \int_{\Gamma_B} \beta K_{\beta \nu}(\eta) \big((w_1(x, t) - w_2(x, t)) - 2h \big) \psi(x, 0) \, dx$$

$$\quad + \int_{\Omega_1} (E_1 \psi_x u_{1x} + G_1 \psi_y u_{1y}) \, dx \, dy.$$

To write this equality as an abstract equation, we define the operator L_1 by

$$\langle L_1 u, \psi \rangle = \int_{\Omega_1} (E_1 \psi_x u_{1x} + G_1 \psi_y u_{1y}) \, dx \, dy,$$

and note that it is linear, coercive, and bounded as a map from V_1 to V_1'. Next, let B_1 be a Lipschitz function, given by

$$
\begin{aligned}
\langle B_1(w_1, w_2, \eta, \beta), \psi \rangle = &\int_0^{l_2} p_+ \psi(x, 2h)\, dx \\
&+ \int_{\Gamma_B} \beta K_{\beta v}(\eta)\big((w_1(x,t) - w_2(x,t)) - 2h\big)\psi(x, 0)\, dx.
\end{aligned}
$$

Therefore, we write the equation for u_1, see (2.4) ($i = 1$), in abstract form as

$$
u_1'' + L_1 u_1 = B_1(w_1, w_2, \eta, \beta), \quad u_1(0) = u_{10} \in V_1, u_1'(0) = u_{11} \in H. \tag{3.1}
$$

Here, the prime denotes a weak time derivative.

The same reasoning applies to u_2, see (2.4) ($i = 2$), with $\psi \in V_2$, so we let

$$
\langle L_2 u, \psi \rangle = \int_{\Omega_2} (E_2 \psi_x u_{2x} + G_2 \psi_y u_{2y})\, dx\, dy,
$$

and note again that it is linear, coercive (up to a constant), and bounded as a map from V_2 to V_2'. Next, let B_2 be a Lipschitz function, given by

$$
\begin{aligned}
\langle B_2(w_1, w_2, \eta, \beta), \psi \rangle = &-\int_{l_1}^{1} p_- \psi(x, -2h)\, dx + \int_{-2h}^{0} p_* \psi(1, y)\, dy \\
&- \int_{\Gamma_B} \beta K_{\beta v}(\eta)\big((w_1(x,t) - w_2(x,t)) - 2h\big)\psi(x, 0)\, dx.
\end{aligned}
$$

Then, the abstract form of (2.4) ($i = 2$) is

$$
u_2'' + L_2 u_2 = B_2(w_1, w_2, \eta, \beta), \quad u_2(0) = u_{20} \in V_2, u_2'(0) = u_{21} \in H. \tag{3.2}
$$

We note that, thanks to the compact embedding theorems in Sobolev spaces, if $w_{in} \to w_i$ in $L^2(0, T; W_i)$ and w_{in}' is bounded in $L^2(0, T; W_i')$, one may conclude that $B(w_{1n}, w_{2n}, \theta_{1n}) \to B(w_1, w_2, \theta_1)$ in $L^2(0, T; V_1')$.

We turn to the equations for w_1 and w_2. Since the Signorini condition of perfect nonpenetration enters as a body force into the equations, rather than as a boundary condition, the standard methods used previously, do not apply. This may reflect the fact that such a fictitious perfect force does not make sense physically and, indeed, there is no such thing as a perfectly rigid body. Therefore, we replace it with a more physically realistic condition. To that end, we introduce the function R such that $R(s) = 0$ if $s \geq 0$ and $R(s) = s$ if $s < 0$. Then, we replace $\partial I_{[0,\infty)}(s)$ with $\frac{1}{\varepsilon}R(s)$, assuming that $0 < \varepsilon \ll 1$. Thus, we allow some interpenetration, but penalize it heavily. Then we replace the term $(w_1 - w_2)$, which originally is nonnegative, with $|w_1 - w_2|$, which is nonnegative.

To proceed, we multiply (2.5) ($i = 1$) with $\psi \in W_1$ such that $\psi(l_2) = 0$, integrate over Ω_1, use integration by parts, note that ψ and w_1 do not depend on y, and use algebraic

manipulations to obtain

$$4h \int_0^{l_2} w_1'' \psi \, dx + 2h \int_0^{l_2} G_1 w_{1x} \psi_x \, dx + \int_0^{l_2} (E_1 - G_1)\big(u_1(x, 2h) - u_1(x, 0)\big) \psi_x \, dx$$
$$+ \frac{2h}{\varepsilon} \int_0^{l_2} R\big(w_1(x, t) - w_2(x, t) - 2h\big) \psi \, dx = 0.$$

