Ali and Adam *Fixed Point Theory Algorithms Sci Eng* (2022) 2022:9
https://doi.org/10.1186/s13663-022-00719-6

Fixed Point Theory and Algorithms for Sciences and Engineering

RESEARCH **Open Access**

An inertial s-iteration process for a common fixed point of a family of quasi-Bregman nonexpansive mappings

Bashir Ali[1*] and Aisha A. Adam[2]

*Correspondence:
bashiralik@yahoo.com
[1]Department of Mathematical Sciences, Bayero University, Kano, Nigeria
Full list of author information is available at the end of the article

Abstract

In this paper, an inertial S-iteration iterative process for approximating a common fixed point of a finite family of quasi-Bregman nonexpansive mappings is introduced and studied in a reflexive Banach space. A strong convergence theorem is proved. Some applications of the theorem are presented. The results presented here improve, extend, and generalize some recent results in the literature.

MSC: 47H09; 47J25

Keywords: Innertial iterative process; S-iterative process; Quasi-Bregman nonexpansive mapping; Fixed point

1 Introduction

Let E be a real reflexive Banach space with dual space E^*. Throughout this paper we shall assume that $f : E \to (-\infty, +\infty)$ is a proper, lower semicontinuous, and convex function. We denote by $\operatorname{dom} f := \{x \in E : f(x) < +\infty\}$, the domain of f. Let $x \in \operatorname{int} \operatorname{dom} f$, then the subdifferential of f at x is the convex function defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in E\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

It is known that the Young–Fenchel inequality,

$$\langle x^*, x\rangle \leq f(x) + f(x^*), \quad \forall x \in E, x^* \in E^*,$$

holds. A function f is coercive [12] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

A function f is said to be strongly coercive if

$$\lim_{\|x\|\to\infty}\frac{f(x)}{\|x\|}=+\infty.$$

For any $x\in \operatorname{int}\operatorname{dom}f$ and $y\in E$, the derivative of f at x in the direction of y is defined by

$$f^0(x,y)=\lim_{t\to 0}\frac{f(x+ty)-f(x)}{t}. \tag{1.1}$$

The function f is said to be Gâteaux differentiable at x if the limit (1.1) exists for any y. In this case, the gradient of f at x is the function $\nabla f(x): E\to(-\infty,+\infty]$ defined by $\langle\nabla f(x),y\rangle=f^0(x,y)$ for any $y\in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at every point $x\in \operatorname{int}\operatorname{dom}f$. Furthermore, f is said to be Fréchet differentiable at x if this limit (1.1) is attained uniformly in y, $\|y\|=1$; f is said to be uniformly Fréchet differentiable on a subset C of E if the limit (1.1) is attained uniformly for $x\in C$ and $\|y\|=1$. It is well known that if f is Gâteaux differentiable (respectively Fréchet differentiable) on $\operatorname{int}\operatorname{dom}f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (respectively continuous) on $\operatorname{int}\operatorname{dom}f$, see, for example, [2, 3, 6]. Let $f:E\to(-\infty,+\infty)$ be a convex and Gâteaux differentiable function. The Bregman distance with respect to f, $D_f:\operatorname{dom}f\times\operatorname{int}\operatorname{dom}f\to[0,+\infty)$ is defined as

$$D_f(x,y)=f(x)-f(y)-\langle\nabla f(y),x-y\rangle.$$

Let C be a nonempty closed and convex subset of E. Let $T:C\to E$ be a mapping, then

- A point $v\in C$ is said to be an asymptotic fixed point of T if for any sequence $\{x_n\}\subset C$ which converges weakly to v, $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$;
- T is said to be Bregman relatively nonexpansive if $F(T)\neq\emptyset, F(T)=\hat{F}(T)$, and $D_f(x,Ty)\le D_f(x,y)$ for any $x\in C, y\in F(T)$;
- T is said to be quasi-Bregman nonexpansive if $F(T)\neq\emptyset$ and $D_f(x,Ty)\le D_f(x,y)$ for any $x\in C, y\in F(T)$;
- $(I-T)$ is demiclosed at $y\in E$ if having a sequence $\{v_n\}$ in C converging weakly to u and $\{v_n-Tv_n\}$ converging strongly to y implies that $(I-T)u=y$ where I is the identity mapping. From this we get that $(I-T)$ is demiclosed at zero if whenever a sequence $\{v_n\}$ in C converges weakly to u and $\{v_n-Tv_n\}$ converges strongly to 0 then $u\in F(T)$.

