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Tykhonov well-posedness of fixed point problems in contact mechanics

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Abstract

We consider a fixed point problem $\mathcal{S}u = u$ where $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ is an almost history-dependent operator. First, we recall the unique solvability of the problem. Then, we introduce the concept of Tykhonov triple, provide several relevant examples, and prove the corresponding well-posedness results for the considered fixed point problem. This allows us to deduce various consequences which illustrate the stability of the solution with respect to perturbations of the operator $\mathcal{S}$. Our results provide mathematical tools in the analysis of a large number of history-dependent problems which arise in solid and contact mechanics. To give some examples, we consider two mathematical models which describe the equilibrium of a viscoelastic body in frictionless contact with a foundation. We state the mechanical problems, list the assumptions on the data, and derive their associated fixed point formulation. Then, we illustrate the use of our abstract results in order to deduce the continuous dependence of the solution with respect to the data and parameters. We also provide the corresponding mechanical interpretations.

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1 Introduction

The concept of well-posedness in the sense of Tykhonov was introduced in [26] for a minimization problem. It is based on two main ingredients: the existence and uniqueness of solution to the problem and the convergence to it of any approximating sequence. The well-posedness in the sense of Tykhonov (well-posedness, for short) has been extended for various optimization problems, see [2, 3, 9, 12] for instance. In addition, it was introduced in the study of variational and hemivariational inequalities in [13, 14] and [6], respectively. References in the field include [5, 8, 22, 27, 28], where general results concerning the well-posedness of various classes of inequalities can be found. A Tykhonov well-posedness result for a fixed point problem can be found in [18]. A general framework which unifies the view on well-posedness of abstract problems in metric spaces was recently considered in [23, 25]. The approach used in [25] was based on the concept of

Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ where I is a set of parameters, Ω represents a family approximating sets, and $\mathcal{C}$ is a set of sequences of elements of I.

History-dependent and almost history-dependent operators represent a special class of nonlinear operators defined on a space of continuous functions. They arise both in functional analysis, solid mechanics, and contact mechanics. In contact mechanics such operators could be involved either in the constitutive law, which describes the material behavior, or in the interface boundary conditions. Such kind of operators have a number of relevant properties, including fixed point properties and, for this reason, they have been intensively studied in the recent years. A reference in the field is the book [21]. Existence and uniqueness results on variational and hemivariational inequalities with history-dependent operators can be found in [20, 21]. General results on their numerical analysis could be found in [7].

Processes of contact between a deformable solid and a foundation are ubiquitous, and they can be found in many industrial settings, in transportation, in various scientific experimental settings, and in everyday life. This is the reason for the very large engineering literature dedicated to the modeling, numerical approximations, and computer simulations of such processes. On the other hand, the mathematical theory of contact mechanics has expanded substantially in the last decades and is quickly maturing. A sample of references are, e.g., the books [1, 4, 10, 15–17, 20, 21]. The concepts of Tykhonov triple and Tykhonov well-posedness have been used in [11, 24], in the study of quasistatic contact problems with viscoelastic or viscoplastic materials.

In this current paper we deal with the well-posedness of a class of fixed problems with emphasis to the study of history-dependent models of contact. To this end we need the notation that we introduce in what follows. First, we use $\mathbb{N}$ and $\mathbb{R}_+$ to represent the set of positive integers and the set of real nonnegative numbers, i.e., $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{R}_+ = [0, +\infty)$. Unless stated otherwise, X will be a real Banach space equipped with the norm $\|\cdot\|_X$. We also denote by $C(\mathbb{R}_+; X)$ the space of continuous functions on $\mathbb{R}_+$ with values in X and use notation 0_X and $\mathbf{0}_X$ for the zero element of the spaces X and $C(\mathbb{R}_+; X)$, respectively. Recall that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e., as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. The convergence of a sequence $\{v_n\}$ to an element v can be described as follows:

$$\begin{cases} v_n \to v \quad \text{in } C(\mathbb{R}_+; X) \text{ as } n \to \infty \quad \text{if and only if} \\ \max_{t \in [0,m]} \|v_n(t) - v(t)\|_X \to 0 \quad \text{as } n \to \infty, \text{for all } m \in \mathbb{N}. \end{cases} \tag{1}$$

We also mention that, unless stated otherwise, all the limits below are considered as $n \to \infty$, even if we do not mention it explicitly.

With these notation, the fixed point problem we study in this paper is stated as follows.

Problem $\mathcal{P}$ Given an operator $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$, find a function $u \in C(\mathbb{R}_+; X)$ such that

$$\mathcal{S}u(t) = u(t) \quad \text{for all } t \in \mathbb{R}_+. \tag{2}$$

Note that here and below in this paper we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point t, i.e., $\mathcal{S}u(t) = (\mathcal{S}u)(t)$. Moreover, for an element

$u_0 \in X$, we still write u_0 for the constant function $t \mapsto u_0$, for all $t \in \mathbb{R}_+$. Therefore, notation $\mathcal{S}u_0$ used below defines an element of $C(\mathbb{R}_+;X)$.

Our aim in this paper is twofold. The first one is to deduce stability results for Problem $\mathcal{P}$ with respect to perturbations of the operator $\mathcal{S}$. To this end we use a strategy based on the well-posedness of the fixed point problem $\mathcal{P}$ with respect to various Tykhonov triples. This represents, at the best of our knowledge, the first trait of novelty of our work. The second aim is to illustrate how these abstract results can be applied in the study of mathematical models of contact. To this end we consider two models of contact for which, in contrast with various references in the literature, we provide a variational formulation governed by fixed point problems. Using such formulation allows us to deduce some convergence results, which represents the second trait of novelty of this work.

The rest of the paper is structured as follows. In Sect. 2 we introduce the concept of well-posedness for the fixed point problem (2), then we prove its well-posedness with respect to four relevant Tykhonov triples. The proofs are based on various estimates, the properties of almost history-dependent operators, and the Gronwall argument. Next, we use these well-posedness results in Sect. 3. There, we prove various convergence results which show the stability of the solution with respect to perturbations of the operator $\mathcal{S}$. In Sect. 4 we present our first example of contact problem. It concerns a model of quasistatic frictionless contact between a viscoelastic body and a deformable foundation. In Sect. 5, we present our second example which, in contrast, models the quasistatic contact between a viscoelastic body with a rigid foundation covered by a layer of elastic material, say asperities. We end this paper with Sect. 6 in which we present some concluding remarks.

2 Tykhonov well-posedness

We start with an existence and uniqueness result in the study of Problem $\mathcal{P}$. To this end, we recall the following definition.

Definition 2.1 An operator $\mathcal{S}\colon C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X)$ is called an almost history-dependent (a.h.d.) operator if for any $m \in \mathbb{N}$ there exist $l^m \in [0,1)$ and $L^m > 0$ such that

$$\left\|\mathcal{S}u(t) - \mathcal{S}v(t)\right\|_X \le l^m \left\|u(t) - v(t)\right\|_X + L^m \int_0^t \left\|u(s) - v(s)\right\|_X ds \tag{3}$$

for all $u, v \in C(\mathbb{R}_+;X)$, $t \in [0,m]$.

An operator $\mathcal{S}\colon C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X)$ is called a history-dependent (h.d.) operator if inequality (3) holds with $l^m = 0$ for all $m \in \mathbb{N}$.

In order to avoid any confusion, we underline that in Definition 2.1 and everywhere in this paper the superscript m does not represent a power.

The unique solvability of Problem $\mathcal{P}$ is guaranteed by the following existence and uniqueness result.

Theorem 2.2 *Assume that $\mathcal{S}$ is an almost history-dependent operator. Then Problem $\mathcal{P}$ has a unique solution $u \in C(\mathbb{R}_+;X)$.*

A proof of Theorem 2.2 can be found in [19] and [21, p. 42], based on the Banach fixed point principle. There, various examples of a.h.d. and h.d. operators are provided and general properties of these classes of operators are proved.

We now introduce the concepts of Tykhonov triple and well-posedness of the fixed point problem (2) with respect to a given Tykhonov triple. These concepts have been introduced in [25] in the framework of abstract problems in metric spaces. Here we specify them to the particular setting of Problem $\mathcal{P}$ and, to this end, we denote by $\mathcal{X}$ the set of nonempty subsets of the space $C(\mathbb{R}_+;X)$. Moreover, for any set J, we use the notation $\mathcal{R}(J)$ for the set of sequences with elements in J.

Definition 2.3 A Tykhonov triple is a mathematical object of the form $\mathcal{T}=(I,\Omega,\mathcal{C})$ where I is a given nonempty set, $\Omega: I\to\mathcal{X}$ is a multivalued mapping, and $\mathcal{C}$ is a nonempty subset of the set $\mathcal{R}(I)$.

Definition 2.4 Given a Tykhonov triple $\mathcal{T}=(I,\Omega,\mathcal{C})$, a sequence $\{u_n\}\subset C(\mathbb{R}_+;X)$ is called a $\mathcal{T}$-approximating sequence if there exists a sequence $\{\omega_n\}\in\mathcal{C}$ such that $u_n\in\Omega(\omega_n)$ for each $n\in\mathbb{N}$.

Definition 2.5 Given a Tykhonov triple $\mathcal{T}=(I,\Omega,\mathcal{C})$, Problem $\mathcal{P}$ is said to be well-posed with $\mathcal{T}$ (or $\mathcal{T}$-well-posed, for short) if it has a unique solution and every $\mathcal{T}$-approximating sequence converges in the space $C(\mathbb{R}_+;X)$ to its solution.

We now complete the definition above with the following comments.

First, we refer to I as the set of parameters. In the examples we consider below in this paper we shall take either $I=\mathbb{R}_+$ or $I=\mathcal{R}(\mathbb{R}_+)$. In the case when $I=\mathbb{R}_+$ we use the notation θ for an element of I and $\boldsymbol{\theta}=\{\theta_n\}_n$ (or $\boldsymbol{\theta}=\{\theta_n\}$, for short) for an element of $\mathcal{R}(I)$. In the case when $I=\mathcal{R}(\mathbb{R}_+)$ we shall use the notation $\boldsymbol{\theta}=\{\theta^m\}_m$ for an element of I and $\boldsymbol{\theta}=\{\boldsymbol{\theta}_n\}_n$ (or $\boldsymbol{\theta}=\{\boldsymbol{\theta}_n\}$) with $\boldsymbol{\theta}_n=\{\theta_n^m\}_m$ for an element of $\mathcal{R}(I)$.