Here, we used the fact that $\psi(l_2) = 0$, hence,

$$\int_{\Omega_1} u_{1xy} \psi \, dx \, dy = - \int_0^{l_2} \big(u_1(x, 2h) - u_1(x, 0)\big) \psi_x \, dx.$$

This is an important observation because in the weak solutions u_{xy} is not well defined. We define the operators $\widehat{L}_1$ and $\widehat{B}_1$ by

$$\langle \widehat{L}_1 w_1, \psi \rangle = \frac{1}{2} \int_0^{l_2} G_1 w_{1x} \psi_x \, dx,$$

$$\big(\widehat{B}_1(u_1, w_1, w_2), \psi\big) = \frac{1}{4h} \int_0^{l_2} (E_1 - G_1)\big(u_1(x, 2h) - u_1(x, 0)\big) \psi_x \, dx.$$

Thus, the abstract formulation of (2.5) ($i = 1$) is

$$w_1'' + \widehat{L}_1 w_1 + \frac{1}{\varepsilon} R\big(w_1(x, t) - w_2(x, t) - 2h\big) = \widehat{B}_1(u_1, w_1, w_2). \tag{3.3}$$

Similarly for (2.5) ($i = 2$), using $\psi \in W_2$, such that $\psi(l_1) = \psi(1) = 0$, we define $\widehat{L}_2$ and $\widehat{B}_2$ by

$$\langle \widehat{L}_2 w_2, \psi \rangle = \frac{1}{2} \int_{l_1}^1 G_2 w_{2x} \psi_x \, dx,$$

$$\big(\widehat{B}_2(u_2, w_1, w_2), \psi\big) = \frac{1}{4h} \int_{l_1}^1 (E_2 - G_2)\big(u_2(x, -2h) - u_2(x, 0)\big) \psi_x \, dx.$$

Thus, the abstract formulation of (2.5) ($i = 2$) is

$$w_2'' + \widehat{L}_2 w_2 + \frac{1}{\varepsilon} R\big(w_1(x, t) - w_2(x, t) - 2h\big) = \widehat{B}_2(u_2, w_1, w_2). \tag{3.4}$$

We note that the $\widehat{L}_i$ are coercive, bounded, and linear.

To complete the mechanical model, the initial conditions need to be added. We do it below, when we discuss the smoothed mechanical problem.

Next, we consider the set-inclusion for the bonding field β,

$$\beta_t - k_\beta \beta_{xx} + \Phi\big(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta\big) \in -\partial I_{[0,1]}(\beta),$$

with the initial condition $\beta(x, 0) = \beta_0(x)$, for $x \in [l_1, l_2]$. The boundary conditions are $\beta_x = 0$. Let $\psi \in H^1(l_1, l_2)$, we define the operator $L_\beta : H^1(l_1, l_2) \to H^1(l_1, l_2)'$ by

$$\langle L_\beta \beta, \psi \rangle \equiv k_\beta \int_{l_1}^{l_2} \beta_x \psi_x \, dx = -k_\beta \int_{l_1}^{l_2} \beta_{xx} \psi \, dx,$$

and note that the operator L_β, is the subgradient of the function

$$\phi(\beta) = \begin{cases} \frac{k_\beta}{2} \int_{l_1}^{l_2} \beta_x^2 \, dx, & \text{if } \beta \in H^1(l_1, l_2), \\ \infty & \text{if } \beta \in L^2(l_1, l_2) \setminus H^1(l_1, l_2). \end{cases}$$

Next, multiplying the equation with ψ as a test function, and using integration by parts, we obtain the weak form

$$\beta' + k_\beta L_\beta \beta + \partial I_{[0,1]}(\beta) \ni -\Phi\big(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta\big), \quad \beta(0) = \beta_0. \tag{3.5}$$

We assume that Φ is bounded and Lipschitz in all its variables. Also, we let $H = L^2(l_1, l_2)$. We have the following preliminary result.

Proposition 2 *Assume that* $\Phi \in L^2((0, T); L^2(l_1, l_2))$ *and its arguments are sufficiently smooth. Then, there exists a unique solution* $\beta \in L^2(0, T; H^1(l_1, l_2))$ *of the problem*

$$\beta' + k_\beta L \beta + \partial I_{[0,1]}(\beta) \ni \Phi, \tag{3.6}$$

with $\beta(0) = \beta_0 \in H^1(l_1, l_2)$.
The solution satisfies $\beta', \beta \in L^2((0, T); H)$, $L\beta \in L^2((0, T); H)$, *and* $\beta(t) \in (0, 1)$, *for a.a. t.*

Proof The proof is based on a sequence of approximations as follows. Let $\Psi(\beta)$ be a piecewise linear and decreasing function on $(-\infty, \infty)$, with slope -1 for $\beta < 0$, equal to 0 on $[0, 1]$, and increasing with slope 1 for $\beta > 1$.