Agarwal et al. [1] introduced and studied a two-step iterative process called the S-iteration process. They proved a convergence theorem for fixed points of nearly asymptotically nonexpansive mappings. Since then various modifications of the S-iteration scheme and also multistep schemes were studied by many authors for solutions of some nonlinear problems, see, for example, [10, 11, 15] and the references therein.

Suparatulatorn et al. [24] introduced and studied an iteration method called modified S-iteration process which is defined by

$$\begin{cases} x_0\in C;\\ y_n=(1-\beta_n)x_n+\beta_n S_1x_n;\\ x_{n+1}=(1-\alpha_n)S_1x_n+\alpha_n S_2y_n,\end{cases}$$

where C is a nonempty closed convex subset of a real Banach space, S_1, S_2 are G-nonexpansive mappings, and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. They proved that the sequence generated by the iterative algorithm converges weakly to a common fixed point of two G-nonexpansive mappings in a uniformly convex Banach space.

Recently, Phon-on et al. [17] studied the following inertial modified S-iteration process by combining the inertial extrapolation and modified S-iteration process to speed up the convergence of the modified S-iteration process:

$$\begin{cases} w_n = x_n + \gamma_n(x_n - x_{n-1}); \\ y_n = (1-\beta_n)w_n + \beta_n S_1 w_n; \\ x_{n+1} = (1-\alpha_n)S_1 w_n + \alpha_n S_2 y_n, \end{cases}$$

$n \geq 1$, where S_1, S_2 are nonexpansive mappings, $\{S_i w_n - w_n\}$ bounded for $i = 1,2$, $\{S_i w_n - y\}$ is bounded for $i = 1,2$, and for any $y \in F(S_1) \cap F(S_2)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\{\gamma_n\} \subset [0,\gamma]$, $0 \leq \gamma < 1$, $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 0.5)$.

They proved, under some assumptions, that the sequence generated by the algorithm converges weakly to a common fixed point of two nonexpansive mappings in a uniformly convex Banach space. Several inertial algorithms were studied by numerous authors to speed up the convergence processes of iterative schemes, see, for example, [13, 18–20] and the references contained therein.

Motivated by the results of Phon-on et al. [17] and Suparatulatorn et al. [24], we raised the following interesting questions:

1. Can one iteratively approximate solutions of inertial modified S-iteration process in real Banach spaces more general than uniformly convex spaces?
2. Can the result also be proved for a common fixed point of a finite family of quasi-Bregman nonexpansive mappings?
3. Can a strong convergence theorem be proved without assuming that the operator is semicompact?

In this paper, we answer the questions in the affirmative. We introduce and study the following algorithm:

$$\begin{cases} x_0, x_1 \in C, \quad C = C_1; \\ w_n = x_n + \gamma_n(x_n - x_{n-1}); \\ y_{1n} = \nabla f^*(\beta_n \nabla f w_n + (1-\beta_n)\nabla f S_1 w_n); \\ y_{in} = \nabla f^*(\beta_n \nabla f S_{i-1} w_n + (1-\beta_n)\nabla f S_i y_{(i-1)n}), \quad 2 \leq i \leq m; \\ C_{in} = \{v \in C_n : D_f(v, y_{in}) \leq D_f(v, w_n)\}; \\ C_{n+1} = \bigcap_{i=1}^{m} C_{in}; \\ x_{n+1} = \Pi_{C_{n+1}}{}^{f} x_0, \end{cases} \tag{1.2}$$

where C is a nonempty, closed, and convex subset of a reflexive Banach space E, for some natural number $m \geq 2$, $\{S_i\}_{i=1}^{m}$ is a finite family of quasi-Bregman nonexpansive self-mappings of C, $\{\gamma_n\}, \{\beta_n\} \subset (a,b)$ are sequences such that $0 < a < b < 1$. Then we prove that the sequence generated by the algorithm (1.2) converges to a common fixed point of a finite family of quasi-Bregman nonexpansive mappings. Furthermore, we apply our theorem to solution of some equilibrium problem and zeros of some maximal monotone operators.

2 Preliminaries

Let $f : E \to (-\infty, +\infty)$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in \operatorname{int}\operatorname{dom} f$ is the function $\nu_f(x,\cdot) : [0,+\infty) \to [0,+\infty)$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f, \|y-x\| = t\}.$$

The function f is called totally convex at x if $\nu_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at every point $x \in \operatorname{int}\operatorname{dom} f$ and is said to be totally convex on bounded subsets if $\nu_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0,+\infty) \to [0,+\infty)$ defined by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap \operatorname{dom} f\}.$$

The function f is said to be Legendre if it satisfies the following conditions:

(1) $\operatorname{int}\operatorname{dom} f \neq \emptyset$ and the subdifferential ∂f is single-valued on its domain;

(2) $\operatorname{int}\operatorname{dom} f^* \neq \emptyset$ and ∂f^* is single-valued on its domain.

