Fixed Point Theory and Algorithms for Sciences and Engineering

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Fixed point theorems for generalized (α,ψ)-contraction mappings in rectangular quasi b-metric spaces

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Abstract
In this paper, we introduce the class of rectangular quasi b-metric spaces as a generalization of rectangular metric spaces, rectangular quasi-metric spaces, rectangular b-metric spaces, define generalized (α,ψ)-contraction mappings and study fixed point results for the maps introduced in the setting of rectangular quasi b-metric spaces. Our results extend and generalize related fixed point results in the literature, in particular, the works of Karapinar and Lakzian (J. Funct. Spaces 2014:914398, 2014), Alharbi et al. (J. Math. Anal. 9(3):47–60, 2018), and Khuangsatung et al. (Thai J. Math. 2020:89–101, 2020) from rectangular quasi metric space and rectangular *b*-metric space to rectangular quasi *b*-metric spaces. We also provide examples in support of our main findings. Furthermore, we applied one of our results to determine the existence of a solution to an integral equation.

Keywords: Fixed points; Rectangular quasi b-metric spaces; Generalized (α,ψ)-contraction mappings

1 Introduction

Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics, and almost all Engineering areas. The contraction mapping principle introduced by Banach [4] has a wide range of applications in fixed point theory. The Banach contraction principle has been extended and generalized in different directions by different researchers. For more details see ([5–22]). In 2014, Lin et al. [23] introduced the concept of rectangular quasi metric space and proved the fixed point theorem for the Meir-Keeler contractive mappings in the setting of rectangular quasi metric Spaces.

In 2014, Karapinar and Lakzian [1] defined (α,ψ)-contractive mapping in rectangular quasi metric space and proved fixed point theorems for the maps introduced. In 2015, George et al. [24] announced the notion of rectangular b-metric space as a generalization of metric, b-metric space, and rectangular metric space; many authors initiated and studied a lot of existing fixed point theorems in such spaces. Alharbi et al. [2] defined (α)-contractive mapping and proved fixed point theorems in rectangular b-metric space.

Afterward, several research papers were published on the existence of fixed point results for single-valued and multi-valued mappings in the setting of rectangular quasi metric Spaces.

Very recently, Khuangsatung et al. [3] introduced the notion of ψ-contraction and studied fixed point results for ψ-contraction mappings in complete rectangular quasi metric spaces and proved the existence and uniqueness of fixed points.

Inspired and motivated by the works of Karapinar and Lakzian [1], Alharbi et al. [2], and Khuangsatung et al. [3], the main purpose of this paper is to establish fixed point results for generalized (α, ψ)-contraction mappings in the setting of rectangular quasi b-metric spaces.

2 Preliminaries

In what follows, we recall basic definition and results on the topics for the sake of completeness.

Notation We need the following symbols and class of functions to prove certain results of this section:

- $R^+ = [0, \infty)$;
- R is the set of all real numbers;
- N is the set of all natural numbers;
- $\Psi = \{\psi : R^+ \to R^+$, such that, ψ is non-decreasing, continuous, $\sum_{k=1}^{\infty} s^k \psi^k(t) < \infty$, $s\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$ if and only if $t = 0$, where ψ^k is the kth iterate of ψ and $s \geq 1\}$.

Definition 1 ([6]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces since a b-metric is a metric when $s = 1$.

Definition 2 ([7]) Let X be a nonempty set and $d : X \times X \to R^+$ be a function satisfying the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct point $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular metric on X, and the pair (X, d) is called a rectangular metric space.

Definition 3 ([11]) Let X be a nonempty set, $s \geq 1$ be a given real number, and $d : X \times X \to R^+$ be a function satisfying the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;

(iii) $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ for all $x, y \in X$ and all distinct point $u, v \in X \backslash \{x, y\}$.

Then d is called a rectangular b-metric on X, and the pair (X, d) is called a rectangular b-metric space.

Note: Every metric space is a rectangular metric space, and every rectangular metric space is a rectangular b-metric space with coefficient $s = 1$.

It is evident that any rectangular metric space is a rectangular b-metric space, but the converse is not true in general.

We give an example to show that not every rectangular b-metric space is a rectangular metric space.

Example 1 ([11]) Let $X = N$, $\alpha > 0$ and $d : X \times X \to R^+$ such that:

(i) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(ii) $d(x,y) = 0$, if and only if $x = y$;

(iii) $d(x,y) = 4\alpha$, if $x, y \in \{1, 2\}$ and $x \neq y$;

(iv) $d(x,y) = \alpha$, if $x, y \notin \{1, 2\}$ and $x \neq y$.

Then (X, d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (X, d) is not a rectangular metric space, as $d(1,2) = 4\alpha \nleq 3\alpha = d(1,3) + d(3,4) + d(4,2)$.

The following is the definition of the notion of rectangular quasi metric space.

Definition 4 ([23]) Let X be a nonempty set and $d : X \times X \to R^+$ be a function satisfying the following conditions:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all $x, y \in X$ and all distinct point $u, v \in X \backslash \{x, y\}$.

Then d is called a rectangular quasi metric on X, and the pair (X, d) is called a rectangular quasi metric space.

Note: Any rectangular metric space is a rectangular quasi metric space, but the converse is not true in general.

We give an example to show that not every rectangular quasi metric on a set X is a rectangular metric space on X.

Example 2 ([23]) Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ as a constant, $\alpha > 0$ and define $d : X \times X \to R^+$ by

(i) $d(x,x) = 0$ for all $x \in X$;

(ii) $d(t, 2t) = d(2t, t) = 3\alpha$;

(iii) $d(t, 3t) = d(2t, 3t) = d(3t, t) = d(3t, 2t) = \alpha$;

(iv) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = d(4t, t) = d(4t, 2t) = d(4t, 3t) = 2\alpha$;

(v) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = d(4t, 5t) = \frac{3}{2}\alpha$;

(vi) $d(5t, t) = d(5t, 2t) = d(5t, 3t) = d(5t, 4t) = \frac{5}{4}\alpha$.

Then (X, d) is a rectangular quasi metric space, but for the fact that $d(t, 5t) = \frac{3}{2}\alpha \nleq \frac{5}{4}\alpha = d(5t, t)$, (X, d) is not a rectangular metric space.

Definition 5 ([25]) Let X be a nonempty set, $T : X \longrightarrow X$ be a self-mapping of a set X and $\alpha : X \times X \to R^+$. Then T is called an α-admissible if $x, y \in X$,

$$\alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 6 ([3]) Let (X,d) be a rectangular quasi metric space and $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is called convergent to $x \in X$ if $\lim_{n\longrightarrow\infty} d(x_n,x) = 0 = \lim_{n\longrightarrow\infty} d(x,x_n)$ and this fact is represented by $\lim_{n\longrightarrow\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is called the Cauchy sequence in (X,d) if $\lim_{n\longrightarrow\infty} d(x_n,x_{n+p}) = 0 = \lim_{n\longrightarrow\infty} d(x_{n+p},x_n)$ for all $p > 0$.

(iii) (X,d) is called complete rectangular quasi metric space if every Cauchy sequence in X converges to some $x \in X$.

3 Main results

In this section, we introduce rectangular quasi b-metric spaces, define generalized (α,ψ)-contraction mappings, and study fixed point results for the mappings introduced in the setting of rectangular quasi b-metric spaces.

We start by introducing the notion of a rectangular quasi b-metric space as follows:

Definition 7 Let X be a nonempty set, $s \geq 1$, and suppose that the mapping $d : X \times X \to R^+$ satisfies the following conditions:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ for all $x,y \in X$ and all distinct points $u,v \in X\backslash\{x,y\}$.

Then d is called a rectangular quasi b-metric on X, and the pair (X,d) is called a rectangular quasi b-metric space.