We consider the following sequence of approximate problems, which for $n = 1, 2, 3, \ldots,$ consists of finding a solution to

$$\beta_n' + L_\beta \beta_n + n \partial \Psi(\beta_n) = \Phi, \quad \beta_n(0) = \beta_0.$$

It follows from the theory of maximal monotone operators (see Brezis [5]) that for each n, there exists a solution β_n to the problem. This solution satisfies $\beta_n' \in L^2((0, T) : L^2(l_1, l_2))$ and $\beta_{nxx} \in L^2((0, T) : L^2(l_1, l_2))$. Multiplying both sides by β_n' and integrating yields ($H \equiv L^2(l_1, l_2)$)

$$\int_0^t \int_{l_1}^{l_2} (\beta_n')^2 \, dx \, ds + \frac{k_\beta}{2} |\beta_{nx}(t)|_H^2 + n \int_{l_1}^{l_2} \Psi(\beta_n(t)) \, dx$$

$$\leq C_\Phi + \frac{1}{2} \int_0^t \int_{l_1}^{l_2} (\beta_n')^2 \, dx \, dt,$$

where C_Φ is a positive constant that depends on Φ. Hence, we obtain the bound

$$\frac{1}{2} \int_0^t |\beta_n'|_H^2 \, ds + \frac{k_\beta}{2} |\beta_{nx}(t)|_H^2 + n \int_{l_1}^{l_2} \Psi(\beta_n(t)) \, dx \leq C_\Phi.$$

Similarly, multiplying by β_n and integrating yields

$$\frac{1}{2} |\beta_n(t)|_H^2 \leq C_\Phi(\beta_0),$$

where $C_\Phi(\beta_0)$ depends also on β_0. Therefore, we find

$$\frac{1}{2}\int_0^t |\beta_n'|_H^2\, ds + \frac{k_\beta}{2}\|\beta_n(t)\|_{H^1}^2 + n\int_{l_1}^{l_2} \Psi(\beta_n(t))\, dx \le C_\Phi(\beta_0).$$

Thus, there is a subsequence, still denoted by β_n, such that

$\beta_n' \to \beta'$ weakly in $L^2((0,T);H)$,

$\beta_n \to \beta$ strongly in $C([0,T];H)$,

$\beta_n \to \beta$ weak $*$ in $L^\infty((0,T);H^1)$,

$n\partial\Psi(\beta_n) \to \xi \in \partial I_{[0,1]}(\beta)$ weakly in $L^2((0,T);H)$,

$\beta(t) \in [0,1]$ a.a. t.

Passing to the limit as $n \to \infty$, we obtain the desired solution. However, we lose regularity of β because of the subgradient term. The weak solution is unique because of monotonicity considerations. $\qquad\square$

We turn the equation for η (see (2.7))

$$\eta_t - \left(D(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta)\eta_x\right)_x = 0, \quad \eta(x,0) = \eta_0(x),$$

with the boundary conditions (2.8), $\eta(l_1, t) = \eta_L$, $\eta(l_2, t) = \eta_R$. To obtain zero boundary conditions, we let $\widehat{\eta} = \eta - (\frac{x - l_1}{l_2 - l_1}\eta_R + \frac{l_2 - x}{l_2 - l_1}\eta_L)$, and the equation in terms of $\widehat{\eta}$ has the form

$$\widehat{\eta}_t - \left(D(|u_2 - u_1|, |w_1 - w_2|, \beta, \widehat{\eta})\widehat{\eta}_x\right)_x = \Theta(x, |u_2 - u_1|, |w_1 - w_2|, \beta, \widehat{\eta}),$$

$$\widehat{\eta}(x,0) = \widehat{\eta}_0(x).$$

Thus, without loss of generality, we assume that the initial-value problem has zero boundary conditions, and the equation is replaced with

$$\eta_t - \left(D(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta)\eta_x\right)_x = \Theta(t, x, |u_2 - u_1|, |w_1 - w_2|, \beta, \eta),$$

$$\eta(x,0) = \widehat{\eta}_0(x),$$

where $\eta \in H_0^1(l_1, l_2)$, and the hat is omitted.