If E is a reflexive Banach space, we have the following:

(i) f is Legendre if and only if f^* is Legendre (see [4, Corollary 55]).

(ii) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*$ and $\operatorname{ran} \nabla f^* = \operatorname{dom} f = \operatorname{int}\operatorname{dom} f$(see [4, Theorem 5.10]).

If the Banach space E is smooth and strictly convex, the function $\frac{1}{p}\|\cdot\|^p$ with $p \in (1,\infty)$ is Legendre.

The Bregman projection [7] with respect to f of $x \in \operatorname{int}\operatorname{dom} f$ onto a nonempty closed convex subset $C \subset \operatorname{int}\operatorname{dom} f$ is defined as the unique vector $\Pi_C{}^f x \in C$, which satisfies

$$D_f\bigl(\Pi_C{}^f x, x\bigr) = \inf\{D_f(y,x), y \in C\}.$$

Lemma 2.1 ([8]) *Let C be a nonempty closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(1) $z = \Pi_C{}^f x$ *if and only if* $\langle \nabla f x - \nabla f z, y - z\rangle \le 0, \forall y \in C$;

(2) $D_f(y, \Pi_C{}^f x) + D_f(\Pi_C{}^f x, x) \le D_f(y,x), \forall x \in E, y \in C$.

Lemma 2.2 ([8, 14]) *Let E be a reflexive Banach space. Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and let V be the function defined by*

$$V_f\bigl(x,x^*\bigr) = f(x) - \bigl\langle x,x^*\bigr\rangle + f^*\bigl(x^*\bigr), \quad x \in E, x^* \in E^*.$$

Then the following hold:

(1) $D_f(x, \nabla f^*(x^*)) = V(x,x^*), \forall x \in E, x^* \in E^*$;

(2) $V_f(x,x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \le V_f(x, x^* + y^*)$.

Lemma 2.3 ([22]) *If $f: E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from strong topology of E to the strong topology of E^*.*

Theorem 2.4 ([25]) *Let E be a reflexive Banach space and let $f: E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following are equivalent*:

(1) *f is strongly coercive and uniformly convex on bounded subsets of E.*
(2) $\operatorname{dom} f^* = E^*$, *$f^*$ is bounded and uniformly smooth on bounded subsets of E^*.*
(3) $\operatorname{dom} f^* = E^*$, *$f^*$ is Fréchet differentiable and ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*.*

Theorem 2.5 ([25]) *Let E be a reflexive Banach space and let $f: E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following are equivalent*:

(1) *f is bounded and uniformly smooth on bounded subsets of E.*
(2) *f^* is Fréchet differentiable and f^* is norm-to-norm uniformly continuous on bounded subsets of E^*.*
(3) $\operatorname{dom} f^* = E^*$, *$f^*$ is strongly coercive and uniformly convex on bounded subsets of E^*.*

Lemma 2.6 *Let E be a reflexive Banach space, let $r > 0$ be a constant, let ρ_r be the gauge of uniform convexity of f, and let $f: E \to \mathbb{R}$ be a convex function which is bounded and uniformly convex on bounded subsets of E. Then, for any $x \in E$, $y^*, z^* \in B_r$ and $\alpha \in (0,1)$,*

$$V_f\big(x, \alpha y^* + (1-\alpha)z^*\big) \le \alpha V_f\big(x, y^*\big) + (1-\alpha)V_f\big(x, z^*\big) - \alpha(1-\alpha)\rho_r{}^*\big(\|y^* - z^*\|\big).$$

Lemma 2.7 ([16]) *Let E be a Banach space and $f: E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then*

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.8 ([21]) *Let $f: E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, the sequence $\{x_n\}$ is bounded, too.*

The function f is called sequentially consistent if for any two sequences $\{u_n\}$ and $\{v_n\}$ in E such that the first one is bounded:

$$\lim_{n\to\infty} D_f(u_n, v_n) = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|u_n - v_n\| = 0.$$

Lemma 2.9 ([9]) *The function f is totally convex on bounded subsets if and only if the function f is sequentially consistent.*

3 Main results

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E, and let $f: E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{S_i\}_{i=1}^m$ be a finite family of quasi-Bregman nonexpansive self mappings of C such that S_i is L_i-Lipschitz and $(I - S_i)$ is demiclosed at 0 for each $i \in \{1, 2, \dots, m\}$. Assume $\Gamma = \bigcap_{i=1}^m F(S_i) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated by (1.2), then the sequence $\{x_n\}$ converges to $\Pi_\Gamma{}^f x_0$.*

Proof We divide the proof into six steps.