Now, we give an example of a rectangular quasi b-metric space.

Example 3 Let $X = A \cup B$, where $A = \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\}$ and $B = [1,2]$. Define the generalized metric d on X as follows

$d(\frac{1}{2},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{5}) = 0.3$; $d(\frac{1}{3},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{4}) = 0.1$;
$d(\frac{1}{2},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{5}) = 0.6$; $d(\frac{1}{4},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{3}) = 0.4$;
$d(\frac{1}{2},\frac{1}{5}) = 1.05$; $d(\frac{1}{5},\frac{1}{2}) = d(\frac{1}{4},\frac{1}{3}) = 0.5$;
$d(\frac{1}{2},\frac{1}{2}) = d(\frac{1}{3},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{4}) = d(\frac{1}{5},\frac{1}{5}) = 0$;

and $d(x,y) = |x-y|$ if $x,y \in B$ or $x \in A$, $y \in B$ or $x \in B$, $y \in A$.

Then (X,d) is a rectangular quasi b-metric space with coefficient $s = \frac{3}{2} > 1$. Indeed Condition (i) in Definition 7 trivially holds.

Now, we show condition (ii) in Definition 7 holds:

Case (i) If $x,y \in A$, then

$d(x,y) = d(\frac{1}{2},\frac{1}{3}) = 0.3 \leq s[d(\frac{1}{2},u) + d(u,v) + d(v,\frac{1}{3})]$ when $u,v \in \{\frac{1}{4},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{3},\frac{1}{2}) = 0.1 \leq s[d(\frac{1}{3},u) + d(u,v) + d(v,\frac{1}{2})]$ when $u,v \in \{\frac{1}{4},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{3},\frac{1}{4}) = 0.1 \leq s[d(\frac{1}{3},u) + d(u,v) + d(v,\frac{1}{4})]$ when $u,v \in \{\frac{1}{2},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{4},\frac{1}{3}) = 0.5 \leq s[d(\frac{1}{4},u) + d(u,v) + d(v,\frac{1}{3})]$ when $u,v \in \{\frac{1}{2},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{4},\frac{1}{5}) = 0.3 \leq s[d(\frac{1}{4},u) + d(u,v) + d(v,\frac{1}{4})]$ when $u,v \in \{\frac{1}{2},\frac{1}{3}\}$.
$d(x,y) = d(\frac{1}{5},\frac{1}{4}) = 0.1 \leq s[d(\frac{1}{5},u) + d(u,v) + d(v,\frac{1}{4})]$ when $u,v \in \{\frac{1}{2},\frac{1}{3}\}$.
$d(x,y) = d(\frac{1}{2},\frac{1}{4}) = 0.6 \leq s[d(\frac{1}{2},u) + d(u,v) + d(v,\frac{1}{4})]$ when $u,v \in \{\frac{1}{3},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{4},\frac{1}{2}) = 0.4 \leq s[d(\frac{1}{4},u) + d(u,v) + d(v,\frac{1}{2})]$ when $u,v \in \{\frac{1}{3},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{2},\frac{1}{5}) = 1.05 \leq s[d(\frac{1}{2},u) + d(u,v) + d(v,\frac{1}{5})]$ when $u,v \in \{\frac{1}{3},\frac{1}{4}\}$.
$d(x,y) = d(\frac{1}{5},\frac{1}{2}) = 0.5 \leq s[d(\frac{1}{5},u) + d(u,v) + d(v,\frac{1}{2})]$ when $u,v \in \{\frac{1}{3},\frac{1}{4}\}$.
$d(x,y) = d(\frac{1}{3},\frac{1}{5}) = 0.6 \leq s[d(\frac{1}{3},u) + d(u,v) + d(v,\frac{1}{5})]$ when $u,v \in \{\frac{1}{2},\frac{1}{4}\}$.

$d(x,y) = d(\frac{1}{5},\frac{1}{3}) = 0.4 \leq s[d(\frac{1}{5},u) + d(u,v) + d(v,\frac{1}{3})]$ when $u,v \in \{\frac{1}{2},\frac{1}{4}\}$.

Case (ii) If $x,y \in B$ or $x \in A$, $y \in B$ or $x \in B$, $y \in A$, then

$d(x,y) = |x-y| \leq s|x-u| + |u-v| + |v-y|$ for all distinct points $u,v \in X \setminus \{x,y\}$.

But (X,d) is neither a metric space, a rectangular metric space nor a rectangular quasi metric space because the triangle inequality, symmetry, and rectangular inequality fail respectively as follows:

$d(\frac{1}{2},\frac{1}{4}) = 0.6 \nleq 0.4 = d(\frac{1}{2},\frac{1}{3}) + d(\frac{1}{3},\frac{1}{4}) = 0.3 + 0.1$,
$d(\frac{1}{2},\frac{1}{4}) = 0.6 \neq 0.4 = d(\frac{1}{4},\frac{1}{2})$, and
$d(\frac{1}{2},\frac{1}{5}) = 1.05 \nleq 0.7 = d(\frac{1}{2},\frac{1}{3}) + d(\frac{1}{3},\frac{1}{4}) + d(\frac{1}{4},\frac{1}{5})$.

We next give the definitions of rectangular quasi b-convergence of a sequence and completeness of rectangular quasi b-metric spaces.

Definition 8 Let (X,d) be a rectangular quasi b-metric space and $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is said to be convergent to x if $\lim_{n\longrightarrow\infty} d(x_n,x) = 0 = \lim_{n\longrightarrow\infty} d(x,x_n)$ and this fact is represented by $\lim_{n\longrightarrow\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is called the Cauchy sequence in (X,d) if $\lim_{n\longrightarrow\infty} d(x_n,x_{n+p}) = 0 = \lim_{n\longrightarrow\infty} d(x_{n+p},x_n)$ for all $p > 0$.

(iii) (X,d) is called complete rectangular quasi b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Remark 1 Let (X,d) be a rectangular quasi b-metric space. For $x \in X$, we define the open ball with center x and radius $r > 0$ by

$$B_r(x,r) = \left\{y \in X : \max\left\{d(x,y), d(y,x)\right\} < r\right\}.$$

In general, an open ball in a rectangular metric space need not be an open set. A rectangular quasi b-metric space need not be continuous. A convergent sequence in a rectangular quasi b-metric space need not be a Cauchy. A rectangular quasi b-metric space need not be a Hausdorff, and hence the uniqueness of limits cannot be guaranteed.

Now, we give an example to support Remark 1.

Example 4 Let $X = A \cup B$, where $A = \{\frac{1}{n}, n \in N\}$, and $B = \{0,3\}$. Define the function $d : X \times X \to R^+$ such that

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ \frac{9}{2} & \text{if } x,y \in A; \\ \frac{1}{n} & \text{if } x \in A, y \in B; \\ \frac{1}{n+1} & \text{if } x \in B, y \in A; \\ 2 & \text{if } x,y \in B. \end{cases}$$

The function d is a rectangular quasi b-metric space with $s = 2$. But d is neither a rectangular quasi metric nor a rectangular b-metric space because

$d(\frac{1}{2},\frac{1}{3}) = \frac{9}{2} \nleq \frac{11}{4} = d(\frac{1}{2},0) + d(0,3) + d(3,\frac{1}{3})$ and
$d(\frac{1}{3},0) = \frac{1}{3} \neq \frac{1}{4} = d(0,\frac{1}{3})$.

It is also clear that
$\lim_{n\longrightarrow\infty} d(\frac{1}{2n},0) = \lim_{n\longrightarrow\infty} \frac{1}{n} = 0 = \lim_{n\longrightarrow\infty} d(0,\frac{1}{2n}) = \lim_{n\longrightarrow\infty} \frac{1}{n+1}$ and
$\lim_{n\longrightarrow\infty} d(\frac{1}{2n},3) = \lim_{n\longrightarrow\infty} \frac{1}{n} = 0 = \lim_{n\longrightarrow\infty} d(3,\frac{1}{2n}) = \lim_{n\longrightarrow\infty} \frac{1}{n+1}$,
that is, the sequence $\{\frac{1}{2n}\}$ has two different limits the numbers 0 and 3.