Second, refer to the family of sets $\{\Omega(\omega)\}_{\omega\in I}$ as the family of approximating sets and, moreover, we say that $\mathcal{C}$ defines the criterion of convergence. Note that approximating sequences always exist since, by assumption, $\mathcal{C}\neq\emptyset$ and for any sequence $\{\omega_n\}\in\mathcal{C}$ and any $n\in\mathbb{N}$, the set $\Omega(\omega_n)$ is not empty.

Next, we underline that the concept of well-posedness above is not an intrinsic concept for Problem $\mathcal{P}$. Indeed, it is defined through the convergence of the approximating sequences which, in turn, depends on the Tykhonov triple $\mathcal{T}$ we consider. For this reason, we use the terminology "$\mathcal{T}$-approximating sequence" and "well-posedness with $\mathcal{T}$" or "$\mathcal{T}$-well-posedness", for short.

Assume now that $\mathcal{S}$ is an almost history-dependent operator and, therefore, Problem $\mathcal{P}$ has a unique solution $u\in C(\mathbb{R}_+;X)$. Denote

$$\mathcal{R}_{\mathcal{P}}=\big\{\{u_n\}\subset C(\mathbb{R}_+;X): u_n\to u \text{ in } C(\mathbb{R}_+;X)\big\}. \tag{4}$$

Moreover, for any Tykhonov triple $\mathcal{T}$, denote by $\mathcal{R}_{\mathcal{T}}$ the set of $\mathcal{T}$-approximating sequences, i.e.,

$$\mathcal{R}_{\mathcal{T}}=\big\{\{u_n\}\subset C(\mathbb{R}_+;X): \{u_n\} \text{ is a } \mathcal{T}\text{-approximating sequence}\big\}. \tag{5}$$

Then it is easy to see that the well-posedness of a Problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}$ is equivalent with the inclusion $\mathcal{R}_{\mathcal{T}} \subset \mathcal{R}_{\mathcal{P}}$, i.e.,

$$\text{Problem } \mathcal{P} \text{ is } \mathcal{T}\text{-well-posed if and only if } \mathcal{R}_{\mathcal{T}} \subset \mathcal{R}_{\mathcal{P}}. \tag{6}$$

We now introduce the following definition.

Definition 2.6 Given two Tykhonov triples $\mathcal{T} = (I, \Omega, \mathcal{C})$ and $\mathcal{T}' = (I', \Omega', \mathcal{C}')$, we say that:
(a) $\mathcal{T}$ and $\mathcal{T}'$ are equivalent (and we write $\mathcal{T} \sim \mathcal{T}'$) if $\mathcal{R}_{\mathcal{T}} = \mathcal{R}_{\mathcal{T}'}$.
(b) $\mathcal{T}$ is smaller than $\mathcal{T}'$ (and we write $\mathcal{T} \leq \mathcal{T}'$) if $\mathcal{R}_{\mathcal{T}} \subset \mathcal{R}_{\mathcal{T}'}$.

It is easy to see that "$\sim$" represents an equivalence relation on the set of Tykhonov triples while "$\leq$" defines a relation of order on the same set. Moreover, using (6) we deduce that the following statements hold.

$$\text{If } \mathcal{T} \sim \mathcal{T}' \text{ then } \mathcal{P} \text{ is } \mathcal{T}\text{-well-posed if and only if } \mathcal{P} \text{ is } \mathcal{T}'\text{-well-posed.} \tag{7}$$

$$\text{If } \mathcal{T} \leq \mathcal{T}' \text{ and } \mathcal{P} \text{ is } \mathcal{T}'\text{- well-posed, then } \mathcal{P} \text{ is } \mathcal{T}\text{- well-posed, too.} \tag{8}$$

We now construct four relevant triples in the study of Problem $\mathcal{P}$.

Example 2.7 Assume that $\mathcal{S}$ is an a.h.d. operator and take $\mathcal{T}_1 = (I_1, \Omega_1, \mathcal{C}_1)$ where

$$I_1 = \mathbb{R}_+, \tag{9}$$

$$\Omega_1(\theta) = \big\{ u \in C(\mathbb{R}_+; X) : \big\| \mathcal{S}u(t) - u(t) \big\|_X \leq \theta \ \forall t \in \mathbb{R}_+ \big\} \quad \text{for all } \theta \in I_1, \tag{10}$$

$$\mathcal{C}_1 = \big\{ \{\theta_n\}_n : \theta_n \in I_1 \ \forall n \in \mathbb{N}, \theta_n \to 0 \text{ as } n \to \infty \big\}. \tag{11}$$

Note that for each $\theta \in I_1$ the fixed point u obtained in Theorem 2.2 belongs to $\Omega_1(\theta)$, which shows that $\Omega_1(\theta) \neq \emptyset$. Therefore, according to Definition 2.3, $\mathcal{T}_1$ is a Tykhonov triple.

Example 2.8 Assume that $\mathcal{S}$ is an a.h.d. operator and take $\mathcal{T}_2 = (I_2, \Omega_2, \mathcal{C}_2)$ where

$$I_2 = \mathbb{R}_+, \tag{12}$$

$$\Omega_2(\theta) = \big\{ u \in C(\mathbb{R}_+; X) : \big\| \mathcal{S}u(t) - u(t) \big\|_X \leq \theta \big(\big\| u(t) \big\|_X + 1 \big) \ \forall t \in \mathbb{R}_+ \big\} \quad \text{for all } \theta \in I_2, \tag{13}$$

$$\mathcal{C}_2 = \big\{ \{\theta_n\}_n : \theta_n \in I \ \forall n \in \mathbb{N}, \theta_n \to 0 \text{ as } n \to \infty \big\}. \tag{14}$$

Note that, again, using Theorem 2.2 it follows that $\Omega_2(\theta) \neq \emptyset$ for each $\theta \in I_2$.

Example 2.9 Assume that $\mathcal{S}$ is an a.h.d. operator and take $\mathcal{T}_3 = (I_3, \Omega_3, \mathcal{C}_3)$, where

$$I_3 = \big\{ \boldsymbol{\theta} = \big\{ \theta^m \big\}_m : \theta^m \in \mathbb{R}_+ \ \forall m \in \mathbb{N} \big\}, \tag{15}$$

$$\Omega_3(\boldsymbol{\theta}) = \big\{ u \in C(\mathbb{R}_+; X) : \big\| \mathcal{S}u(t) - u(t) \big\|_X \leq \theta^m \ \forall t \in [0, m], m \in \mathbb{N} \big\} \tag{16}$$

$$
\begin{aligned}
&\qquad \text{for all } \boldsymbol{\theta} = \{\theta^m\}_m \in I_3, \\
&\mathcal{C}_3 = \Big\{\{\boldsymbol{\theta}_n\}_n : \boldsymbol{\theta}_n = \{\theta_n^m\}_m \in I_3 \ \forall n \in \mathbb{N}, \\
&\quad \theta_n^m \to 0 \text{ as } n \to \infty, \forall m \in \mathbb{N}\Big\}.
\end{aligned} \tag{17}
$$

Note that $\Omega_3(\boldsymbol{\theta}) \neq \emptyset$ for each $\boldsymbol{\theta} \in I_3$.

Example 2.10 Assume that $\mathcal{S}$ is an a.h.d. operator and take $\mathcal{T}_4 = (I_4, \Omega_4, \mathcal{C}_4)$ where

$$I_4 = \big\{\boldsymbol{\theta} = \{\theta^m\}_m : \theta^m \in \mathbb{R}_+ \ \forall m \in \mathbb{N}\big\}, \tag{18}$$

$$
\begin{aligned}
\Omega_4(\boldsymbol{\theta}) = \big\{&u \in C(\mathbb{R}_+; X) : \\
&\|\mathcal{S}u(t) - u(t)\|_X \le \theta^m \big(\|u(t)\|_X + 1\big) \ \forall t \in [0,m], m \in \mathbb{N}\big\}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&\text{for all } \boldsymbol{\theta} = \{\theta^m\}_m \in I_3, \\
&\mathcal{C}_4 = \Big\{\{\boldsymbol{\theta}_n\}_n : \boldsymbol{\theta}_n = \{\theta_n^m\}_m \in I_3 \ \forall n \in \mathbb{N}, \\
&\quad \theta_n^m \to 0 \text{ as } n \to \infty, \forall m \in \mathbb{N}\Big\}.
\end{aligned} \tag{20}
$$

Note that $\Omega_4(\boldsymbol{\theta}) \neq \emptyset$ for each $\boldsymbol{\theta} \in I_4$.

The following result is useful in order to compare the Tykhonov triples $\mathcal{T}_3$ and $\mathcal{T}_4$.