Step 1. We show that C_n is closed and convex for any $n \geq 1$.

Since $C = C_1$, C_1 is closed and convex.

Assume C_n is closed and convex for some $n \geq 1$. Since for any $y \in C_n$, $i = 1$,

$$\begin{aligned} &D_f(y, y_{1n}) \leq D_f(y, w_n) \\ &\quad \Leftrightarrow \quad f(w_n) - f(y_{1n}) + \langle \nabla f(w_n), y - w_n \rangle - \langle \nabla f(y_{1n}), y - y_{1n} \rangle \leq 0 \\ &\quad \Leftrightarrow \quad f(w_n) - f(y_{1n}) + \langle \nabla f(y_{1n}), y_{1n} \rangle - \langle \nabla f(w_n), w_n \rangle \leq \langle \nabla f(y_{1n}) - \nabla f(w_n), y \rangle \end{aligned}$$

and, for $2 \leq i \leq m$,

$$\begin{aligned} &D_f(y, y_{in}) \leq D_f(y, w_n) \\ &\quad \Leftrightarrow \quad f(w_n) - f(y_{in}) + \langle \nabla f(w_n), y - w_n \rangle - \langle \nabla f(y_{in}), y - y_{in} \rangle \leq 0 \\ &\quad \Leftrightarrow \quad f(w_n) - f(y_{in}) + \langle \nabla f(y_{in}), y_{in} \rangle - \langle \nabla f(w_n), w_n \rangle \leq \langle \nabla f(y_{in}) - \nabla f(w_n), y \rangle, \end{aligned}$$

we have that C_{n+1} is closed and convex. Therefore, C_n is closed and convex for any $n \geq 1$.

Step 2. We show that $\Gamma \subset C_n$ for any $n \geq 1$.

For $n = 1$, $\Gamma \subset C = C_1$.

Now assume $\Gamma \subset C_n$ for some $n \geq 1$. Let $u \in \Gamma$, then by Lemma 2.6, we have for $i = 1$,

$$\begin{aligned} D_f(u, y_{1n}) &= D_f\bigl(u, \nabla f^*\bigl(\beta_n \nabla f(w_n) + (1 - \beta_n)\nabla f\bigl(S_1(w_n)\bigr)\bigr)\bigr) \\ &= V_f\bigl(u, \beta_n \nabla f(w_n) + (1 - \beta_n)\nabla f\bigl(S_1(w_n)\bigr)\bigr) \\ &= f(u) - \bigl\langle u, \beta_n \nabla f(w_n) + (1 - \beta_n)\nabla f\bigl(S_1(w_n)\bigr)\bigr\rangle \\ &\quad + f^*\bigl(\beta_n \nabla f(w_n) + (1 - \beta_n)\nabla f\bigl(S_1(w_n)\bigr)\bigr) \\ &= \beta_n f(u) + (1 - \beta_n) f(u) - \beta_n \bigl\langle u, \nabla f(w_n)\bigr\rangle - (1 - \beta_n)\bigl\langle u, \nabla f\bigl(S_1(w_n)\bigr)\bigr\rangle \\ &\quad + f^*\bigl(\beta_n \nabla f(w_n) + (1 - \beta_n)\nabla f\bigl(S_1(w_n)\bigr)\bigr) \\ &\leq \beta_n f(u) + (1 - \beta_n) f(u) - \beta_n \bigl\langle u, \nabla f(w_n)\bigr\rangle - (1 - \beta_n)\bigl\langle u, \nabla f\bigl(S_1(w_n)\bigr)\bigr\rangle \\ &\quad + \beta_n f^*\bigl(\nabla f(w_n)\bigr) + (1 - \beta_n) f^*\bigl(\nabla f\bigl(S_1(w_n)\bigr)\bigr) \\ &= \beta_n\bigl[f(u) - \bigl\langle u, \nabla f(w_n)\bigr\rangle + f^*\bigl(\nabla f(w_n)\bigr)\bigr] \\ &\quad + (1 - \beta_n)\bigl[f(u) - \bigl\langle u, \nabla f\bigl(S_1(w_n)\bigr)\bigr\rangle + f^*\bigl(\nabla f\bigl(S_1(w_n)\bigr)\bigr)\bigr] \\ &= \beta_n D_f(u, w_n) + (1 - \beta_n) D_f\bigl(u, S_1(w_n)\bigr) \\ &\leq \beta_n D_f(u, w_n) + (1 - \beta_n) D_f(u, w_n) \\ &= D_f(u, w_n) \end{aligned} \tag{3.1}$$