Again, we assume that D, Θ are Lipschitz continuous and bounded. Also, we assume that $D(\cdot) \ge \delta > 0$, so the diffusion coefficient does not vanish and thus the problem is nondegenerate. Assume, in addition, that the functions $a, b \in L^\infty([l_1, l_2] \times [0, T])$, and $a(t, x) \ge \delta > 0$, and define the operator A as

$$\langle A\eta, \sigma\rangle \equiv \int_0^T \int_{l_1}^{l_2} a(t, x)\eta_x \sigma_x\, dx\, dt.$$

Proposition 3 *Under the above assumptions, if* $f \in L^2((0,T) : H^{-1})$, *there exists a unique weak solution to the problem of finding* $\eta \in L^2((0,T) : H_0^1(l_1, l_2))$, *such that*

$$\eta' + A\eta = f, \quad \eta(0) = \eta_0.$$

Moreover, $\eta \in L^2((0,T) : H_0^1(l_1, l_2))$, *and* $\eta' \in L^2((0,T) : H^{-1})$.

Proof This is a standard result since A is monotone, bounded, hemicontinuous, and coercive as a map from $L^2((0, T) : H_0^1(l_1, l_2))$ to its dual space. To include the dependence of D on η, one can use the Schauder fixed point argument or to retard in time the functions on which the diffusion coefficients depend and then passing to a limit using compactness. □

We conclude that the initial value problem for η has a weak solution.

We now revisit the mechanical system and modify it by adding viscosity, which acts as smoothing operator of the solutions, and is needed for the a priori estimates below. The modified system is

$$u_1'' + \gamma K_1 u_1' + L_1 u_1 = B_1(w_1, w_2),$$

$$u_2'' + \gamma K_2 u_2' + L_2 u_1 = B_2(w_1, w_2),$$

$$w_1'' + \gamma \widehat{K}_1 w_1' + \widehat{L}_1 w_1 + \frac{1}{\varepsilon} R\big(w_1(x, t) - w_2(x, t) - 2h\big) = \widehat{B}_1(\theta_1, w_1, w_2),$$

$$w_2'' + +\gamma \widehat{K}_2 w_2' + \widehat{L}_2 w_2 + \frac{1}{\varepsilon} R\big(w_2(x, t) - w_1(x, t), +2h\big) = \widehat{B}_2(\theta_2, w_1, w_2),$$

(3.7)

along with initial conditions. Here, γ is a small positive number, representing material viscosity. Also, K_i, $\hat{K}_i$ are suitable linear maps from an appropriate Sobolev space S to its dual space S', which ensure that u_i' and w_i' have two square integrable derivatives. This Sobolev space is a closed subspace of $H^2(\Omega_i)$ such that $u_{ix} = 0$ on the boundary of I_i, and is dense in V_i. Then, the K_i can be considered as Riesz maps from these new spaces to their duals. We assume sufficient regularity on the initial conditions to justify all of the following argument and, in particular, the initial data u_{i0} lie in S.

The regularized abstract form of the full model follows.

Problem 4 Find the functions u_i, w_i, η and β, (for $i = 1, 2$) such that

$$u_1'' + \gamma K_1 u_1' + L_1 u_1 = B_1(w_1, w_2),$$

$$u_2'' + \gamma K_2 u_2' + L_2 u_1 = B_2(w_1, w_2),$$

$$w_1'' + \gamma \widehat{K}_1 w_1' + \widehat{L}_1 w_1 + \frac{1}{\varepsilon} R\big(w_1(x, t) - w_2(x, t) - 2h\big) = \widehat{B}_1(w_1, w_2),$$

$$w_2'' + +\gamma \widehat{K}_2 w_2' + \widehat{L}_2 w_2 + \frac{1}{\varepsilon} R\big(w_2(x, t) - w_1(x, t) + 2h\big) = \widehat{B}_2(w_1, w_2),$$

(3.8)

$$\eta_t - \big(D\big(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta\big)\eta_x\big)_x = \Theta\big(t, x, |u_2 - u_1|, |w_1 - w_2|, \beta, \eta\big),$$

$$\beta' + k_\beta L_\beta \beta + \partial I_{[0,1]}(\beta) \ni \Phi\big(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta\big).$$

The initial conditions are:

$$u_i(\cdot, 0) = u_{i0}, u_i'(\cdot, 0) = v_{i0}, w_i(\cdot, 0) = w_{i0},$$

$$w_i'(\cdot, 0) = v_{wi0}, \eta(x, 0) = \eta_0(x), \beta(0) = \beta_0, \quad \text{for } i = 1, 2.$$

We have the following existence result.

Theorem 5 *There exists a solution to the system* (3.8), *for each* $0 \leq \gamma$.