Lemma 2.11 *Assume that $\mathcal{S}$ is an a.h.d. operator, and let $\{u_n\}$ be a $\mathcal{T}_4$-approximating sequence. Then, for each $m \in \mathbb{N}$, there exists $Z^m > 0$ such that*

$$\|u_n(t)\|_X \le Z^m \quad \textit{for all } t \in [0,m], n \in \mathbb{N}. \tag{21}$$

Proof Using Definition 2.4, (19), and (20), we deduce that there exists a sequence $\{\boldsymbol{\theta}_n\}_n$ with $\boldsymbol{\theta}_n = \{\theta_n^m\}_m \in \mathcal{S}(\mathbb{R}_+)$ for any $n \in \mathbb{N}$ such that

$$\theta_n^m \to 0 \quad \text{as } n \to \infty, \forall m \in \mathbb{N} \tag{22}$$

and, for any $m, n \in \mathbb{N}$, the following inequality holds:

$$\|\mathcal{S}u_n(t) - u_n(t)\|_X \le \theta_n^m \big(\|u_n(t)\|_X + 1\big) \quad \forall t \in [0,m]. \tag{23}$$

Let $m \in \mathbb{N}$, $t \in [0,m]$ and $n \in \mathbb{N}$. We write

$$\|u_n(t)\|_X \le \|u_n(t) - \mathcal{S}u_n(t)\|_X + \|\mathcal{S}u_n(t) - \mathcal{S}\mathbf{0}_X(t)\|_X + \|\mathcal{S}\mathbf{0}_X(t)\|_X,$$

then we use inequalities (23) and (3) to find that

$$\|u_n(t)\|_X \le \theta_n^m \big(\|u_n(t)\|_X + 1\big) + l^m \|u_n(t)\|_X + L^m \int_0^t \|u_n(s)\|_X \, ds + \|\mathcal{S}\mathbf{0}_X(t)\|_X$$

or, equivalently,

$$\big(1 - \theta_n^m - l^m\big) \|u_n(t)\|_X \le \theta_n^m + L^m \int_0^t \|u_n(s)\|_X \, ds + \|\mathcal{S}\mathbf{0}_X(t)\|_X. \tag{24}$$

Next, (22) and inequality $l^m < 1$ imply that for n large enough we can assume that $\theta_n^m \leq \frac{1-l^m}{2}$ and, with notation

$$F^m = \max_{t\in[0,m]} \left\| \mathcal{S}\mathbf{0}_X(t) \right\|_X, \tag{25}$$

inequality (24) yields

$$\left\| u_n(s) \right\|_X \leq \frac{2(\theta_n^m + F^m)}{1 - l^m} + \frac{2L^m}{1 - l^m} \int_0^t \left\| u_n(s) \right\|_X ds. \tag{26}$$

We now use the Gronwall argument to find that

$$\left\| u_n(t) \right\|_X \leq \frac{2(\theta_n^m + F^m)}{1 - l^m} e^{\frac{2L^m}{1-l^m} t}$$

and, using convergence (22) combined with inequality $t \leq m$, we conclude the proof. □

We now remark that, obviously, $\mathcal{R}_{\mathcal{T}_1} \subset \mathcal{R}_{\mathcal{T}_2} \subset \mathcal{R}_{\mathcal{T}_4}$ and, moreover, $\mathcal{R}_{\mathcal{T}_3} \subset \mathcal{R}_{\mathcal{T}_4}$. In addition, using Lemma 2.11 it is easy to see that $\mathcal{R}_{\mathcal{T}_4} \subset \mathcal{R}_{\mathcal{T}_3}$ and, therefore, $\mathcal{R}_{\mathcal{T}_3} = \mathcal{R}_{\mathcal{T}_4}$. We now use Definition 2.6 to see that

$$\mathcal{T}_1 \leq \mathcal{T}_2 \leq \mathcal{T}_4 \quad \text{and} \quad \mathcal{T}_3 \sim \mathcal{T}_4. \tag{27}$$

We are now in a position to introduce our main result in this section.

Theorem 2.12 *Assume that $\mathcal{S}$ is an almost history-dependent operator. Then Problem $\mathcal{P}$ is well-posed with respect to the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$ in Examples 2.7, 2.8, 2.9, and 2.10, respectively.*

Proof First, we recall that the existence of a unique solution to problem $\mathcal{P}$, needed for the well-posedness of Problem $\mathcal{P}$ with any Tykhonov triple, follows from Theorem 2.2.

Consider now the Tykhonov triple $\mathcal{T}_3$ in Example 2.9, and let $\{u_n\}$ be a $\mathcal{T}_3$-approximating sequence. Then, using Definition 2.4, we deduce that there exists a sequence $\{\boldsymbol{\theta}_n\}_n$ with $\boldsymbol{\theta}_n = \{\theta_n^m\}_m \in \mathcal{R}(\mathbb{R}_+)$ for all $n \in \mathbb{N}$ such that (22) holds and, moreover,

$$\left\| \mathcal{S}u_n(t) - u_n(t) \right\|_X \leq \theta_n^m \quad \forall t \in [0, m]. \tag{28}$$

Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and let $t \in [0, m]$. We use (2) and write

$$\left\| u_n(t) - u(t) \right\|_X \leq \left\| u_n(t) - \mathcal{S}u_n(t) \right\|_X + \left\| \mathcal{S}u_n(t) - \mathcal{S}u(t) \right\|_X,$$

then we use inequalities (28) and (3) to find that

$$\left\| u_n(t) - u(t) \right\|_X \leq \theta_n^m + l^m \left\| u_n(t) - u(t) \right\|_X + L^m \int_0^t \left\| u_n(s) - u(s) \right\|_X ds$$

or, equivalently,

$$\left(1 - \theta_n^m - l^m\right) \left\| u_n(t) - u(t) \right\|_X \leq \theta_n^m + L^m \int_0^t \left\| u_n(s) \right\|_X ds. \tag{29}$$

Next, arguments similar to those used in the proof of Lemma 2.11 show that, for n large enough, inequality (29) yields

$$\|u_n(s) - u(t)\|_X \le \frac{2\theta_n^m}{1 - l^m} + \frac{2L^m}{1 - l^m} \int_0^t \|u_n(s)\|_X \, ds \tag{30}$$

and, therefore,

$$\|u_n(t) - u(t)\|_X \le \frac{2\theta_n^m}{1 - l^m} e^{\frac{2L^m}{1-l^m} t}. \tag{31}$$

We now combine inequality (31) and convergence (22) to see that

$$\max_{t \in [0,m]} \|u_n(t) - u(t)\|_X \to 0 \quad \text{as } n \to \infty$$

and, since $m \in \mathbb{N}$ is arbitrary, using (1) we deduce that

$$u_n \to u \quad \text{in } C(\mathbb{R}_+; X) \text{ as } n \to \infty. \tag{32}$$

Convergence (32) combined with Definition 2.5 implies that Problem $\mathcal{P}$ is well-posed with respect to the Tykhonov triple $\mathcal{T}_3$ in Example 2.9. The well-posedness of Problem $\mathcal{P}$ with the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_4$ is now a direct consequence of relations (27), (7), and (8). □

3 Stability results

In this section we study the stability of the solution of Problem $\mathcal{P}$ with respect to perturbations of the operator $\mathcal{S}$. To this end, for each $n \in \mathbb{N}$, we consider the following fixed point problem.

Problem $\mathcal{P}_n$ Given an operator $\mathcal{S}_n : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$, find a function $u_n \in C(\mathbb{R}_+; X)$ such that

$$\mathcal{S}_n u_n(t) = u_n(t) \quad \text{for all } t \in \mathbb{R}_+. \tag{33}$$

We assume that for each $n \in \mathbb{N}$ the operator $\mathcal{S}_n$ is an almost history-dependent operator, i.e.,

$$\begin{cases} \text{For each } n, m \in \mathbb{N}, \text{there exist } l_n^m \in [0,1) \text{ and } L_n^m > 0 \text{ such that} \\ \|\mathcal{S}_n u(t) - \mathcal{S}_n v(t)\|_X \le l_n^m \|u(t) - v(t)\|_X + L_n^m \int_0^t \|u(s) - v(s)\|_X \, ds \\ \text{for all } u, v \in C(\mathbb{R}_+; X), t \in [0, m]. \end{cases}$$

Then, using Theorem 2.2, it follows that Problem $\mathcal{P}_n$ has a unique solution $u_n \in C(\mathbb{R}_+; X)$ for each $n \in \mathbb{N}$. Our aim in what follows is to state various conditions which link the operators $\mathcal{S}_n$ and $\mathcal{S}$ and guarantee that the solution of Problem $\mathcal{P}_n$ converges to the solution of Problem $\mathcal{P}$, that is,

$$u_n \to u \quad \text{in } C(\mathbb{R}_+; X), \text{as } n \to \infty. \tag{34}$$

To this end we use the following two step strategy.

$$\begin{cases} \text{(i) First, we impose conditions which guarantees that the sequence } \{u_n\} \\ \quad \textit{is a } \mathcal{T}_i\textit{-approximating sequence for some } i \in \{1,2,3\}. \\ \text{(ii) Second, we use the } \mathcal{T}_i\text{-well-posedness of Problem } \mathcal{P}, \\ \quad \textit{obtained in Theorem 2.12, to deduce that convergence (34) holds.} \end{cases} \tag{35}$$

The conditions we consider are the following.

$$\begin{cases} \text{For each } n \in \mathbb{N}, \text{there exists } \theta_n > 0 \text{ such that} \\ \text{(a) } \|\mathcal{S}_n v(t) - \mathcal{S}v(t)\|_X \le \theta_n \quad \text{for all } v \in C(\mathbb{R}_+;X), t \in \mathbb{R}_+. \\ \text{(b) } \theta_n \to 0 \quad \text{as } n \to \infty. \end{cases} \tag{36}$$

$$\begin{cases} \text{For each } n \in \mathbb{N}, \text{there exists } \theta_n > 0 \text{ such that} \\ \text{(a) } \|\mathcal{S}_n v(t) - \mathcal{S}v(t)\|_X \le \theta_n(1 + \|v(t)\|_X) \quad \text{for all } v \in C(\mathbb{R}_+;X), t \in \mathbb{R}_+. \\ \text{(b) } \theta_n \to 0 \quad \text{as } n \to \infty. \end{cases} \tag{37}$$

$$\begin{cases} \text{For each } m \in \mathbb{N} \text{ and } n \in \mathbb{N}, \text{there exists} \theta_n^m > 0 \text{ such that} \\ \text{(a) } \|S_n v(t) - Sv(t)\|_X \le \theta_n^m(1 + \|v(t)\|_X + \int_0^t \|v(s)\|_X\, ds) \\ \quad \text{for all } v \in C(\mathbb{R}_+;X), t \in [0,m]. \\ \text{(b) } \theta_n^m \to 0 \quad \text{as } n \to \infty \text{ for each } m \in \mathbb{N}. \end{cases} \tag{38}$$

We have the following convergence result.

Theorem 3.1 *Assume that $\mathcal{S}$ and $\mathcal{S}_n$ are almost history-dependent operators for each $n \in \mathbb{N}$. Then*:

(a) *Under assumption* (36) *the sequence $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence. Moreover, convergence* (34) *holds*.

(b) *Under assumption* (37) *the sequence $\{u_n\}$ is a $\mathcal{T}_2$-approximating sequence. Moreover, convergence* (34) *holds*.

(c) *Under assumption* (38) *the sequence $\{u_n\}$ is a $\mathcal{T}_3$-approximating sequence. Moreover, convergence* (34) *holds*.