Now for $2 \leq i \leq m$, we have

$$\begin{aligned} &D_f(u, y_{in}) \\ &\quad = D_f\bigl(u, \nabla f^*\bigl(\beta_n \nabla f(S_{i-1} w_n) + (1 - \beta_n)\nabla f(S_i y_{(i-1)n})\bigr)\bigr) \\ &\quad = V_f\bigl(u, \beta_n \nabla f(S_{i-1} w_n) + (1 - \beta_n)\nabla f(S_i y_{(i-1)n})\bigr) \end{aligned}$$

$$
\begin{aligned}
&= f(u) - \langle u, \alpha_n \nabla f(S_{i-1} w_n) + (1-\beta_n)\nabla f(S_2 y_{(i-1)n})\rangle \\
&\quad + f^*\big(\beta_n \nabla f(S_{i-1} w_n) + (1-\beta_n)\nabla f(S_i y_{(i-1)n})\big) \\
&= \beta_n f(u) + (1-\beta_n) f(u) - \beta_n \langle u, \nabla f(S_{i-1} w_n)\rangle \\
&\quad - (1-\beta_n)\langle u, \nabla f(S_i y_{(i-1)n})\rangle \\
&\quad + f^*\big(\beta_n \nabla f(S_{i-1} w_n) + (1-\beta_n)\nabla f(S_i y_{(i-1)n})\big) \\
&\le \beta_n f(u) + (1-\beta_n) f(u) - \beta_n \langle u, \nabla f(S_{i-1} w_n)\rangle \\
&\quad - (1-\beta_n)\langle u, \nabla f(S_i y_{(i-1)n})\rangle \\
&\quad + \beta_n f^*\big(\nabla f(S_{i-1} w_n)\big) + (1-\beta_n) f^*\big(\nabla f(S_i y_{(i-1)n})\big) \\
&= \beta_n \big[f(u) - \langle u, \nabla f(S_{i-1} w_n)\rangle + f^*\big(\nabla f(S_{i-1} w_n)\big)\big] \\
&\quad + (1-\beta_n)\big[f(u) - \langle u, \nabla f(S_i y_{(i-1)n})\rangle + f^*\big(\nabla f(S_i y_{(i-1)n})\big)\big] \\
&= \beta_n D_f(u, S_{i-1} w_n) + (1-\beta_n) D_f(u, S_i y_{(i-1)n}) \\
&\le \beta_n D_f(u, w_n) + (1-\beta_n) D_f(u, y_{(i-1)n}) \\
&\le \beta_n D_f(u, w_n) + (1-\beta_n)\big[\beta_n D_f(u, w_n) + (1-\beta_n) D_f(u, y_{(i-2)n})\big] \\
&= \big(\beta_n + \beta_n(1-\beta_n)\big) D_f(u, w_n) + (1-\beta_n)^2 D_f(u, y_{(i-2)n}) \\
&\le \beta_n\big(1 + (1-\beta_n)\big) D_f(u, w_n) \\
&\quad + (1-\beta_n)^2\big[\beta_n D_f(u, w_n) + (1-\beta_n) D_f(u, y_{(i-3)n})\big] \\
&= \beta_n\big(1 + (1-\beta_n) + (1-\beta_n)^2\big) D_f(u, w_n) + (1-\beta_n)^3 D_f(u, y_{(i-3)n}) \\
&\le \\
&\vdots \\
&\le \beta_n\big(1 + (1-\beta_n) + (1-\beta_n)^2 + \cdots + (1-\beta_n)^{i-1}\big) D_f(u, w_n) \\
&\quad + (1-\beta_n)^i D_f(u, w_n) \\
&= \beta_n \left[\frac{1-(1-\beta_n)^i}{1-(1-\beta_n)}\right] D_f(u, w_n) + (1-\beta_n)^i D_f(u, w_n) \\
&= D_f(u, w_n).
\end{aligned} \tag{3.2}
$$

Hence $\Gamma \subset C_n$ for any $n \ge 1$.