In addition, the sequence $\{\frac{1}{2n}\}$ is rectangular quasi b-convergent, but not a rectangular quasi b-Cauchy sequence, because
$\lim_{n\longrightarrow\infty} d(x_n,x_{n+p}) = \lim_{n\longrightarrow\infty} d(\frac{1}{2n},\frac{1}{2n+p}) = \frac{9}{2} = \lim_{n\longrightarrow\infty} d(x_{n+p},x_n) =$
$\lim_{n\longrightarrow\infty} d(\frac{1}{2n+p},\frac{1}{2n})$.

In the following, we define an (α,ψ)-contraction mapping in the setting of rectangular quasi b-metric space.

Definition 9 Let (X,d) be a rectangular quasi b-metric space and $T:X\longrightarrow X$ be a given mapping. We say that T is a generalized (α,ψ)-contraction mapping if there exist two functions $\alpha: X\times X\to R^+$ and $\psi\in\Psi$ such that

$$\alpha(x,y)d(Tx,Ty)\le\psi\bigl(M(x,y)\bigr)\quad\text{for all } x,y\in X, \tag{1}$$

where $M(x,y)=\max\{d(x,y),\frac{d(x,Tx)d(x,Ty)}{1+d(x,Ty)+d(y,Tx)},d(x,Tx),d(y,Ty)\}$.

Now, we state and prove the following fixed point theorem.

Theorem 1 *Let (X,d) be a complete rectangular quasi b-metric space and $T:X\longrightarrow X$ be generalized (α,ψ)- contraction mapping. Suppose that*

(i) *T is an α-admissible mapping*;
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$, $\alpha(Tx_0,x_0)\ge 1$, $\alpha(x_0,T^2x_0)\ge 1$, and $\alpha(T^2x_0,x_0)\ge 1$*;
(iii) *T is continuous.*

Then T has a fixed point.

Proof By (ii) above, there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$ and $\alpha(Tx_0,x_0)\ge 1$. Now, we construct a sequence $\{x_n\}$ in X by $x_{n+1}=Tx_n=T^{n+1}x_0$ for all $n\ge 0$. Suppose that $x_{n_0}=x_{n_0+1}$ for some $n_0\ge 0$. Since $Tx_{n_0}=x_{n_0+1}$, the point $u=x_{n_0}$ forms a fixed point of T. Hence, that completes the proof. We assume that $x_n\ne x_{n+1}$ for all $n\ge 0$.

Since T is α-admissible, we have $\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\ge 1\Rightarrow\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\ge 1$. Utilizing the expression above, we obtain that

$$\alpha(x_n,x_{n+1})\ge 1,\quad\text{for all } n=0,1,\ldots, \tag{2}$$

and $\alpha(x_1,x_0)=\alpha(Tx_0,x_0)\ge 1\Rightarrow\alpha(Tx_1,Tx_0)=\alpha(x_2,x_1)\ge 1$. Utilizing the expression above, we obtain that

$$\alpha(x_{n+1},x_n)\ge 1,\quad\text{for all } n=0,1,\ldots. \tag{3}$$

In a similar way, we derive that $\alpha(x_0,x_2)=\alpha(x_0,T^2x_0)\ge 1\Rightarrow\alpha(Tx_0,Tx_2)=\alpha(x_1,x_3)\ge 1$.

Recursively, we get that

$$\alpha(x_n,x_{n+2})\ge 1,\quad\text{for all } n=0,1,\ldots. \tag{4}$$

Analogously, we can easily derive that

$$\alpha(x_{n+2},x_n)\geq 1,\quad \text{for all } n=0,1,\ldots. \tag{5}$$

Step 1: We show that $\lim_{n\longrightarrow\infty} d(x_n,x_{n+1})=0=\lim_{n\longrightarrow\infty} d(x_{n+1},x_n)$ and $\lim_{n\longrightarrow\infty} d(x_n,x_{n+2})=0=\lim_{n\longrightarrow\infty} d(x_{n+2},x_n)$.

Regarding (1), we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\ &\leq \alpha(x_{n-1},x_n)d(Tx_{n-1},Tx_n)\\ &\leq \psi\big(M(x_{n-1},x_n)\big),\quad \text{for all } n\geq 1, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &M(x_{n-1},x_n)\\ &\quad=\max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)}{1+d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})},d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)\right\}\\ &\quad=\max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_{n+1})+d(x_n,x_n)},d(x_{n-1},x_n),d(x_n,x_{n+1})\right\}\\ &\quad=\max\big\{d(x_{n-1},x_n),d(x_n,x_{n+1})\big\}. \end{aligned}$$

If $M(x_{n-1},x_n)=d(x_n,x_{n+1})$, then from (6), we get

$$d(x_n,x_{n+1})\leq\psi\big(d(x_n,x_{n+1})\big)\leq s\psi\big(d(x_n,x_{n+1})\big)<d(x_n,x_{n+1}),$$

which is a contradiction. Hence, $M(x_{n-1},x_n)=d(x_{n-1},x_n)$.

We let $e_n=d(x_n,x_{n+1})$, $l_n=d(x_{n+1},x_n)$, $e_n^*=d(x_n,x_{n+2})$ and $l_n^*=d(x_{n+2},x_n)$ for all $n\geq 0$. By using (6), we get

$$\begin{aligned} e_n=d(x_n,x_{n+1})&=d(Tx_{n-1},Tx_n)\leq\psi\big(d(x_{n-1},x_n)\big)\\ &=\psi\big(d(Tx_{n-2},Tx_{n-1})\big)\\ &\leq\psi^2\big(d(x_{n-2},x_{n-1})\big)\\ &\;\vdots\\ &\leq\psi^n\big(d(x_0,x_1)\big)=\psi^n(e_0)\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \end{aligned} \tag{7}$$

Also,

$$\begin{aligned} l_n&=d(x_{n+1},x_n)=d(Tx_n,Tx_{n-1})\\ &\leq\alpha(x_n,x_{n-1})d(Tx_n,Tx_{n-1})\\ &\leq\psi\big(M(x_n,x_{n-1})\big),\quad\text{for all } n\geq 1, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &M(x_n,x_{n-1})\\ &\quad=\max\left\{d(x_n,x_{n-1}),\frac{d(x_n,Tx_n)d(x_n,Tx_{n-1})}{1+d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)},d(x_n,Tx_n),d(x_{n-1},Tx_{n-1})\right\} \end{aligned}$$

$$= \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1})d(x_n, x_n)}{1 + d(x_n, x_n) + d(x_{n-1}, x_{n+1})}, d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right\}$$
$$= \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}.$$

We consider three different cases:

Case (i) If $M(x_n, x_{n-1}) = d(x_{n-1}, x_n)$, then by (8), we get

$$d(x_{n+1}, x_n) \le \psi\big(d(x_{n-1}, x_n)\big) \le \psi^n\big(d(x_0, x_1)\big) = \psi^n(e_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Case (ii) If $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$, then by (8), we get

$$d(x_{n+1}, x_n) \le \psi\big(d(x_n, x_{n+1})\big) \le \psi^n\big(d(x_0, x_1)\big) = \psi^n(e_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Case (iii) If $M(x_n, x_{n-1}) = d(x_n, x_{n-1})$, then by (8), we get

$$\begin{aligned} l_n &= d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le \psi\big(d(x_n, x_{n-1})\big) \\ &= \psi\big(d(Tx_{n-1}, Tx_{n-2})\big) \\ &\le \psi^2\big(d(x_{n-1}, x_{n-2})\big) \\ &\vdots \\ &\le \psi^n\big(d(x_1, x_0)\big) = \psi^n(l_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