Proof (a) Let $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. We use (33) and (36)(a) to write

$$\|\mathcal{S}u_n(t) - u_n(t)\|_X \le \|\mathcal{S}u_n(t) - \mathcal{S}_n u_n(t)\|_X \le \theta_n.$$

This inequality combined with assumption (36)(b) and definition (9)–(11) shows that the sequence $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence. Convergence (34) is now a consequence of the $\mathcal{T}_1$-well-posedness of Problem $\mathcal{P}$, guaranteed by Theorem 2.12.

(b) Let $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. We use (33) and (37)(a) to write

$$\|\mathcal{S}u_n(t) - u_n(t)\|_X \le \|\mathcal{S}u_n(t) - \mathcal{S}_n u_n(t)\|_X \le \theta_n(\|u_n(t)\|_X + 1).$$

This inequality combined with assumption (37)(b) and definition (12)–(14) show that in this case the sequence $\{u_n\}$ is a $\mathcal{T}_2$-approximating sequence. Convergence (34) is now a consequence of the $\mathcal{T}_2$-well-posedness of Problem $\mathcal{P}$, guaranteed by Theorem 2.12.

(c) We first prove that for each $m \in \mathbb{N}$ there exists $U^m > 0$ such that

$$\|u_n(t)\|_X \le U^m \quad \text{for all } t \in [0,m], n \in \mathbb{N}. \tag{39}$$

Indeed, let $m \in \mathbb{N}$, $n \in \mathbb{N}$ and let $t \in [0,m]$. We use (33) and write

$$\|u_n(t)\|_X \le \|\mathcal{S}_n u_n(t) - \mathcal{S}u_n(t)\|_X + \|\mathcal{S}u_n(t) - \mathcal{S}\mathbf{0}_X(t)\|_X + \|\mathcal{S}\mathbf{0}_X(t)\|_X,$$

then we use inequalities (38)(a) and (3) to find that

$$\begin{aligned}\|u_n(t)\|_X &\le \theta_n^m\Big(1 + \|u_n(t)\|_X + \int_0^t \|u_n(s)\|_X\,ds\Big) \\ &\quad + l^m\|u_n(t)\|_X + L^m\int_0^t \|u_n(s)\|_X\,ds + \|\mathcal{S}\mathbf{0}_X(t)\|_X.\end{aligned}$$

Therefore, combining this inequality with notation (25), we obtain

$$\big(1 - \theta_n^m - l^m\big)\|u_n(t)\|_X \le \theta_n^m + \big(L^m + \theta_n^m\big)\int_0^t \|u_n(s)\|_X\,ds + F^m. \tag{40}$$

Next, (38)(b) and the inequality $l^m < 1$ imply that for n large enough we can assume that $\theta_n^m \le \frac{1-l^m}{2} < 1$ and, therefore, (40) yields

$$\|u_n(s)\|_X \le \frac{2(F^m + \theta_n^m)}{1 - l^m} + \frac{2(L^m + \theta_n^m)}{1 - l^m}\int_0^t \|u_n(s)\|_X\,ds. \tag{41}$$

We now use the Gronwall argument to find that

$$\|u_n(t)\|_X \le \frac{2(F^m + \theta_n^m)}{1 - l^m} e^{\frac{2(L^m+\theta_n^m)}{1-l^m}t},$$

and, using inequality $t \le m$, we conclude that (39) holds with

$$U_m = \frac{2(F^m + \theta_n^m)}{1 - l^m} e^{\frac{2(L^m+\theta_n^m)}{1-l^m}m}.$$

We now use (33) and (38)(a) to write

$$\|\mathcal{S}u_n(t) - u_n(t)\|_X = \|\mathcal{S}u_n(t) - \mathcal{S}_n u_n(t)\|_X \le \theta_n\Big(1 + \|u_n(t)\|_X + \int_0^t \|u_n(s)\|_X\,ds\Big)$$

and, using bound (39), we find that

$$\|\mathcal{S}u_n(t) - u_n(t)\|_X \le \theta_n^m\big(1 + U^m + U^m m\big). \tag{42}$$

Consider now the sequence $\boldsymbol{\omega}_n = \{\omega_n^m\}_m \in \mathcal{R}(\mathbb{R}_+)$ where

$$\omega_n^m = \theta_n^m\big(1 + U^m + U^m m\big) \tag{43}$$

for each $m, n \in \mathbb{N}$. Then inequality (42) implies that $u_n \in \Omega_3(\boldsymbol{\omega}_n)$ where, recall, $\Omega_3(\boldsymbol{\omega})$ is the set defined by (16) for each $\boldsymbol{\omega} = \{\omega^m\}_m \in \mathcal{R}(\mathbb{R}_+)$. On the other hand, assumptions (38)(b) and definition (43) imply that $\omega_n^m \to 0$ as $n \to \infty$ for each $m \in \mathbb{N}$. This shows that $\boldsymbol{\omega}_n = \{\omega_n^m\}_m \in \mathcal{C}_3$ where $\mathcal{C}_3$ represents the set given by (17). It follows from the above that the sequence $\{u_n\}$ is a $\mathcal{T}_3$-approximating sequence for Problem $\mathcal{P}$. We now use Theorem 2.12 and Definition 2.5 to deduce convergence (34), which concludes the proof. □

We now remark that assumption (37) does not guarantee that the sequence $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence. Moreover, assumption (38) does not guarantee that the sequence $\{u_n\}$ is a $\mathcal{T}_2$-approximating sequence. An evidence of these statements is provided by the two examples below.

Example 3.2 Let $X = \mathbb{R}$ and let $\mathcal{S}_n, \mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be the operators defined by

$$\mathcal{S}_n u(t) = \int_0^t u(s)\,ds + \frac{1}{n+1} u(t) + \frac{1}{n+1}, \qquad \mathcal{S} u(t) = \int_0^t u(s)\,ds \tag{44}$$

for all $u \in C(\mathbb{R}_+; X)$, $t \in \mathbb{R}_+$, and $n \in \mathbb{N}$. Then, it is easy to see that $\mathcal{S}_n$ and $\mathcal{S}$ are almost history-dependent operators which satisfy condition (37). Moreover, with this choice, the solution of Problem $\mathcal{P}_n$ is given by

$$u_n(t) = \frac{1}{n} e^{\frac{n+1}{n} t} \quad \forall n \in \mathbb{N}, t \in \mathbb{R}_+ \tag{45}$$

and the solution of Problem $\mathcal{P}$ is

$$u(t) = 0 \quad \forall t \in \mathbb{R}_+. \tag{46}$$

Arguing by contradiction, assume that the sequence $\{u_n\}$ is a $\mathcal{T}_1$-approximation sequence. Then there exists $\{\theta_n\} \subset \mathcal{R}(\mathbb{R}_+)$ such that $\theta_n \to 0$ and

$$|\mathcal{S} u_n - u_n| \le \theta_n \quad \forall n \in \mathbb{N}, t \in \mathbb{R}_+. \tag{47}$$

Using now (44)–(47) we deduce that

$$\frac{1}{n(n+1)} e^{\frac{n+1}{n} t} + \frac{1}{n+1} \le \theta_n \quad \forall n \in \mathbb{N}, t \in \mathbb{R}_+.$$

We now take $t = n^2$ in the previous inequality, then we pass to the limit as $n \to \infty$ and arrive at a contradiction. We conclude from here that $\{u_n\}$ is not a $\mathcal{T}_1$-approximation sequence. Nevertheless, using (45) and (46) it is easy to see that convergence (34) holds.

Example 3.3 Let $X = \mathbb{R}$, and let $\mathcal{S}_n, \mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be the operators defined by

$$\mathcal{S}_n u(t) = 1 - \frac{1}{n} \int_0^t u(s)\,ds, \qquad \mathcal{S} u(t) = 1 \tag{48}$$

for all $u \in C(\mathbb{R}_+; X)$, $t \in \mathbb{R}_+$, and $n \in \mathbb{N}$. Then it is easy to see that $\mathcal{S}_n$ and $\mathcal{S}$ are history-dependent operators which satisfy condition (38). Moreover, with this choice, the solution of Problem $\mathcal{P}_n$ is given by

$$u_n(t) = e^{-\frac{t}{n}} \quad \forall n \in \mathbb{N}, t \in \mathbb{R}_+ \tag{49}$$

and the solution of Problem $\mathcal{P}$ is

$$u(t) = 1 \quad \forall t \in \mathbb{R}_+. \tag{50}$$

Arguing by contradiction, assume that the sequence $\{u_n\}$ is a $\mathcal{T}_2$-approximation sequence. Then there exists $\{\theta_n\} \subset \mathcal{R}(\mathbb{R}_+)$ such that $\theta_n \to 0$ and

$$|\mathcal{S}u_n - u_n| \le \theta_n\big(|u_n(t)| + 1\big) \quad \forall n \in \mathbb{N}, t \in \mathbb{R}_+. \tag{51}$$

Using now (48)–(51) we deduce that

$$1 - e^{-\frac{t}{n}} \le \theta_n\big(e^{-\frac{t}{n}} + 1\big) \quad \forall n \in \mathbb{N}, t \in \mathbb{R}_+.$$

We take $t = n$ in the previous inequality, then we pass to the limit as $n \to \infty$ and arrive at a contradiction. We conclude from here that $\{u_n\}$ is not a $\mathcal{T}_2$ approximation sequence. Nevertheless, convergence (34) holds.

It follows from Example 3.2 that, under condition (37), the $\mathcal{T}_1$-well-posedness of Problem $\mathcal{P}$ cannot be used to provide convergence (34) in the framework of the two step strategy (35). Similarly, Example 3.3 shows that, under condition (38), the $\mathcal{T}_2$-well-posedness of Problem $\mathcal{P}$ cannot be used to provide convergence (34) in the framework of the same strategy. We conclude from the above that the choice of the Tykhonov triple plays a crucial role in proving convergence result for Problem $\mathcal{P}$.