Step 3. We shall show that $\{x_n\}$ is a Cauchy sequence.

Since $\Gamma \subset C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n}{}^f x_0 \subset C_n$, by Lemma 2.1, we have that $D_f(x_n, x_0) \le D_f(x_{n+1}, x_0)$ and also $D_f(x_n, x_0) \le D_f(u, x_0), u \in \Gamma$. Hence $D_f(x_n, x_0)$ is nondecreasing and bounded. So, $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. Furthermore, by Lemma 2.8, $\{x_n\}$ is bounded. Also, since $x_n = \Pi_{C_n}{}^f x_0$, it follows from Lemma 2.1 that $D_f(x_k, x_n) = D_f(x_k, \Pi_{C_n}{}^f x_0) \le D_f(x_k, x_0) - D_f(x_n, x_0) \to 0$ as $n, k \to \infty$. Since f is totally convex on bounded subsets of E, f is sequentially consistent. Therefore $\|x_n - x_k\| \to 0$ as $n, k \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence.

Step 4. We show that

$$
\lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|x_n - y_{in}\|
$$

$$= \lim_{n\to\infty} \|y_{(i+1)n} - y_{in}\|$$

$$= \lim_{n\to\infty} \left\|(I - S_1)w_n\right\|$$

$$= \lim_{n\to\infty} \left\|(I - S_i)y_{(i-1)n}\right\| = 0,$$

for each $i \in \{1, 2, \dots, m\}$.

Since $x_{n+1} \in C_{n+1} \subset C_n$, by Lemma 2.1, we have $D_f(x_{n+1}, x_n) \le D_f(x_{n+1}, x_0) - D_f(x_n, x_0)$. Taking the limit as $n \to \infty$, we have $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$.

Since f is totally convex on bounded subsets of E, f is sequentially consistent. Therefore

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

From (1.2) we get

$$\|x_n - w_n\| = \left\|\gamma_n(x_n - x_{n-1})\right\| \le \|x_n - x_{n-1}\|,$$

which implies

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.4}$$

Since $\{x_n\}$ is bounded, (3.4) implies that $\{w_n\}$ is also bounded and

$$\|x_{n+1} - w_n\| \le \|x_{n+1} - x_n\| + \|x_n - w_n\|.$$

Thus, we get

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0.$$

By Lemma 2.7,

$$\lim_{n\to\infty} D_f(x_{n+1}, w_n) = 0.$$

Since $x_{n+1} \in C_n$, for $1 \le i \le m$, from (1.2) we have $D_f(x_{n+1}, y_{in}) \le D_f(x_{n+1}, w_n)$. Hence $\lim_{n\to\infty} D_f(x_{n+1}, y_{in}) = 0$, $\forall i \in \{1, 2, 3, \dots, m\}$. Since f is totally convex on bounded subsets of E, f is sequentially consistent. Therefore

$$\|x_{n+1} - y_{in}\| \to 0 \quad \text{as } n \to \infty, \forall i \in \{1, 2, 3, \dots, m\}. \tag{3.5}$$

Observe that $\|x_n - y_{in}\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_{in}\|, \forall i \in \{1, 2, 3, \dots, m\}$, which implies

$$\lim_{n\to\infty} \|x_n - y_{in}\| = 0, \quad \forall i \in \{1, 2, 3, \dots, m\}. \tag{3.6}$$

Also, $\|y_{in} - w_n\| \le \|y_{in} - x_n\| + \|x_n - w_n\|$. Thus,

$$\lim_{n\to\infty} \|y_{in} - w_n\| = 0, \quad \forall i \in \{1, 2, 3, \dots, m\}. \tag{3.7}$$

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|\nabla f y_{in} - \nabla f w_n\| = 0, \quad \forall i \in \{1,2,3,\dots,m\}. \tag{3.8}$$

Since $\{w_n\}$ is bounded, (3.7) implies that $\{y_{in}\}$ is also bounded.