The theorem guarantees the existence of weak solutions to the equations for two bars that are in adhesive contact, provided the subgradient acting on the bars for w_i is replaced with a penalization operator approximating the subgradient on Γ_B. Thus, we allow for $\gamma = 0$, so the materials are purely elastic. We only give a sketch of the proof since there are many dependent variables and the considerations are mostly routine. The proof is based on the following pair of compactness results, due to [14] and [21], and used below.

Theorem 6 *Let $q > 1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W. Let S be defined by*

$$S_R = \left\{ u \in L^2(a, b, E); \left\| u(t) \right\|_E \leq R; t \in [a, b], \left\| u(s) - u(t) \right\|_X \leq R|t - s|^{1/q} \right\}.$$

Then, S_R is bounded in $L^\infty(a, b, E)$ and, in addition, the functions are uniformly Hölder continuous into X. Also, $S_R \subseteq C([a, b]; W)$ and if $\{u_n\} \subseteq S_R$, there exists a subsequence $\{u_{n_k}\}$ which converges to a function $u \in C([a, b]; W)$, i.e.,

$$\lim_{k \to \infty} \|u_{n_k} - u\|_{\infty, W} = 0.$$

The second result need is the following.

Theorem 7 *Let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W. Let $p \geq 1$, let $q > 1$, and define*

$$S \equiv \left\{ u \in L^p([a, b]; E) : \text{for some } C, \left\| u(t) - u(s) \right\|_X \leq C|t - s|^{1/q} \right.$$

$$\left. \text{and } \|u\|_{L^p([a,b];E)} \leq R \right\}.$$

Thus, S is bounded in $L^p([a, b]; E)$ and Hölder continuous into X. Then, S is precompact in $L^p([a, b]; W)$. This means that if $\{u_n\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\{u_{n_k}\}$ which converges in $L^p([a, b]; W)$.

Proof We provide a sketch of the proof. It is based on time delay. We start with the first two equations in (3.8) for u_i when w_1, w_2 are known, so, we replace the time variable in w_i by the delayed time $t - h$ and for $t < h$, we use the initial conditions. Therefore, w_1, w_2 are known on the time interval $[t, t + h]$. Similarly, in the second pair of equations for w_1, w_2, we use $w_2(t - h)$ in the equation for w_1, and $w_1(t - h)$ in the equation for w_2. In the equation for η, we replace t with $t - h$ in all the variables in $D(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta)$ and use the the initial condition if $t \leq h$. Next, we do the same in all the variables in $\Phi(|u_2 - u_1|, |w_1 - w_2|, \beta, \eta)$, $\Theta(t, x, |u_2 - u_1|, |w_1 - w_2|, \beta, \eta)$. Then, routine considerations, tedious a priori estimates, along with Proposition 2, yield a solution to this time delayed regularized system of equations which has the property that u_{ix}, u_{iy} vanish on the boundaries and w_{ix}, too. Then, we obtain the necessary estimates on this solution by multiplying through by $-E_i u_{ixx} - G_i u_{yy}$ in the equations for u_i and use integration by parts and the first term to argue that there is an estimate on $\|u_{it}(t)\|_{V_i}$ which is independent of γ. A similar process is applied to the equations for w_i.

Next, having the estimates, we pass to the limit as $h \to 0$, using Theorems 6, and 7, and the embedding theorems in Sobolev space, and obtain a solution to the regularized

system without the time delay, along with estimates on $\|u_{it}(t)\|_{V_i}$, which are independent of γ, and similar estimates for w_i. Finally, we use these theorems to pass to the limit in a subsequence for which $\gamma \to 0$. This yields the existence of a weak solution to the original problem in which the idealized subgradient term in the equations for w_i are replaced with a penalized version of this subgradient.

The passage to the limits is straightforward since the higher-order operators are linear and the boundary operators and other functions which are not linear depend on lower-order terms and can be estimated using compactness considerations in the two theorems. □

We consider the uniqueness of the solution very unlikely, and so we skip it.

Now that the existence of weak solutions has been established, some questions of interest for further research are:

- Perform additional analysis of the solutions, especially about their regularity.
- Construct an efficient and convergent algorithm for computer simulations.
- Find numerically a typical debonding process behavior.
- Study how the debonding depends on the excitation frequency.
- Get an estimate of the degree of debonding from the shift in the vibrations spectrum.
- Correlate the computer experiments with experimental data to find a possible structure for the debonding source function Φ and the humidity diffusion coefficient D.

Abbreviations
MTCM, Mathematical Theory of Contact Mechanics; PDE, Partial Differential Equation; 2D, two-dimensional.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MS contributed the model, some analysis, and part of the writing; KLK contributed the analysis and the writing. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA. [2]American Association of Retired Persons: AARP, Grantsville, USA.

Publisher's Note
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