4 A contact problem with normal compliance

In this section we apply Theorem 3.1 in the study of a quasistatic frictionless contact problem with normal compliance. The physical setting is the following. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with a Lipschitz continuous boundary Γ, divided into three measurable disjoint parts Γ_1, Γ_2, and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. The body is fixed on Γ_1, is acted upon by a given surface traction on Γ_2, and could arrive in contact with a deformable foundation on Γ_3. Moreover, there is a gap between the body and the foundation, the mechanical process is quasistatic, the contact is frictionless, and the time interval of interest is $\mathbb{R}_+ = [0, +\infty)$. We denote by $\boldsymbol{\nu}$ the unit outward normal to Γ and by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. Then the classical formulation of the contact problem we study in this section is the following.

Problem $\mathcal{Q}$ Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that, for any $t \in \mathbb{R}_+$, the following equalities hold:

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \int_0^t \mathcal{F}(t-s)\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\,ds \quad \text{in } \Omega, \tag{52}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{53}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{54}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{55}$$

$$-\sigma_\nu(t) = kp\big(u_\nu(t) - g\big) \quad \text{on } \Gamma_3, \tag{56}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{57}$$

We now provide a short description of the equations and boundary conditions in Problem $\mathcal{Q}$ and send the reader to [20, 21] for more details and mechanical interpretations. First, equation (52) represents the viscoelastic constitutive law of the material in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor, $\mathcal{E}$ is a fourth order elasticity tensor, and $\mathcal{F}$ is the relaxation tensor. Equation (53) represents the equilibrium equation in which $\boldsymbol{f}_0$ denotes the density of body forces, (54) is the displacement boundary condition, and (55) is the traction boundary condition in which $\boldsymbol{f}_2$ denotes the density of surface tractions. Condition (56) is the contact condition with normal compliance in which p is a given prescribed function, g represents the gap function, and k is a positive stiffness coefficient. Moreover, u_ν and σ_ν denote the normal components of the displacement and the stress field, respectively. Finally, condition (56) is the frictionless condition in which $\boldsymbol{\sigma}_\tau$ represents the tangential shear.

In the variational analysis of Problem $\mathcal{Q}$ we denote by "$\cdot$", $\|\cdot\|$, and $\boldsymbol{0}$ the inner product, the Euclidean norm, and the zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. We use the standard notation for the Sobolev and Lebesgue spaces associated to Ω and Γ and, for an element $\boldsymbol{v} \in H^1(\Omega)^d$, we write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v} \in L^2(\Gamma)^d$ of $\boldsymbol{v}$ on Γ. Moreover, we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on the boundary, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$, respectively. We also use the following spaces:

$$V = \big\{\boldsymbol{v} = (v_i) \in H^1(\Omega)^d : v_i = 0 \text{ on } \Gamma_1, i = 1, \dots, d\big\},$$
$$Q = \big\{\boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), i, j = 1, \dots, d\big\},$$
$$\mathbf{Q}_\infty = \big\{\mathcal{A} = (a_{ijkl}) \mid a_{ijkl} = a_{jikl} = a_{klij} \in L^\infty(\Omega), i, j, k, l = 1, \dots, d\big\}.$$

The spaces V and Q are real Hilbert spaces endowed with the canonical inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

where, for any $\boldsymbol{v} \in V$, $\boldsymbol{\varepsilon}(\boldsymbol{v})$ represents the symmetric part of the gradient of V. The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. The space $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{A}\|_{\mathbf{Q}_\infty} = \max_{0 \le i,j,k,l \le d} \|a_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, the following inequality holds:

$$\|\mathcal{A}\boldsymbol{\tau}\|_Q \le d\|\mathcal{A}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_Q \quad \text{for all } \mathcal{A} \in \mathbf{Q}_\infty, \boldsymbol{\tau} \in Q. \tag{58}$$

We now list the assumption on the data of the contact problem $\mathcal{Q}$. The elasticity tensor $\mathcal{E}$ is symmetric and positively definite, i.e., it satisfies the conditions

$$\begin{cases} \text{(a) } \mathcal{E} \in \mathbf{Q}_\infty. \\ \text{(b) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\ \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^d, \text{a.e. in } \Omega. \end{cases} \tag{59}$$

The relaxation tensor $\mathcal{F}$ and the normal compliance function p satisfy the following conditions:

$$\mathcal{F} \in C(\mathbb{R}_+, \mathbf{Q}_\infty). \tag{60}$$

$$\begin{cases} \text{(a) } p\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) there exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_p |r_1 - r_2| \\ \qquad\quad \text{for all } r_1, r_2 \in \mathbb{R}, \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } p(\cdot, r) \text{ is measurable on } \Gamma_3, \text{for all } r \in \mathbb{R}. \\ \text{(d) } p(\boldsymbol{x}, r) = 0 \text{ if and only if } r \leq 0, \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{61}$$

We also assume that the densities of the body forces and surface tractions, the stiffness coefficient and the gap function are such that

$$\boldsymbol{f}_0 \in C\big(I; L^2(\Omega)^d\big). \tag{62}$$

$$\boldsymbol{f}_2 \in C\big(I; L^2(\Gamma_2)^d\big). \tag{63}$$

$$k \in L^\infty(\Gamma_3), \qquad k(\boldsymbol{x}) \geq 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{64}$$

$$g \in L^2(\Gamma_3), \qquad g(\boldsymbol{x}) \geq 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{65}$$

We now use assumption (59) to endow the space V with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{E}} = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \tag{66}$$

and the associated norm $\|\cdot\|_{\mathcal{E}}$. Then, using inequalities (59)(b) and (58), it follows that

$$\sqrt{m_{\mathcal{E}}} \|\boldsymbol{v}\|_V \leq \|\boldsymbol{v}\|_{\mathcal{E}} \leq \sqrt{d\|\mathcal{C}\|_{\mathbf{Q}_\infty}} \|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V, \tag{67}$$

which shows that the norms $\|\cdot\|_V$ and $\|\cdot\|_{\mathcal{E}}$ are equivalent norms on the space V. Therefore, V is a Hilbert space endowed with the inner product $(\cdot,\cdot)_{\mathcal{E}}$, too. Moreover, it follows from the Sobolev trace argument and the continuity of the embedding $V \subset L^2(\Omega)^d$ that there exist some constants $c_0 > 0$, $d_2 > 0$, and $e_0 > 0$ such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}\|_{\mathcal{E}}, \tag{68}$$

$$\|\boldsymbol{v}\|_{L^2(\Gamma_2)^d} \leq d_0 \|\boldsymbol{v}\|_{\mathcal{E}}, \qquad \|\boldsymbol{v}\|_{L^2(\Omega)^d} \leq e_0 \|\boldsymbol{v}\|_{\mathcal{E}} \tag{69}$$

for all $\boldsymbol{v} \in V$.

Finally, we assume that the following smallness condition holds:

$$c_0^2 L_p \|k\|_{L^\infty(\Gamma_3)} < 1, \tag{70}$$

where c_0 and L_p are the positive constants in (68) and (61), respectively.

We now turn to construct a fixed point weak formulation of Problem $\mathcal{Q}$. Let $\boldsymbol{v} \in V$ and $t \in \mathbb{R}_+$. We use standard arguments based on the Green formula to see that if $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a smooth solution of Problem $\mathcal{Q}$ then

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,d\Gamma - \int_{\Gamma_3} kp\big(u_\nu(t) - g\big)v_\nu\,d\Gamma.$$

We combine this equality with the constitutive law (52) to see that

$$\begin{aligned} &\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \left(\int_0^t \mathcal{F}(t-s)\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\,ds\right) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \\ &\quad = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,d\Gamma - \int_{\Gamma_3} kp\big(u_\nu(t) - g\big)v_\nu\,d\Gamma. \end{aligned} \tag{71}$$

Next, we use the Riesz representation theorem to define the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ by the equality

$$\begin{aligned} \big(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v}\big)_{\mathcal{E}} &= \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,d\Gamma \\ &\qquad = -\int_{\Gamma_3} kp\big(u_\nu(t) - g\big)v_\nu\,d\Gamma - \int_\Omega \left(\int_0^t \mathcal{F}(t-s)\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\,ds\right)\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \end{aligned} \tag{72}$$

for all $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $t \in \mathbb{R}_+$, and $\boldsymbol{v} \in V$. We now combine equalities (66), (71), and (72) to deduce the following fixed point weak formulation of Problem $\mathcal{Q}$ in terms of displacements.

Problem $\mathcal{Q}^V$ Find a displacement field $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ such that $\mathcal{S}\boldsymbol{u}(t) = \boldsymbol{u}(t)$ for any $t \in \mathbb{R}_+$.

The unique solvability of Problem $\mathcal{Q}^V$ is provided by the following existence and uniqueness result.

Theorem 4.1 *Assume* (59)–(65) *and* (70). *Then Problem* $\mathcal{Q}^V$ *has a unique solution* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$.

Proof Let $\boldsymbol{u}, \boldsymbol{v} \in C(\mathbb{R}_+; X)$, $m \in \mathbb{N}$, $t \in [0, m]$, and $\boldsymbol{w} \in W$. We use the definition (72) to see that

$$\begin{aligned} \big|\big(\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t), \boldsymbol{w}\big)_{\mathcal{E}}\big| &\le \int_{\Gamma_3} k\big|p\big(u_\nu(t) - g\big) - p\big(v_\nu(t) - g\big)\big||w_\nu|\,d\Gamma \\ &\qquad + \left\|\int_0^t \mathcal{F}(t-s)\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big) - \boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\big)\right\|_Q ds \cdot \big\|\boldsymbol{\varepsilon}(\boldsymbol{w})\big\|_Q \end{aligned}$$

and, therefore, assumption (61)(b) and inequality (58) yield

$$\big|\big(\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t), \boldsymbol{w}\big)_{\mathcal{E}}\big| \le L_p \|k\|_{L^\infty(\Gamma_3)} \big\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\big\|_{L^2(\Gamma_3)^d} \|\boldsymbol{w}\|_{L^2(\Gamma_3)^d}$$

$$+ d\left(\int_0^t \left\|\mathcal{F}(t-s)\right\|_{\mathbf{Q}_\infty} \left\|\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big) - \boldsymbol{\varepsilon}\big(\boldsymbol{v}(s)\big)\right\|_Q ds\right) \left\|\boldsymbol{\varepsilon}(\boldsymbol{w})\right\|_Q.$$