Thus, for $1 \le i \le m-1$, we have $\|y_{(i+1)n} - y_{in}\| \le \|y_{(i+1)n} - x_{n+1}\| + \|x_{n+1} - y_{in}\|$, so that

$$\lim_{n\to\infty} \|y_{(i+1)n} - y_{in}\| = 0. \tag{3.9}$$

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|\nabla f y_{(i+1)n} - \nabla f y_{in}\| = 0, \quad \forall i \in \{1,2,3,\dots,m-1\}. \tag{3.10}$$

From (1.2)

$$\|\nabla f y_{1n} - \nabla f w_n\| = (1-\beta_n)\|\nabla f S_1 w_n - \nabla f w_n\|.$$

From (3.7), we have

$$0 = \lim_{n\to\infty} \|\nabla f y_{1n} - \nabla f w_n\| = \lim_{n\to\infty} (1-\beta_n)\|\nabla f S_1 w_n - \nabla f w_n\|.$$

Hence

$$\lim_{n\to\infty} \|\nabla f S_1 w_n - \nabla f w_n\| = 0. \tag{3.11}$$

This implies that as ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*,

$$\lim_{n\to\infty} \|w_n - S_1 w_n\| = 0. \tag{3.12}$$

Now

$$\|y_{1n} - S_1 w_n\| \le \|y_{1n} - w_n\| + \|w_n - S_1 w_n\|,$$

which implies

$$\lim_{n\to\infty} \|y_{1n} - S_1 w_n\| = 0.$$

Thus

$$\|y_{2n} - S_1 w_n\| \le \|y_{2n} - y_{1n}\| + \|y_{1n} - S_1 w_n\|$$

gives

$$\lim_{n\to\infty} \|y_{2n} - S_1 w_n\| = 0.$$

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|\nabla f y_{2n} - \nabla f S_1 w_n\| = 0.$$

Again, from (1.2), we have

$$\|\nabla f y_{2n} - \nabla f S_1 w_n\| = (1-\beta_n)\|\nabla f S_2 y_{1n} - \nabla f S_1 w_n\|.$$

Therefore,

$$\lim_{n\to\infty} \|\nabla f S_2 y_{1n} - \nabla f S_1 w_n\| = 0.$$

Since ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|S_2 y_{1n} - S_1 w_n\| = 0.$$

Thus

$$\|y_{1n} - S_2 y_{1n}\| \leq \|y_{1n} - w_n\| + \|w_n - S_1 w_n\| + \|S_1 w_n - S_2 y_{1n}\|$$

gives

$$\lim_{n\to\infty} \left\|(I - S_2) y_{1n}\right\| = 0. \tag{3.13}$$

Now

$$\begin{aligned}\|y_{3n} - S_2 w_n\| &\leq \|y_{3n} - y_{2n}\| + \|y_{2n} - y_{1n}\| + \|y_{1n} - S_2 y_{1n}\| + \|S_2 y_{1n} - S_2 w_n\| \\ &\leq \|y_{3n} - y_{2n}\| + \|y_{2n} - y_{1n}\| + \|y_{1n} - S_2 y_{1n}\| + L_2\|y_{1n} - w_n\|.\end{aligned}$$

This implies $\lim_{n\to\infty} \|y_{3n} - S_2 w_n\| = 0$.

From this and the fact that ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|\nabla f y_{3n} - \nabla f S_2 w_n\| = 0.$$

Similarly, from (1.2) we have

$$\|\nabla f y_{3n} - \nabla f S_2 w_n\| = (1-\beta_n)\|\nabla f S_3 y_{2n} - \nabla f S_2 w_n\|.$$

Therefore,

$$\lim_{n\to\infty} \|\nabla f S_3 y_{2n} - \nabla f S_2 w_n\| = 0.$$

Since ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|S_3 y_{2n} - S_2 w_n\| = 0.$$

From the following inequality:

$$\begin{aligned}\|y_{2n}-S_3y_{2n}\| &\le \|y_{2n}-y_{1n}\|+\|y_{1n}-S_2y_{1n}\|+\|S_2y_{1n}-S_2w_n\|+\|S_2w_n-S_3y_{2n}\|\\ &\le \|y_{2n}-y_{1n}\|+\|y_{1n}-S_2y_{1n}\|+L_2\|y_{1n}-w_n\|+\|S_2w_n-S_3y_{2n}\|,\end{aligned}$$

we get

$$\lim_{n\to\infty}\big\|(I-S_3)y_{2n}\big\|=0. \tag{3.14}$$

Also,

$$\begin{aligned}\|y_{4n}-S_3w_n\| &\le \|y_{4n}-y_{3n}\|+\|y_{3n}-y_{2n}\|+\|y_{2n}-S_3y_{2n}\|+\|S_3y_{2n}-S_3w_n\|\\ &\le \|y_{4n}-y_{3n}\|+\|y_{3n}-y_{2n}\|+\|y_{2n}-S_3y_{2n}\|+L_3\|y_{2n}-w_n\|,\end{aligned}$$

implies $\lim_{n\to\infty}\|y_{4n}-S_3w_n\|=0$.