From Case (i)-Case (iii), we get

$$l_n = d(x_{n+1}, x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{9}$$

From (7) and (9), we deduce that

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0 = \lim_{n\longrightarrow\infty} d(x_{n+1}, x_n). \tag{10}$$

Now, we show $\lim_{n\longrightarrow\infty} d(x_n, x_{n+2}) = 0 = \lim_{n\longrightarrow\infty} d(x_{n+2}, x_n)$.

$$\begin{aligned} e_n^* &= d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \\ &\le \alpha(x_{n-1}, x_{n+1}) d(Tx_{n-1}, Tx_{n+1}) \\ &\le \psi\big(M(x_{n-1}, x_{n+1})\big), \quad \text{for all } n \ge 1 \end{aligned} \tag{11}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\Bigg\{d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_{n+1})}{1 + d(x_{n-1}, Tx_{n+1}) + d(x_{n+1}, Tx_{n-1})}, d(x_{n-1}, Tx_{n-1}), \\ &\qquad d(x_{n+1}, Tx_{n+1})\Bigg\} \\ &= \max\Bigg\{d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+2})}{1 + d(x_{n-1}, x_{n+2}) + d(x_{n+1}, x_n)}, d(x_{n-1}, x_n), \\ &\qquad d(x_{n+1}, x_{n+2})\Bigg\} \end{aligned}$$

$$= \max\bigl\{d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2})\bigr\}.$$

We consider three different cases:

Case (i) If $M(x_n,x_{n-1}) = d(x_{n-1},x_{n+1})$, then by (11), we get

$$d(x_n,x_{n+2}) \le \psi\bigl(d(x_{n-1},x_{n+1})\bigr) \le \psi^{n-1}\bigl(d(x_0,x_2)\bigr) = \psi^{n-1}\bigl(e_0^*\bigr) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Case (ii) If $M(x_{n-1},x_{n+1}) = d(x_{n-1},x_n)$, then by (11), we get

$$d(x_n,x_{n+2}) \le \psi\bigl(d(x_{n-1},x_n)\bigr) \le \psi^{n-1}\bigl(d(x_0,x_1)\bigr) = \psi^{n-1}(e_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Case (iii) If $M(x_{n-1},x_{n+1}) = d(x_{n+1},x_{n+2})$, then by (11), we get $d(x_n,x_{n+2}) \le \psi(d(x_{n+1}, x_{n+2})) \le \psi^{n+1}(d(x_0,x_1)) = \psi^{n+1}(e_0) \longrightarrow 0$ as $n \longrightarrow \infty$.

From Case (i)-Case (iii), we get

$$e_n^* = d(x_n,x_{n+2}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{12}$$

$$\begin{aligned} &d(x_{n+2},x_n) = d(Tx_{n+1}, Tx_{n-1}) \le \alpha(x_{n+1},x_{n-1})d(Tx_{n+1}, Tx_{n-1}) \le \psi\bigl(M(x_{n+1},x_{n-1})\bigr), \\ &\quad \text{for all } n \ge 1, \end{aligned} \tag{13}$$

where

$$\begin{aligned} M(x_{n+1},x_{n-1}) &= \max\biggl\{d(x_{n+1},x_{n-1}), \frac{d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_{n-1})}{1 + d(x_{n+1}, Tx_{n-1}) + d(x_{n-1}, Tx_{n+1})}, d(x_{n+1}, Tx_{n+1}), \\ &\quad d(x_{n-1}, Tx_{n-1})\biggr\} \\ &= \max\biggl\{d(x_{n+1},x_{n-1}), \frac{d(x_{n+1},x_{n+2})d(x_{n+1},x_n)}{1 + d(x_{n+1},x_n) + d(x_{n-1},x_{n+2})}, d(x_{n+1},x_{n+2}), \\ &\quad d(x_{n-1},x_n)\biggr\} \\ &= \max\bigl\{d(x_{n+1},x_{n-1}),d(x_{n+1},x_{n+2}),d(x_{n-1},x_n)\bigr\}. \end{aligned}$$

We consider three different cases:

Case (i) If $M(x_{n+1},x_{n-1}) = d(x_{n+1},x_{n-1})$, then by (13), we get

$$d(x_{n+2},x_n) \le \psi\bigl(d(x_{n+1},x_{n-1})\bigr) \le \psi^{n+1}\bigl(d(x_2,x_0)\bigr) = \psi^{n+1}\bigl(l_0^*\bigr) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Case (ii) If $M(x_{n+1},x_{n-1}) = d(x_{n+1},x_{n+2})$, then by (13), we get

$$d(x_{n+2},x_n) \le \psi\bigl(d(x_{n+1},x_{n+2})\bigr) \le \psi^{n+1}\bigl(d(x_0,x_1)\bigr) = \psi^{n+1}(e_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Case (iii) If $M(x_{n+1},x_{n-1}) = d(x_{n-1},x_n)$, then by (13), we get

$$d(x_{n+2},x_n) \le \psi\bigl(d(x_{n-1},x_n)\bigr) \le \psi^{n-1}\bigl(d(x_0,x_1)\bigr) = \psi^{n-1}(e_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{14}$$

From (12) and (14), we deduce that

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n+2}) = 0 = \lim_{n\longrightarrow\infty} d(x_{n+2}, x_n).$$

Step 2: We shall prove that $\{x_n\}$ is a rectangular quasi b-Cauchy sequence, that is,

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n+p}) = 0 = \lim_{n\longrightarrow\infty} d(x_{n+p}, x_n) \quad \text{for all } p \in N.$$

Case (i) Suppose that for some $n, m \in N$ with $m > n$, we have $x_n = x_m$, by (10)

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, Tx_n) = d(x_m, Tx_m) = d(x_m, x_{m+1}) \\ &\leq \psi^{m-n}\big(d(x_n, x_{n+1})\big) \\ &\leq s\psi\big(d(x_n, x_{n+1})\big) < d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction.

Case (ii) Suppose that for some $n, m \in N$ with $n > m$, we have $x_n = x_m$, by (10)

$$\begin{aligned} d(x_{m+1}, x_m) = d(Tx_m, x_m) = d(Tx_n, x_n) = d(x_{n+1}, x_n) \\ &\leq \psi^{n-m}\big(d(x_{m+1}, x_m)\big) \\ &\leq s\psi\big(d(x_{m+1}, x_m)\big) < d(x_{m+1}, x_m), \end{aligned}$$

which is a contradiction.

Therefore, from Case (i) and Case (ii) $x_n \neq x_m$ for $m \neq n$.

The case $p = 1$ and $p = 2$ is proved. Now we take $p \geq 3$; arbitrary, we distinguish four different cases:

Case (i) Let $p = 2m$, where $m \geq 2$. By the rectangular inequality, we get

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq s\big[d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+2m})\big] \\ &\leq sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + s^2\big[d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+5}) \\ &\quad + d(x_{n+5}, x_{n+2m})\big] \\ &= sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) + s^2 d(x_{n+4}, x_{n+5}) \\ &\quad + s^2 d(x_{n+5}, x_{n+2m}) \\ &\quad \vdots \\ &\leq sd(x_n, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + s^4 d(x_{n+3}, x_{n+4}) + s^5 d(x_{n+4}, x_{n+5}) + \cdots \\ &\quad + s^{2m} d(x_{n+2m-1}, x_{n+2m}) \\ &= sd(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^{k-n+1} d(x_k, x_{k+1}) \\ &\leq sd(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^k \psi^k(e_0) \\ &\leq sd(x_n, x_{n+2}) + \sum_{k=n+2}^{\infty} s^k \psi^k(e_0). \end{aligned}$$

By (14), $\lim_{n\longrightarrow\infty} d(x_n, x_{n+2}) = 0$ and $\sum_{k=n+2}^{\infty} s^k \psi^k(e_0) \longrightarrow 0$ as $n \longrightarrow \infty$.