We now use the trace inequality (68) to deduce that

$$\begin{aligned}\left|\big(\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t), \boldsymbol{w}\big)_{\mathcal{E}}\right| &\le c_0^2 L_p \|k\|_{L^\infty(\Gamma_3)} \left\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\right\|_{\mathcal{E}} \|\boldsymbol{w}\|_{\mathcal{E}} \\ &\quad + d \max_{r\in[0,m]} \left\|\mathcal{F}(r)\right\|_{\mathbf{Q}_\infty} \left(\int_0^t \left\|\boldsymbol{u}(s)\right) - \boldsymbol{v}(s)\right\|_V ds)\|\boldsymbol{w}\|_V.\end{aligned}$$

Finally, we use (67), then we take $\boldsymbol{w} = \mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}(t)$ in the resulting inequality to find that

$$\begin{aligned}\left\|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t)\right\|_{\mathcal{E}} &\le c_0^2 L_p \|k\|_{L^\infty(\Gamma_3)} \left\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\right\|_{\mathcal{E}} \\ &\quad + \frac{d}{m_{\mathcal{E}}} \max_{r\in[0,m]} \left\|\mathcal{F}(r)\right\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}(s)) - \boldsymbol{v}(s)\|_{\mathcal{E}}\, ds.\end{aligned} \tag{73}$$

Inequality (73) combined with the smallness assumption (70) shows that the operator (72) is an a.h.d. operator. Theorem 4.1 is now a direct consequence of Theorem 2.2. □

We now prove a continuous dependence result of the solution with respect to the data. To this end we consider the sequences $\{\mathcal{F}_n\}$, $\{\boldsymbol{f}_{0n}\}$, $\{\boldsymbol{f}_{2n}\}$, $\{k_n\}$ such that, for each $n \in \mathbb{N}$, the following conditions hold:

$$\mathcal{F}_n \in C(\mathbb{R}_+, \mathbf{Q}_\infty), \tag{74}$$

$$\boldsymbol{f}_{0n} \in C\big(I; L^2(\Omega)^d\big), \tag{75}$$

$$\boldsymbol{f}_{2n} \in C\big(I; L^2(\Gamma_2)^d\big), \tag{76}$$

$$k_n \in L^\infty(\Gamma_3), \qquad k_n(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{77}$$

$$c_0^2 L_p \|k_n\|_{L^\infty(\Gamma_3)} < 1. \tag{78}$$

With these data we define the operator $\mathcal{S}_n : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ by the equality

$$\begin{aligned}\big(\mathcal{S}_n\boldsymbol{u}(t), \boldsymbol{v}\big)_{\mathcal{E}} &= \int_\Omega \boldsymbol{f}_{0n}(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_{2n}(t)\cdot\boldsymbol{v}\,d\Gamma \\ &= -\int_{\Gamma_3} k_n p\big(u_\nu(t) - g\big) v_\nu\, d\Gamma - \int_\Omega \left(\int_0^t \mathcal{F}_n(t-s)\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\,ds\right)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx\end{aligned} \tag{79}$$

for all $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $t \in \mathbb{R}_+$, and $\boldsymbol{v} \in V$. Then we consider the following variational problem.

Problem $\mathcal{Q}_n^V$ Find a displacement field $\boldsymbol{u}_n \in C(\mathbb{R}_+; V)$ such that $\mathcal{S}_n\boldsymbol{u}_n(t) = \boldsymbol{u}_n(t)$ for any $t \in \mathbb{R}_+$.

Using Theorem 4.1 it follows that Problem $\mathcal{Q}_n$ has a unique solution for each $n \in \mathbb{N}$. Consider now the following assumptions:

$$\mathcal{F}_n \to \mathcal{F} \quad \text{in } C(\mathbb{R}_+, \mathbf{Q}_\infty) \text{ as } n \to \infty, \tag{80}$$

$$\boldsymbol{f}_{0n} \to \boldsymbol{f}_0 \quad \text{in } C\big(I; L^2(\Omega)^d\big) \text{ as } n \to \infty, \tag{81}$$

$$\boldsymbol{f}_{2n} \to \boldsymbol{f}_2 \quad \text{in } C\big(I; L^2(\Gamma_3)^d\big) \text{ as } n \to \infty, \tag{82}$$

$$k_n \to k \quad \text{in } L^\infty(\Gamma_3) \text{ as } n \to \infty. \tag{83}$$

We have the following convergence result.

Theorem 4.2 *Assume* (59)–(65), (70), (74)–(78), *and* (80)–(83). *Then the solution* $\boldsymbol{u}_n$ *of Problem* $\mathcal{Q}_n^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{Q}^V$, *i.e.*,

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \textit{in } C(\mathbb{R}_+, V) \textit{ as } n \to \infty. \tag{84}$$

Proof Let $n, m \in \mathbb{N}$, $t \in [0, m]$, and $\boldsymbol{v} \in C(\mathbb{R}_+; V)$. Then a simple calculation based on definitions (72), (79), inequalities (69), and arguments similar to those used in the proof of inequality (73) imply that there exists a constant $C_0 > 0$ which does not depend on m, n, and t, such that

$$\begin{aligned}
&\big\|\mathcal{S}_n\boldsymbol{v}(t) - \mathcal{S}\boldsymbol{v}(t)\big\|_{\mathcal{E}} \\
&\quad \le C_0\Big(\max_{r\in[0,m]}\big\|\boldsymbol{f}_{0n}(r) - \boldsymbol{f}_0(r)\big\|_{L^2(\Omega)^d} + \max_{r\in[0,m]}\big\|\boldsymbol{f}_{2n}(r) - \boldsymbol{f}_2(r)\big\|_{L^2(\Gamma_2)^d}\Big) \\
&\qquad + c_0^2 L_p\|k_n - k\|_{L^\infty(\Gamma_3)}\big\|\boldsymbol{v}(t)\big\|_{\mathcal{E}} \\
&\qquad + \frac{d}{m_{\mathcal{E}}}\max_{r\in[0,m]}\big\|\mathcal{F}_n(r) - \mathcal{F}(r)\big\|_{\mathbf{Q}_\infty}\int_0^t\big\|\boldsymbol{v}(s)\big\|_{\mathcal{E}}\,ds.
\end{aligned} \tag{85}$$

Denote

$$\alpha_n^m = \max_{r\in[0,m]}\big\|\boldsymbol{f}_{0n}(r) - \boldsymbol{f}_0(r)\big\|_{L^2(\Omega)^d}, \tag{86}$$

$$\beta_n^m = \max_{r\in[0,m]}\big\|\boldsymbol{f}_{2n}(r) - \boldsymbol{f}_2(r)\big\|_{L^2(\Gamma_2)^d}, \tag{87}$$

$$\gamma_n^m = \max_{r\in[0,m]}\big\|\mathcal{F}_n(r) - \mathcal{F}(r)\big\|_{\mathbf{Q}_\infty}, \tag{88}$$

$$\theta_n^m = \max\Big\{C_0\big(\alpha_n^m + \beta_n^m\big), c_0^2 L_p\|k_n - k\|_{L^\infty(\Gamma_3)}, \frac{d}{m_{\mathcal{E}}}\gamma_n^m\Big\}. \tag{89}$$

Then inequality (85) shows that

$$\big\|\mathcal{S}_n\boldsymbol{v}(t) - \mathcal{S}\boldsymbol{v}(t)\big\|_{\mathcal{E}} \le \theta_n^m\Big(1 + \big\|\boldsymbol{v}(t)\big\|_{\mathcal{E}} + \int_0^t\big\|\boldsymbol{v}(s)\big\|_{\mathcal{E}}\,ds\Big). \tag{90}$$

Moreover, assumptions (80)–(83) and notation (86)–(89) imply that

$$\theta_n^m \to 0 \quad \text{as } n \to \infty \text{ for each } m \in \mathbb{N}. \tag{91}$$

It follows now from (90) and (91) that condition (38) is satisfied. Theorem 4.2 is now a direct consequence of Theorem 3.1(c). □

We end this section with the following comments.

(1) We refer to the solution $\boldsymbol{u}$ of Problem $\mathcal{Q}^V$ as the weak solution for the contact Problem $\mathcal{Q}$. We conclude from the above that Theorem 4.1 provides the unique weak solvability of this contact problem and Theorem 4.2 shows the continuous dependence of the weak solution with respect to the data.

(2) In addition to the mathematical interest in convergence (84), it is important from the mechanical point of view since it shows that small perturbation on the relaxation tensor $\mathcal{F}$, the density of body forces $\boldsymbol{f}_0$, the density of traction forces $\boldsymbol{f}_2$, and the stiffness coefficient k lead to small perturbation of the weak solution of the contact problem $\mathcal{Q}$.

5 A contact problem with unilateral constraints

For the contact problem we consider in this section, we keep the physical setting and the notation in Sect. 4. Nevertheless, we now assume that the foundation is made by a rigid obstacle covered by a layer of deformable material, say asperities. Moreover, there is no gap between the body and the foundation, and we model the material's behavior with a different viscoelastic constitutive law. The classical formulation of the problem is the following.

Problem $\mathcal{M}$ Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\varepsilon\big(\dot{\boldsymbol{u}}(t)\big) + \beta\big(\boldsymbol{\sigma}(t) - \mathcal{G}\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big)\big) \quad \text{in } \Omega, \tag{92}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{93}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{94}$$

$$\boldsymbol{\sigma}(t) \cdot \boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{95}$$

$$\left.\begin{array}{l} u_\nu(t) \le g, \sigma_\nu(t) + p(u_\nu(t)) \le 0, \\ (u_\nu(t) - g)(\sigma_\nu(t) + p(u_\nu(t)) = 0 \end{array}\right\} \quad \text{on } \Gamma_3, \tag{96}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on } \Gamma_3 \tag{97}$$

for any $t \in \mathbb{R}_+$ and, moreover,

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{98}$$

The equations and boundary conditions in the statement of Problem $\mathcal{M}$ have a similar meaning as those used in the statement of Problem $\mathcal{Q}$. The novelty arises in the fact that now we use the constitutive law (92), in which β represents a viscosity coefficient, $\mathcal{G}$ is a nonlinear relaxation operator, and the dot above denotes the derivative with respect to the time. In addition, we use a contact condition with unilateral constraint (96). Here g represents the thickness of the deformable layer. Finally, note that the functions $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ in (98) are the initial displacement and the initial stress, respectively. Details on similar viscoelastic and viscoplastic contact models can be found in [20, 21].

Besides the assumptions on $\mathcal{E}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, p, and g already listed in Sect. 4, we consider the following assumptions.

$$\begin{cases} \text{(a) } \mathcal{G}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{G}}>0 \text{ such that} \\ \qquad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\tau}_1)-\mathcal{G}(\boldsymbol{x},\boldsymbol{\tau}_2)\|\le L_{\mathcal{G}}(\|\boldsymbol{\tau}_1-\boldsymbol{\tau}_2\|) \\ \qquad\quad \text{for all } \boldsymbol{\tau}_1,\boldsymbol{\tau}_2\in\mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c) } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{\tau}) \text{ is measurable on } \Omega \text{ for any} \boldsymbol{\tau}\in\mathbb{S}^d. \\ \text{(d) } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{0})\in Q. \end{cases} \tag{99}$$

$$\beta\in L^\infty(\Omega),\qquad \beta(\boldsymbol{x})\ge 0\quad \text{a.e. } \boldsymbol{x}\in\Omega. \tag{100}$$

$$\big(p(\boldsymbol{x},r_1)-p_\nu(\boldsymbol{x},r_2)\big)(r_1-r_2)\ge 0\quad \text{for all } r_1,r_2\in\mathbb{R}, \text{a.e. } \boldsymbol{x}\in\Gamma_3. \tag{101}$$

$$\boldsymbol{\sigma}_0\in Q,\qquad \boldsymbol{u}_0\in V. \tag{102}$$

We now turn to construct a weak formulation of the problem. To that end, we consider the set $K\subset V$, the operators $A:V\to V$, $\Lambda:C(\mathbb{R}_+;Q)\to C(\mathbb{R}_+;Q)$, and the function $\boldsymbol{f}:\mathbb{R}_+\to V$ defined by

$$K=\{\boldsymbol{v}\in V: v_\nu\le g \text{a.e. on} \Gamma_3\}, \tag{103}$$

$$(A\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx+\int_{\Gamma_3}p(u_\nu)v_\nu\,d\Gamma\quad \forall\boldsymbol{u},\boldsymbol{v}\in V, \tag{104}$$

$$\Lambda(\boldsymbol{\sigma},\boldsymbol{\tau})(t)=\int_0^t\beta\big(\boldsymbol{\sigma}(s)-\mathcal{G}\big(\boldsymbol{\tau}(s)\big)\big)\,ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{105}$$
$$\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in C(\mathbb{R}_+;Q), t\in\mathbb{R}_+,$$

$$\big(\boldsymbol{f}(t),\boldsymbol{v}\big)_V=\int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx+\int_{\Gamma_3}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,d\Gamma\quad \forall\boldsymbol{v}\in V, t\in\mathbb{R}_+. \tag{106}$$

Assume now that $(\boldsymbol{u},\boldsymbol{\sigma})$ is a smooth solution of Problem $\mathcal{M}$. Let $\boldsymbol{v}\in K$, $t\in\mathbb{R}_+$ and note that

$$\boldsymbol{u}(t)\in K. \tag{107}$$

We integrate the constitutive law (92) with initial conditions (98) to find that

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)+\int_0^t\beta(\boldsymbol{\sigma}(s)-\mathcal{G}\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\big)\,ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{108}$$

Moreover, we use standard arguments based on the Green formula to see that

$$\int_\Omega\boldsymbol{\sigma}(t)\cdot\big(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big)\,dx+\int_{\Gamma_3}p(u_\nu(t)(v_\nu-u_\nu)\,d\Gamma \tag{109}$$
$$\ge\int_\Omega\boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u})\,dx+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u})\,d\Gamma.$$

Let

$$\boldsymbol{\eta}(t) = \int_0^t \beta\big(\boldsymbol{\sigma}(s) - \mathcal{G}\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big)\big)\big)\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{110}$$

and note that $\boldsymbol{\eta}(t)$ represents the anelastic part of the stress field at the moment t. We now combine relations (107)–(110), then use definitions (103)–(106) to deduce the following weak formulation of Problem $\mathcal{M}$.

Problem $\mathcal{M}^V$ Find a displacement field $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, a stress field $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$, and an anelastic stress field $\boldsymbol{\eta} \in C(\mathbb{R}_+; Q)$ such that, for any $t \in \mathbb{R}_+$, the following hold:

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \boldsymbol{\eta}(t), \tag{111}$$

$$\begin{aligned} &\boldsymbol{u}(t) \in K, \quad \big(A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\big)_V + \big(\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big)_Q \\ &\qquad \geq \big(\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\big)_V \quad \forall \boldsymbol{v} \in K, \end{aligned} \tag{112}$$

$$\boldsymbol{\eta}(t) = \Lambda\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big). \tag{113}$$

We now state the following existence and uniqueness result.

Theorem 5.1 *Assume* (59), (61)–(63), (65), (99)–(102). *Then Problem* $\mathcal{M}^V$ *has a unique solution.*

Proof The proof of Theorem 5.1 is carried out in four steps, based on a fixed point argument. The steps are as follows.

(i) *An intermediate stress-displacement problem.* We claim that for each $\boldsymbol{\eta} \in C(\mathbb{R}_+; Q)$ there exists a unique couple of functions $(\boldsymbol{\sigma}_\eta, \boldsymbol{u}_\eta)$ such that $\boldsymbol{u}_\eta \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma}_\eta \in C(\mathbb{R}_+; Q)$ and

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_\eta(t)\big) + \boldsymbol{\eta}(t), \tag{114}$$

$$\begin{aligned} &\boldsymbol{u}_\eta(t) \in K, \quad \big(A\boldsymbol{u}_\eta(t), \boldsymbol{v} - \boldsymbol{u}_\eta(t)\big)_V + \big(\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\eta(t)\big)\big)_Q \\ &\qquad \geq \big(\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\eta(t)\big)_V \quad \forall \boldsymbol{v} \in K \end{aligned} \tag{115}$$

for and $t \in \mathbb{R}_+$.

Indeed, let $\boldsymbol{\eta} \in C(\mathbb{R}_+; Q)$. We use assumptions (59), (61), and (101) to see that the operator $A : V \to V$ defined by (104) is strongly monotone and Lipschitz continuous. Moreover, assumption (65) implies that the set K is a closed nonempty convex subset of V and regularities (62), (63) guarantee that $\boldsymbol{f} \in C(\mathbb{R}_+; V)$. Then the existence of a unique function $\boldsymbol{u}_\eta \in C(\mathbb{R}_+; V)$ which solves (115) follows from standard arguments on time-dependent elliptic variational inequalities. We now use equality (114) to obtain the existence and uniqueness part in Lemma 2.11.

(ii) *A Lipschitz continuous dependence.* We claim that if $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{\eta_i}$ and $\boldsymbol{u}_i = \boldsymbol{u}_{\eta_i}$ for $\boldsymbol{\eta}_i \in C(\mathbb{R}_+; Q)$, $i = 1, 2$, then there exists a constant $C_0 > 0$ such that

$$\big\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\big\|_Q + \big\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\big\|_V \leq C_0\big\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\big\|_Q \quad \forall t \in \mathbb{R}_+. \tag{116}$$

Estimate (116) follows from standard arguments applied to system (114), (115) and, therefore, we skip its proof.

(iii) *A fixed point problem for the anelastic stress field.* Note that step (i) allows us to introduce the operator $\mathcal{S}: C(\mathbb{R}_+;Q) \to C(\mathbb{R}_+;Q)$ defined by

$$\mathcal{S}\boldsymbol{\eta}(t) = \Lambda\big(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\eta(t)\big)\big) \quad \forall \boldsymbol{\eta} \in C(\mathbb{R}_+;Q), t \in \mathbb{R}_+. \tag{117}$$

We now consider the intermediate fixed point problem of finding an anelastic stress field $\boldsymbol{\eta} \in C(\mathbb{R}_+;Q)$ such that

$$\mathcal{S}\boldsymbol{\eta}(t) = \boldsymbol{\eta}(t) \quad \forall t \in \mathbb{R}_+ \tag{118}$$

and claim that this problem has a unique solution $\boldsymbol{\eta}^* \in C(\mathbb{R}_+;Q)$.

For the proof we consider two elements $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(\mathbb{R}_+;X)$. Let $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{\eta_i}$, $\boldsymbol{u}_i = \boldsymbol{u}_{\eta_i}$ for $i = 1,2$ and let $t \in \mathbb{R}_+$. We use equalities (117), (105) and assumptions (99), (100) to see that

$$\begin{aligned} &\big\|\mathcal{S}\boldsymbol{\eta}_1(t) - \mathcal{S}\boldsymbol{\eta}_2(t)\big\|_Q \\ &\quad \le \|\beta\|_{L^\infty(\Omega)} \int_0^t \big(\big\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\big\|_Q + L_G\big\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\big\|_V\big)\, ds. \end{aligned} \tag{119}$$

It follows now from inequality (116) that $\mathcal{S}$ is a history-dependent operator. We now end the proof of this step by using Theorem 2.2.

(iv) *End of proof.* It follows from (118), (117) that

$$\boldsymbol{\eta}^*(t) = \Lambda\big(\boldsymbol{\sigma}_{\eta^*}(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\eta^*}(t)\big)\big) \quad \forall t \in \mathbb{R}_+. \tag{120}$$

We now write (114) and (115) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and then we combine the resulting relations with (120) to see that the triple $(\boldsymbol{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}\ \boldsymbol{\eta}^*)$ represents a solution of Problem $\mathcal{M}^V$. This proves the existence part in Theorem 5.1. The uniqueness part results from the uniqueness of the fixed point of the history-dependent operator (117) guaranteed by Theorem 2.2. □

We now prove the continuous dependence of the solution with respect to part of the data. To this end we consider the sequences $\{\beta_n\}$, $\{\boldsymbol{\sigma}_{0n}\}$, $\{\boldsymbol{u}_{0n}\}$ such that, for each $n \in \mathbb{N}$, the following conditions hold:

$$\beta_n \in L^\infty(\Omega), \qquad \beta_n(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Omega, \tag{121}$$

$$\boldsymbol{\sigma}_{0n} \in Q, \qquad \boldsymbol{u}_{0n} \in V. \tag{122}$$

With these data we define the operator $\Lambda_n : C(\mathbb{R}_+;Q) \to C(\mathbb{R}_+;Q)$ by equality

$$\begin{aligned} &\Lambda_n(\boldsymbol{\sigma}, \boldsymbol{\tau})(t) = \int_0^t \beta_n\big(\boldsymbol{\sigma}(s) - \mathcal{G}\big(\boldsymbol{\tau}(s)\big)\big)\, ds + \boldsymbol{\sigma}_{0n} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0n}) \\ &\forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in C(\mathbb{R}_+;Q), t \in \mathbb{R}_+. \end{aligned} \tag{123}$$

Then, we consider the following variational problem.