Therefore,

$$\lim_{n,m\longrightarrow\infty} d(x_n, x_{n+2m}) = 0.$$

Case (ii) Let $p = 2m + 1$, where $m \geq 1$. By the rectangular inequality, we get

$$\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq s\big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})\big] \\
&\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + s^2\big[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\
&\quad + d(x_{n+4}, x_{n+2m+1})\big] \\
&= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) \\
&\quad + s^2 d(x_{n+4}, x_{n+2m+1}) \\
&\quad \vdots \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + s^4 d(x_{n+3}, x_{n+4}) + \cdots \\
&\quad + s^{2m+1} d(x_{n+2m}, x_{n+2m+1}) \\
&= \sum_{k=n}^{n+2m} s^{k-n+1} d(x_k, x_{k+1}) \\
&= \sum_{k=n}^{n+2m} s^{k-n+1} \psi^k(e_0) \\
&\leq \sum_{k=n}^{n+2m} s^k \psi^k(e_0) \\
&\leq \sum_{k=n}^{\infty} s^k \psi^k(e_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Thus, we obtain

$$\lim_{n,m\longrightarrow\infty} d(x_n, x_{n+2m+1}) = 0.$$

Case (iii) Let $p = 2m$, where $m \geq 2$. By the rectangular inequality, we get

$$\begin{aligned}
d(x_{n+2m}, x_n) &\leq s\big[d(x_{n+2m}, x_{n+2m-2}) + d(x_{n+2m-2}, x_{n+2m-3}) + d(x_{n+2m-3}, x_n)\big] \\
&\leq sd(x_{n+2m}, x_{n+2m-2}) + sd(x_{n+2m-2}, x_{n+2m-3}) + s^2\big[d(x_{n+2m-3}, x_{n+2m-4}) \\
&\quad + d(x_{n+2m-4}, x_{n+2m-5}) + d(x_{n+2m-5}, x_n)\big] \\
&= sd(x_{n+2m}, x_{n+2m-2}) + sd(x_{n+2m-2}, x_{n+2m-3}) + s^2 d(x_{n+2m-3}, x_{n+2m-4}) \\
&\quad + s^2 d(x_{n+2m-4}, x_{n+2m-5}) + s^2 d(x_{n+2m-5}, x_n) \\
&\quad \vdots \\
&\leq sd(x_{n+2m}, x_{n+2m-2}) + s^{n+2m-2} d(x_{n+2m-2}, x_{n+2m-3}) \\
&\quad + s^{n+2m-3} d(x_{n+2m-3}, x_{n+2m-4}) + s^{n+2m-4} d(x_{n+2m-4}, x_{n+2m-5}) + \cdots \\
&\quad + s^{n-1} d(x_{n-1}, x_n)
\end{aligned}$$

$$
\begin{aligned}
&= sd(x_{n+2m}, x_{n+2m-2}) + \sum_{k=n-1}^{n+2m-1} s^k d(x_k, x_{k+1}) \\
&\leq sd(x_{n+2m}, x_{n+2m-2}) + \sum_{k=n-1}^{n+2m-1} s^k \psi^k\bigl(l_0^*\bigr) \\
&\leq sd(x_{n+2m}, x_{n+2m-2}) + \sum_{k=n-1}^{\infty} s^k \psi^k\bigl(l_0^*\bigr).
\end{aligned}
$$

Since $\lim_{m,n\longrightarrow\infty} d(x_{n+2m}, x_{n+2m-2}) = 0$ and $\sum_{k=n-1}^{\infty} s^k \psi^k(l_0^*) \longrightarrow 0$ as $n \longrightarrow \infty$, we have

$$\lim_{n,m\longrightarrow\infty} d(x_{n+2m}, x_n) = 0.$$

Case (iv): Let $p = 2m + 1$, where $m \geq 1$. By the rectangular inequality, we get

$$
\begin{aligned}
d(x_{n+2m+1}, x_n) &\leq s\bigl[d(x_{n+2m+1}, x_{n+2m}) + d(x_{n+2m}, x_{n+2m-1}) + d(x_{n+2m-1}, x_n)\bigr] \\
&\leq sd(x_{n+2m+1}, x_{n+2m}) + sd(x_{n+2m}, x_{n+2m-1}) + s^2\bigl[d(x_{n+2m-1}, x_{n+2m-2}) \\
&\quad + d(x_{n+2m-2}, x_{n+2m-3}) + d(x_{n+2m-3}, x_n)\bigr] \\
&= sd(x_{n+2m+1}, x_{n+2m}) + sd(x_{n+2m}, x_{n+2m-1}) + s^2 d(x_{n+2m-1}, x_{n+2m-2}) \\
&\quad + s^2 d(x_{n+2m-2}, x_{n+2m-3}) + s^2 d(x_{n+2m-3}, x_n) \\
&\quad \vdots \\
&\leq s^{n+2m+1} d(x_{n+2m+1}, x_{n+2m}) + s^{n+2m} d(x_{n+2m}, x_{n+2m-1}) \\
&\quad + s^{n+2m-1} d(x_{n+2m-1}, x_{n+2m-2}) + s^{n+2m-2} d(x_{n+2m-2}, x_{n+2m-3}) \\
&\quad + \cdots + s^{n+1} d(x_{n+1}, x_n) \\
&= \sum_{k=n+1}^{n+2m} s^k d(x_{k+1}, x_k) \\
&= \sum_{k=n+1}^{n+2m} s^{k-n+1} \psi^k(l_0) \\
&\leq \sum_{k=n+1}^{n+2m} s^k \psi^k(l_0) \\
&\leq \sum_{k=n+1}^{\infty} s^k \psi^k(l_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
$$

Thus, we obtain

$$\lim_{n,m\longrightarrow\infty} d(x_{n+2m+1}, x_n) = 0.$$

Finally, from Case (i)-Case (iv), we get $\lim_{n\longrightarrow\infty} d(x_n, x_{n+p}) = 0 = \lim_{n\longrightarrow\infty} d(x_{n+p}, x_n)$ for all $p \geq 3$.

Thus, $\{x_n\}$ is a rectangular quasi b-Cauchy sequence in (X, d).

Since X is a complete rectangular quasi b-metric space, there exists $u \in X$ such that

$$\lim_{n\longrightarrow\infty} x_n = u, \quad \text{i.e.,} \lim_{n\longrightarrow\infty} d(x_n,u) = 0 = \lim_{n\longrightarrow\infty} d(u,x_n). \tag{15}$$

Now, we shaw that u is a fixed point of T.

Since T is continuous, from (15), we have $u = \lim_{n\longrightarrow\infty} x_{n+1} = \lim_{n\longrightarrow\infty} T(x_n) = T(\lim_{n\longrightarrow\infty} x_n) = Tu$, which gives $u = Tu$.

Thus, u is a fixed point of T. □

Now, we state the following fixed point theorem by removing the continuity assumption of T from Theorem 1.

Theorem 2 *Let (X,d) be a complete rectangular quasi b-metric space and $T : X \longrightarrow X$ be generalized (α,ψ)-contraction mapping. Suppose that*

(i) *T is an α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$, $\alpha(Tx_0,x_0) \geq 1$, $\alpha(x_0,T^2x_0) \geq 1$, and $\alpha(T^2x_0,x_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then $\alpha(x_n,x) \geq 1$ for all $n \geq 0$.*

Then, T has a fixed point.