Problem $\mathcal{M}_n^V$ Find a displacement field $\boldsymbol{u}_n \in C(\mathbb{R}_+;V)$, a stress field $\boldsymbol{\sigma}_n \in C(\mathbb{R}_+;Q)$, and an anelastic stress field $\boldsymbol{\eta}_n \in C(\mathbb{R}_+;Q)$ such that, for any $t \in \mathbb{R}_+$, the following hold:

$$\boldsymbol{\sigma}_n(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_n(t)\big) + \boldsymbol{\eta}_n(t), \tag{124}$$

$$\begin{aligned}\boldsymbol{u}_n(t) \in K, \quad &\big(A\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\big)_V + \big(\boldsymbol{\eta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}\big(\boldsymbol{u}_n(t)\big)\big)_Q \\ &\geq \big(\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\big)_V \quad \forall \boldsymbol{v} \in K,\end{aligned} \tag{125}$$

$$\boldsymbol{\eta}_n(t) = \Lambda\big(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}_n(t)\big)\big). \tag{126}$$

Using Theorem 5.1 it follows that Problem $\mathcal{M}_n$ has a unique solution for each $n \in \mathbb{N}$. Consider now the following additional assumptions:

$$\beta_n \to \beta \quad \text{in } L^\infty(\Omega) \text{ as } n \to \infty, \tag{127}$$

$$\boldsymbol{\sigma}_{0n} \to \boldsymbol{\sigma}_0 \quad \text{in } Q, \qquad \boldsymbol{u}_{0n} \to \boldsymbol{u}_0 \quad \text{in } V \text{ as } n \to \infty. \tag{128}$$

We have the following convergence result.

Theorem 5.2 *Assume* (59), (61)–(63), (65), (99)–(102), (121), (122), (127), *and* (128). *Then the solution* $(\boldsymbol{u}_n, \boldsymbol{\sigma}_n, \boldsymbol{\eta}_n)$ *of Problem* $\mathcal{M}_n^V$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\eta})$ *of Problem* $\mathcal{M}^V$, *i.e.*,

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \textit{in } C(\mathbb{R}_+, V), \textit{as } n \to \infty. \tag{129}$$

$$\boldsymbol{\sigma}_n \to \boldsymbol{\sigma} \quad \textit{in } C(\mathbb{R}_+, Q), \textit{as } n \to \infty. \tag{130}$$

$$\boldsymbol{\eta}_n \to \boldsymbol{\eta} \quad \textit{in } C(\mathbb{R}_+, Q), \textit{as } n \to \infty. \tag{131}$$

Proof Let $n \in \mathbb{N}$ and let $\mathcal{S}_n : C(\mathbb{R}_+;Q) \to C(\mathbb{R}_+;Q)$ defined by

$$\mathcal{S}_n\boldsymbol{\eta}(t) = \Lambda_n\big(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}_\eta(t)\big)\big) \quad \forall \boldsymbol{\eta} \in C(\mathbb{R}_+;Q), t \in \mathbb{R}_+. \tag{132}$$

Recall that here and below we use the notation $(\boldsymbol{\sigma}_\eta, \boldsymbol{u}_\eta)$ for the solution of problem (114), (115).

Let $\boldsymbol{\eta} \in C(\mathbb{R}_+;V)$, $m \in \mathbb{N}$, and $t \in [0,m]$. Then a simple calculation based on definitions (132), (117), (123), (105) combined with bound (58) and arguments similar to those used in the proof of inequality (119) shows that

$$\begin{aligned}&\big\|\mathcal{S}_n\boldsymbol{\eta}(t) - \mathcal{S}\boldsymbol{\eta}(t)\big\|_Q \\ &\quad\leq \|\beta_n - \beta\|_{L^\infty(\Omega)} \int_0^t \big(\big\|\boldsymbol{\sigma}_\eta(s)\big\|_Q + \big\|\mathcal{G}\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}_\eta(s)\big)\big)\big\|_Q\big)\,ds \\ &\qquad + \|\boldsymbol{\sigma}_{0n} - \boldsymbol{\sigma}_0\|_Q + d\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{u}_{0n} - \boldsymbol{u}_0\|_V).\end{aligned} \tag{133}$$

Denote

$$\varepsilon_n = \max\big\{\|\beta_n - \beta\|_{L^\infty(\Omega)}, \|\boldsymbol{\sigma}_{0n} - \boldsymbol{\sigma}_0\|_Q + d\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{u}_{0n} - \boldsymbol{u}_0\|_V\big\}. \tag{134}$$

Then inequality (133) shows that

$$\|\mathcal{S}_n\boldsymbol{\eta}(t)-\mathcal{S}\boldsymbol{\eta}(t)\|_Q \le \varepsilon_n\Big(1+\int_0^t \big(\|\boldsymbol{\sigma}_\eta(s)\|_Q+\|\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\|_Q\big)\,ds\Big). \tag{135}$$

Next, let $(\boldsymbol{u}^0,\boldsymbol{\sigma}^0)$ denote the solution obtained in Step i) for $\boldsymbol{\eta}=\mathbf{0}_Q$. Then, using assumption (99) and inequality (116), we find that

$$\begin{aligned}
&\|\boldsymbol{\sigma}_\eta(s)\|_Q+\|\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\|_Q \\
&\quad\le \|\boldsymbol{\sigma}_\eta(s)-\boldsymbol{\sigma}^0(s)\|_Q+\|\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))-\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}^0(s)))\|_Q \\
&\qquad+\|\boldsymbol{\sigma}^0(s)\|_Q+\|\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}^0(s)))\|_Q \\
&\quad\le C_0\max\{1,L_G\}\|\boldsymbol{\eta}(s)\|_Q+\|\boldsymbol{\sigma}^0(s)\|_Q+\|\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}^0(s)))\|_Q \quad \forall s\in[0,t].
\end{aligned} \tag{136}$$

Let

$$B_m=\max_{s\in[0,m]}\big(\|\boldsymbol{\sigma}^0(s)\|_Q+\|\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}^0(s)))\|_Q\big). \tag{137}$$

Then, combining (135), (136), and (137), we deduce that

$$\|\mathcal{S}_n\boldsymbol{\eta}(t)-\mathcal{S}\boldsymbol{\eta}(t)\|_Q \le \varepsilon_n\Big(1+C_0\max\{1,L_G\}\int_0^t\|\boldsymbol{\eta}(s)\|_Q\,ds+mB_m\Big)$$

and using notation

$$\theta_n^m=\varepsilon_n\big(1+C_0\max\{1,L_G\},mB_m\big) \tag{138}$$

we find that

$$\|\mathcal{S}_n\boldsymbol{\eta}(t)-\mathcal{S}\boldsymbol{\eta}(t)\|_Q \le \theta_n^m\Big(1+\int_0^t\|\boldsymbol{\eta}(s)\|_Q\,ds\Big). \tag{139}$$

Moreover, assumptions (127), (128) and notation (134), (138) show that

$$\theta_n^m\to 0 \quad \text{as } n\to\infty. \tag{140}$$

It follows now from (139) and (140) that condition (38) is satisfied. We now use Theorem 3.1(c) to deduce convergence (131). Finally, inequalities (116) and (131) imply convergences (129) and (130), which concludes the proof. □

We end this section with the following comments.

(1) The solution of Problem $\mathcal{M}_V$ is called a weak solution for the contact Problem $\mathcal{M}$. We conclude from the above that Theorem 5.1 provides the unique weak solvability of this contact problem and Theorem 5.2 shows the continuous dependence of the weak solution with respect to part of data.

(2) In addition to the mathematical interest in convergences (129), (130), they are important from the mechanical point of view since they show that small perturbations on the viscosity coefficient β, the initial stress field $\boldsymbol{\sigma}_0$, and the initial displacement field $\boldsymbol{u}_0$ lead to small perturbations of the weak solution of the contact problem $\mathcal{M}$.

6 Conclusions

We considered a fixed point problem governed by an almost history-dependent operator $\mathcal{S}$ for which we recalled an existence and uniqueness result. Then, we proved its well-posedness with respect to four different Tykhonov triples, the so called $\mathcal{T}$-well posedness. This property led us to elaborate a strategy in order to obtain various stability results under different assumptions on the perturbations of $\mathcal{S}$. The choice of the Tykhonov triples plays a crucial role in using this strategy. Moreover, we presented two applications in the study of boundary value problems arising in contact mechanics and provided the mechanical interpretation of the corresponding convergence results. In this way we illustrated that the fixed point arguments can be successfully used in the variational analysis of mathematical models of contact.

The method used in Sects. 4 and 5 of this paper consists in associating to a boundary value problem an intermediate fixed point problem of the form (2), and in carrying out its analysis by using this fixed point formulation. This method can be extended in the study of a large number of problems, including problems formulated in abstract spaces. In particular, it can be used in the study of various classes of history-dependent variational and hemivariational inequalities, and differential hemivariational inequalities, as well. Consequently, it allows to obtain sensitivity analysis results for the corresponding inequality problems. This, in turn, opens the possibility to use the abstract results in this paper in the study of various mathematical models of contact.

Besides the novelty of the results we presented here, we illustrated the use of new mathematical tools in the variational analysis of viscoelastic contact problems with or without unilateral constraints. This proves, once more, that one of the main features of the mathematical theory of contact mechanics is the cross fertilization between the models and applications, on one hand, and the nonlinear functional analysis, on the other hand.

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