Proof Following the proof of Theorems 1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is rectangular quasi b- converges to a point u in X. It is sufficient to show that T admits a fixed point. By the rectangular inequality of rectangular quasi b-metric space, property of ψ, and (iii), we have

$$\begin{aligned} d(u,Tu) &\leq sd(u,x_n) + sd(x_n,x_{n+1}) + sd(x_{n+1},Tu) \\ &= sd(u,x_n) + sd(x_n,x_{n+1}) + sd(Tx_n,Tu) \\ &\leq sd(u,x_n) + sd(x_n,x_{n+1}) + s\alpha(x_n,u)d(Tx_n,Tu) \\ &\leq sd(u,x_n) + sd(x_n,x_{n+1}) + s\psi\bigl(M(x_n,u)\bigr), \end{aligned} \tag{16}$$

where

$$\begin{aligned} M(x_n,u) &= \max\left\{d(x_n,u), \frac{d(x_n,Tx_n)d(x_n,Tu)}{1+d(x_n,Tu)+d(u,Tx_n)}, d(x_n,Tx_n), d(u,Tu)\right\} \\ &= \max\left\{d(x_n,u), \frac{d(x_n,x_{n+1})d(x_n,Tu)}{1+d(x_n,Tu)+d(u,x_{n+1})}, d(x_n,x_{n+1}), d(u,Tu)\right\} \\ &= \max\bigl\{d(x_n,u), d(x_n,x_{n+1}), d(u,Tu)\bigr\}. \end{aligned}$$

We consider three different cases:

Case (i) If $M(x_n,u) = d(x_n,u)$, then by (16), we get

$$\begin{aligned} d(u,Tu) &\leq sd(u,x_n) + sd(x_n,x_{n+1}) + s\psi\bigl(d(x_n,u)\bigr) \\ &< sd(u,x_n) + sd(x_n,x_{n+1}) + d(x_n,u). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, from (10) and (15), we get that

$$d(u, Tu) \leq 0.$$

Case (ii) If $M(x_n, u) = d(x_n, x_{n+1})$, then by (16), we get

$$\begin{aligned} d(u, Tu) &\leq sd(u, x_n) + sd(x_n, x_{n+1}) + s\psi\big(d(x_n, x_{n+1})\big)s \\ &< sd(u, x_n) + sd(x_n, x_{n+1}) + d(x_n, x_{n+1}). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, from (10) and (15), we get that

$$d(u, Tu) \leq 0.$$

Case (iii) If $M(x_n, u) = d(u, Tu)$, then by (16), we get

$$\begin{aligned} d(u, Tu) &\leq sd(u, x_n) + sd(x_n, x_{n+1}) + s\psi\big(d(u, Tu)\big) \\ &< sd(u, x_n) + sd(x_n, x_{n+1}) + d(u, Tu). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, from (10) and (15), we get that $0 \leq 0$ it is general truth.

Clearly $d(u, Tu) \geq 0$, from Case (i)-Case (iii), we can obtain

$$d(u, Tu) = 0. \tag{17}$$

Also,

$$\begin{aligned} d(Tu, u) &\leq sd(Tu, x_n) + sd(x_n, x_{n+1}) + sd(x_{n+1}, u) \\ &= sd(Tu, Tx_{n-1}) + sd(x_n, x_{n+1}) + sd(x_{n+1}, u) \\ &\leq s\alpha(u, x_{n-1})d(Tu, Tx_{n-1}) + sd(x_n, x_{n+1}) + sd(x_{n+1}, u) \\ &\leq s\psi\big(M(u, x_{n-1})\big) + sd(x_n, x_{n+1}) + sd(x_{n+1}, u)), \end{aligned} \tag{18}$$

where

$$\begin{aligned} M(u, x_{n-1}) &= \max\left\{d(u, x_{n-1}), \frac{d(u, Tu)d(u, Tx_{n-1})}{1 + d(u, Tx_{n-1}) + d(x_{n-1}, Tu)}, d(u, Tu), d(x_{n-1}, Tx_{n-1})\right\} \\ &= \max\left\{d(u, x_{n-1}), \frac{d(u, Tu)d(u, x_n)}{1 + d(u, x_n) + d(x_{n-1}, Tu)}, d(u, Tu), d(x_{n-1}, x_n)\right\} \\ &= \max\big\{d(u, x_{n-1}), d(x_n, x_{n+1}), d(u, Tu)\big\}. \end{aligned}$$

We consider three different cases:

Case (i) If $M(u, x_{n-1}) = d(u, x_{n-1})$, then by (18), we get

$$\begin{aligned} d(Tu, u) &\leq s\psi\big(d(u, x_{n-1})\big) + sd(x_n, x_{n+1}) + sd(x_{n+1}, u) \\ &< d(u, x_{n-1}) + sd(x_n, x_{n+1}) + sd(x_{n+1}, u). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above equality, from (10) and (15), we get that

$$d(Tu,u) \leq 0.$$

Case (ii) If $M(u,x_{n-1}) = d(x_n,x_{n+1})$, then by (18), we get

$$\begin{aligned} d(Tu,u) &\leq s\psi\big(d(x_n,x_{n+1})\big) + sd(x_n,x_{n+1}) + sd(x_{n+1},u) \\ &< d(x_n,x_{n+1}) + sd(x_n,x_{n+1}) + sd(x_{n+1},u). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, from (10) and (15), we get that

$$d(Tu,u) \leq 0.$$

Case (iii) If $M(u,x_{n-1}) = d(u,Tu)$, then by (18), we get

$$\begin{aligned} d(Tu,u) &\leq s\psi\big(d(u,Tu)\big) + sd(x_n,x_{n+1}) + sd(x_{n+1},u) \\ &< d(u,Tu) + sd(x_n,x_{n+1}) + sd(x_{n+1},u). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, from (10), (15), and (17), we get that

$$d(Tu,u) \leq 0.$$

Clearly $d(u,Tu) \geq 0$, from Case (i)-Case (iii), we can obtain

$$d(Tu,u) = 0. \tag{19}$$

From (17) and (19), it follows that $d(u,Tu) = 0 = d(Tu,u)$. So that, $Tu = u$.
Thus, u is a fixed point of T. □

To assure the uniqueness of the fixed point of T, we will consider the following condition.

Property U For all $x,y \in \mathrm{Fix}(T)$, we have $\alpha(x,y) \geq 1$ and $\alpha(y,x) \geq 1$, where $\mathrm{Fix}(T)$ denotes the set of all fixed points of T.

Theorem 3 *Adding condition* (U) *to the hypothesis of Theorem* 1 (*res. Theorem* 2), *one obtains uniqueness of the fixed point of T.*

Proof From the proofs of Theorem 1 and Theorem 2, $\mathrm{Fix}(T) \neq \emptyset$. Suppose that u and v are two distinct fixed points of T.
By condition (U), $\alpha(Tu,Tv) = \alpha(u,v) \geq 1$ and $\alpha(Tv,Tu) = \alpha(v,u) \geq 1$.
Thus, by α-admissibility of T and the above relation, we can obtain

$$d(u,v) \leq \alpha(u,v)d(u,v) = \alpha(Tu,Tv)d(Tu,Tv) \leq \psi\big(M(u,v)\big),$$

where

$$\begin{aligned}M(u,v) &= \max\left\{d(u,v), \frac{d(u.Tu)d(u,Tv)}{1+d(u,Tv)+d(v,Tu)}, d(u,Tu), d(v,Tv)\right\}\\ &= \max\left\{d(u,v), \frac{d(u.u)d(u,v)}{1+d(u,v)+d(v,u)}, d(u,u), d(v,v)\right\}\\ &= d(u,v).\end{aligned}$$

On account of the fact that $s\psi(t) < t$, for all $t > 0$, and inequality above, we get that

$$d(u,v) \leq \psi\bigl(d(u,v)\bigr) \leq s\psi\bigl(d(u,v)\bigr) < d(u,v), \tag{20}$$

which is a contradiction. Similarly,

$$d(v,u) \leq \alpha(v,u)d(v,u) = \alpha(Tv,Tu)d(Tv,Tu) \leq \psi\bigl(M(v,u)\bigr),$$

where

$$\begin{aligned}M(v,u) &= \max\left\{d(v,u), \frac{d(v.Tv)d(v,Tu)}{1+d(v,Tu)+d(u,Tv)}, d(v,Tv), d(u,Tu)\right\}\\ &= \max\left\{d(v,u), \frac{d(v.v)d(v,u)}{1+d(v,u)+d(u,v)}, d(v,v), d(u,u)\right\}\\ &= d(v,u).\end{aligned}$$

On account of the fact that $s\psi(t) < t$, for all $t > 0$, and inequality above, we get that

$$d(v,u) \leq \psi\bigl(d(v,u)\bigr) \leq s\psi\bigl(d(v,u)\bigr) < d(v,u), \tag{21}$$

which is a contradiction. From (20) and (21), we get that $d(u,v) = 0 = d(v,u)$. Therefore, $u = v$.

Thus, T has a unique fixed point. □

Now, we give an example in support of Theorem 2.

Example 5 Let $X = A \cup B$, where $A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$. We define d on X as follows

$d(0,\frac{1}{2}) = d(\frac{1}{2},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{5}) = 0.3$; $d(0,\frac{1}{3}) = d(\frac{1}{3},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{4}) = 0.1$;
$d(0,\frac{1}{4}) = d(\frac{1}{4},\frac{1}{2}) = d(\frac{1}{3},\frac{1}{5}) = 0.6$; $d(0,\frac{1}{5}) = d(\frac{1}{2},\frac{1}{4}) = d(\frac{1}{5},\frac{1}{3}) = 0.4$;
$d(\frac{1}{2},0) = d(\frac{1}{4},0) = d(\frac{1}{2},\frac{1}{5}) = 1.05$; $d(\frac{1}{3},0) = d(\frac{1}{5},0) = d(\frac{1}{5},\frac{1}{2}) = d(\frac{1}{4},\frac{1}{3}) = 0.5$;
$d(0,0) = d(\frac{1}{2},\frac{1}{2}) = d(\frac{1}{3},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{4}) = d(\frac{1}{5},\frac{1}{5}) = 0$;
and $d(x,y) = |x-y|$ if $x,y \in B$ or $x \in A$, $y \in B$ or $x \in B$, $y \in A$.

Then, (X,d) is a complete rectangular quasi b-metric space with coefficient $s = \frac{3}{2} > 1$.

We define $T: X \to X$, $\psi: R^+ \to R^+$ and $\alpha: X \times X \to R^+$ by

$$Tx = \begin{cases}\frac{1}{3}, & \text{if } x \in A,\\ \frac{1+x}{2}, & \text{if } x \in B;\end{cases}$$

$\psi(t) = \frac{1}{2}t$ for all $t \in R^+$ and $\alpha : X \times X \to R^+$ as

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in B, \\ 0, & \text{otherwise.} \end{cases}$$

1. First we show that T is an α-admissible mapping.

To show this assume that $x, y \in X$ such that $\alpha(x,y) \geq 1$. It yields that $x, y \in B$. Owing to the definition of T, we have $Tx, Ty \in B$ and hence $\alpha(Tx, Ty) \geq 1$. Thus, T is α-admissible.

2. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$, $\alpha(T^2x_0, x_0) \geq 1$. In fact for $x_0 = 2$, we have $\alpha(2, T2) = \alpha(2, \frac{3}{2}) = 1$, $\alpha(T2, 2) = \alpha(\frac{3}{2}, 2) = 1$ and $\alpha(2, T^2 2) = \alpha(2, \frac{5}{4}) = 1$, $\alpha(T^2 2, 2) = \alpha(\frac{5}{4}, 2) = 1$.

3. Now, we show that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in N$, then $\{x_n\} \subset B$.

If $x_n \longrightarrow u$ as $n \longrightarrow \infty$, we have $d(x_n, u) = |x_n - u| \longrightarrow 0$ as $n \longrightarrow \infty$ and $d(u, x_n) = |u - x_n| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $u \in B$ and hence $\alpha(x_n, u) = 1 = \alpha(u, x_n)$.

4. Now, we show that T is a generalized (α, ψ)-contraction mapping.

Case (i) Let $x, y \in X$ such that $\alpha(x,y) \geq 1$, so $x, y \in A$, We have $Tx = \frac{1}{3}$ and $Ty = \frac{1}{3}$. Then $\alpha(x,y)d(Tx, Ty) = 0.|\frac{1}{3} - \frac{1}{3}| = 0 \leq \psi(M(x,y))$.

Case (ii) Let $x, y \in B$, we have $Tx = \frac{1+x}{2}$ and $Ty = \frac{1+y}{2}$. Then

$$\alpha(x,y)d(Tx, Ty) = |Tx - Ty| = \left| \frac{1+x}{2} - \frac{1+y}{2} \right| = \frac{1}{2}|x-y| = \psi\bigl(d(x,y)\bigr) \leq \psi\bigl(M(x,y)\bigr).$$

Case (iii) Let $x \in A$, $y \in B$ or $x \in B$, $y \in A$, we have $Tx = \frac{1}{3}$ and $Ty = \frac{1+y}{2}$ or $Tx = \frac{1+x}{2}$ and $Ty = \frac{1}{3}$. Then

$$\alpha(x,y)d(Tx, Ty) = 0, \quad |Tx - Ty| = 0 \leq \psi\bigl(M(x,y)\bigr).$$

Note that for $s = \frac{3}{2}$ and $\psi(t) = \frac{1}{2}t$, we have $\sum_{n=1}^{\infty} s^n \psi^n(t) = t \sum_{n=1}^{\infty} (\frac{3}{4})^n < \infty$ and $\frac{3}{2}\psi(t) < t$ for all $t > 0$.

Hence all the conditions of Theorem 2 are satisfied. Here $\{\frac{1}{3}, 1\}$ is the set of fixed point of T, that is, we have two fixed points.

Now we give an example in support of Theorem 3.

Example 6 Let $X = \{0, 1, 2, 3\}$ and $d : X \times X \to R^+$ defined by

$d(0,1) = d(2,0) = d(1,0) = 1$, $d(0,2) = d(3,0) = 26$,

$d(1,3) = d(2,1) = d(3,2) = 6$, $d(0,3) = d(3,1) = 7$,

$d(1,2) = d(2,3) = 8$,

$d(0,0) = d(1,1) = d(2,2) = d(3,3) = 0$.

Then, (X, d) is a complete rectangular quasi b-metric space with $s = 2 > 1$. Note that d is neither rectangular b-metric nor a rectangular quasi metric on X because

$d(0,1) = 1 \neq d(1,0) = 7$ and

$d(0,2) = 26 \nleq d(0,1) + d(1,3) + d(3,2) = 1 + 6 + 6 = 13$.

Define the map $T : X \longrightarrow X$ by $T0 = T1 = T2 = 0$ and $T3 = 1$. Let $\psi(t) = \frac{t}{4}$ for $t \in R^+$ and

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in \{0,1,2\}, \\ 0, & \text{otherwise.} \end{cases}$$

For $s = 2$, we have $\sum_{n=1}^{\infty} s^n \psi^n(t) = t \sum_{n=1}^{\infty} (\frac{1}{2})^n < \infty$ and $2\psi(t) < t$ for all $t > 0$.

We show that T is an α-admissible mapping.

Notice also that T is α-admissible. To show this assume that $x, y \in X$ such that $\alpha(x,y) \geq 1$. It yields that $x, y \in \{0,1,2\}$. Owing to the definition of T. We have $Tx, Ty \in \{0,1,2\}$ and hence $\alpha(Tx,Ty) \geq 1$. Thus, T is an α-admissible.

We shall show that $\alpha(x,y)d(T_x,T_y) \leq \frac{1}{4}M(x,y)$, for all $x,y \in X$. For this we consider the following cases.

Case (i) For $x, y \in \{0,1,2\}$, We have

$$\alpha(x,y)d(Tx,Ty) = 1, \quad d(0,0) = 0 \leq \frac{1}{4}M(x,y).$$

Case (ii) For $x \in \{0,1,2\}$ and $y = 3$, we have

$$\alpha(x,y)d(Tx,Ty) = 0 \leq \frac{1}{4}\max\bigl\{M(x,y)\bigr\}.$$

Case (iii) For $y \in \{0,1,2\}$ and $x = 3$, we have

$$\alpha(x,y)d(Tx,Ty) = 0 \leq \frac{1}{4}\max\bigl\{M(x,y)\bigr\}.$$

Case (iv) For $x = y = 3$, We have

$$\alpha(x,y)d(Tx,Ty) = 0, \quad d(1,1) = 0 \leq \frac{1}{4}M(3,3).$$

From Case (i)-Case (iv) all the required hypothesis of Theorem 3 are satisfied. Here $x = 0$ is the unique fixed point of T.

In the following, we give some corollary to our main results.

Corollary 1 *Let (X,d) be a complete rectangular quasi b-metric space and $T : X \longrightarrow X$ be an (α,ψ)-contraction mapping, that is,*

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr) \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point.

Proof The result follows by taking $M(x,y) = d(x,y)$ for all $x,y \in X$ in the proof of Theorem 1 (or Theorem 2). □

Remark 2 By taking $s = 1$ in Corollary 1, we get the work by Karapinar and Lakzian [1]. Thus, this work generalizes the work by Karapinar and Lakzian [1].

Corollary 2 *Let* (X,d) *be a complete rectangular quasi b-metric space and* $T : X \longrightarrow X$ *be a continuous mapping if there exist functions* $\psi \in \Psi$ *such that*

$$d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr) \quad \textit{for all } x,y \in X.$$

Then T *has a unique fixed point.*

Proof The result follows by taking $\alpha(x,y) = 1$ and $M(x,y) = d(x,y)$ for all $x,y \in X$ in the proof of Theorem 1. □

Remark 3 By taking $s = 1$ in Corollary 2, we get the work by Khuangsatung et al. [3]. Thus, this work generalizes the work by Khuangsatung et al. [3].

Corollary 3 *Let* (X,d) *be a complete rectangular quasi b-metric space and* $T : X \longrightarrow X$ *be a continuous mapping. Suppose that there exists* $k \in [0,1)$ *such that*

$$d(Tx,Ty) \leq k\bigl(d(x,y)\bigr) \quad \textit{for all } x,y \in X.$$

Proof The result follows by taking $\psi(t) = kt$, where $k \in [0,1)$ and $t \geq 0$ in Corollary 2. □

4 Application to integral equation

In this section, we give an existence theorem for a solution of the following integral equation.

$$x(t) = \int_0^1 K\bigl(t,r,x(r)\bigr)\,dr, \tag{22}$$

where $K : [0,1] \times [0,1] \times R \longrightarrow R$ are continuous functions.

Throughout this section, let $X = C([0,1],R)$ be the set of real continuous functions defined on $[0,1]$. Take the rectangular quasi b-metric $d : X \times X \longrightarrow [0,\infty)$ defined by

$$d(x,y) = \begin{cases} \|(x-y)^2\|_\infty + \|x\|_\infty, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$

where $\|u\|_\infty = \max_{r\in[0,1]} |u(s)|$ for all $u \in X$. It is known that (X,d) is a complete rectangular quasi b- metric space (with $s = \frac{3}{2}$). Now, we prove the following result.

Theorem 4 *Suppose the following hypotheses hold*:

(i) *there exist* $k \in (0,1)$ *and* $g : X \times X \longrightarrow [0,\infty)$ *such that for all* $x,y \in X$ *with* $x(t) \leq y(t)$ *for all* $t \in [0,1]$ *and for every* $r \in [0,1]$, *we have*

$$0 \leq \bigl|K\bigl(t,r,x(r)\bigr) - K\bigl(t,r,y(r)\bigr)\bigr| \leq g(t,r)|x-y|,$$

and

$$\sup_{t\in[0,1]} \int_0^1 g(t,r)\,dr = k.$$

(ii) *K is non-decreasing with respect to its third variable;*
(iii) *there exists $x_0 \in X$ such that for all $t \in [0,1]$, we have*

$$x_0(t) \le \int_0^1 K\big(t,r,x_0(r)\big)\,dr$$

and

$$x_0(t) \le \int_0^1 K\left(t,r,\int_0^1 K\big(t,r,x_0(r)\big)\,dr\right)dr.$$

Then the integral Eq. (22) has a solution in X.

Proof For all $x \in X$ and $t \in [0,1]$, define the mapping $T : X \to X$ by $Tx(t) = \int_0^1 K(t,r,x(r))\,dr$ and $\alpha : X \times X \longrightarrow [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

Take $\psi(t) = kt$, so $\psi(t) < \frac{t}{s}$ for all $t > 0$ (since $s = \frac{3}{2}$). We define $x, y \in X$, $x \le y$ if and only if $x(t) \le y(t)$ for all $t \in [0,1]$, where $\le$ denotes the usual order of real numbers. Let $x, y \in X$ such that $\alpha(x,y) \ge 1$, so $x \le y$, hence $x(t) \le y(t)$ for all $t \in [0,1]$. Thus, by condition (i)

$$\begin{aligned}
\big|Tx(t) - Ty(t)\big| &\le \int_1^0 \big|K\big(t,r,x(r)\big) - K\big(t,r,y(r)\big)\big|\,dr \\
&\le \int_1^0 g(t,r)\big|x(r) - y(r)\big|\,dr \\
&= \int_1^0 g(t,r)\sqrt{\big(x(r) - y(r)\big)^2}\,dr \\
&\le k\sqrt{\|x - y\|_\infty^2}.
\end{aligned}$$

Again

$$\begin{aligned}
\big|Tx(t)\big| &\le \int_1^0 \big|K\big(t,r,x(r)\big)\big|\,dr \\
&\le \int_1^0 g(t,r)\big|x(r)\big|\,dr \\
&\le k\|x\|_\infty.
\end{aligned}$$

We deduce that for all $x, y \in X$ such that $x \le y$

$$\begin{aligned}
d(Tx - Ty) &= \|Tx - Ty\|_\infty^2 + \|x\|_\infty \\
&\le k^2\|x - y\|_\infty^2 + k\|x\|_\infty \\
&\le kd(x,y) = \psi\big(d(x,y)\big) \\
&\le \psi\big(M(x,y)\big).
\end{aligned}$$

Since K is non-decreasing with respect to its third variable, so for all $x, y \in X$ with $x \le y$, we get $Tx(t) \le T(y)(t)$ for all $t \in [0,1]$, that is, if $\alpha(x,y) \ge 1$, we obtained $\alpha(Tx, Ty) \ge 1$. Moreover, the condition (iii) yields that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$, $\alpha(Tx_0, x_0) \ge 1$, $\alpha(x_0, T^2x_0) \ge 1$ and $\alpha(T^2x_0, Tx_0) \ge 1$. Therefore, all conditions of Theorem 2 are verified with $s = \frac{3}{2}$ and hence the operator T has a fixed point, which is a solution to the integral Eq. (22) in X. □

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Authors' contributions
BNA contributed to the conceptualization, formal analysis, methodology, writing, editing, and approving the manuscript. KKT involved in formal analysis, methodology and writing the original draft. MAM supervised the work and critically revised the manuscript. All authors read and approved the final manuscript.

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