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have $\lim_{n\to\infty}\|\nabla f y_{4n}-\nabla f S_3w_n\|=0$.

From (1.2) we have

$$\|\nabla f y_{4n}-\nabla f S_3w_n\|=(1-\beta_n)\|\nabla f S_4y_{3n}-\nabla f S_3w_n\|.$$

Therefore,

$$\lim_{n\to\infty}\|\nabla f S_4y_{3n}-\nabla f S_3w_n\|=0.$$

Since ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty}\|S_4y_{3n}-S_3w_n\|=0.$$

From the inequality

$$\begin{aligned}\|y_{3n}-S_4y_{3n}\| &\le \|y_{3n}-y_{2n}\|+\|y_{2n}-S_3y_{2n}\|+\|S_3y_{2n}-S_3w_n\|+\|S_3w_n-S_4y_{3n}\|\\ &\le \|y_{3n}-y_{2n}\|+\|y_{2n}-S_3y_{2n}\|+L_3\|y_{2n}-w_n\|+\|S_3w_n-S_4y_{3n}\|,\end{aligned}$$

we get

$$\lim_{n\to\infty}\big\|(I-S_4)y_{3n}\big\|=0. \tag{3.15}$$

Continuing in this fashion, we get

$$\begin{aligned}\lim_{n\to\infty}\big\|(I-S_1)w_n\big\| &= \lim_{n\to\infty}\big\|(I-S_2)y_{1n}\big\|\\ &= \lim_{n\to\infty}\big\|(I-S_3)y_{2n}\big\|\\ &= \lim_{n\to\infty}\big\|(I-S_4)y_{3n}\big\|\end{aligned}$$

$$\vdots$$

$$= \lim_{n\to\infty} \left\|(I - S_m)y_{(m-1)n}\right\| = 0.$$

Step 5. We show that $\{x_n\}$ converges to an element of Γ.

Since $\{x_n\}$ is a Cauchy sequence, we assume that $x_n \to x^*$ as $n \to \infty$. From the fact that

$$\lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|x_n - y_{in}\| = 0, \quad \forall i \in \{1,2,3,\dots,m\},$$

we have that

$$w_n \to x^*, \qquad y_{in} \to x^* \quad \text{as } n \to \infty, \forall i \in \{1,2,3,\dots,m\}.$$

Since $I - S_i$, $i \in \{1,2,3,\dots,m\}$ are demiclosed at 0 and

$$\lim_{n\to\infty} \left\|(I - S_1)w_n\right\| = \lim_{n\to\infty} \left\|(I - S_i)y_{(i-1)n}\right\| = 0 \quad \text{for } 2 \le i \le m,$$

we have $x^* \in \bigcap_{i=1}^m F(S_i)$. Therefore, $x^* \in \Gamma$.

Step 6. We show that $x^* = \Pi_\Gamma{}^f x_0$.

Let $y = \Pi_\Gamma{}^f x_0$. Since $x^* \in \Gamma$, we have that

$$D_f(y, x_0) \le D_f\left(x^*, x_0\right). \tag{3.16}$$

Since $y \in \Gamma \subset C_n$ and $x_n = \Pi_{C_n}{}^f x_0$, we have

$$D_f(x_n, x_0) \le D_f(y, x_0)$$

and, taking into account that $x_n \to x^*$, obtain

$$D_f\left(x^*, x_0\right) \le D_f(y, x_0). \tag{3.17}$$

Combining (3.16) and (3.17) yields

$$D_f(y, x_0) = D_f\left(x^*, x_0\right).$$

Hence, $x^* = y = \Pi_\Gamma{}^f x_0$. □

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E, and let $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{S_i\}_{i=1}^m$ be a finite family of Bregman relatively nonexpansive mappings such that S_i, $i = 1,2,3,\dots,m$ are L_i-Lipschitz and $(I - S_i), i = 1,2,\dots,m$ are demiclosed at 0. Assume $\Gamma = \bigcap_{i=1}^m F(S_i) \neq \emptyset$. Let a* *sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0, x_1 \in C, \quad C = C_1; \\ w_n = x_n + \gamma_n(x_n - x_{n-1}); \\ y_{1n} = \nabla f^*(\beta_n \nabla f w_n + (1-\beta_n)\nabla f S_1 w_n); \\ y_{in} = \nabla f^*(\beta_n \nabla f S_{i-1} w_n + (1-\beta_n)\nabla f S_i y_{(i-1)n}); \\ C_{in} = \{v \in C_n : D_f(v, y_{in}) \leq D_f(v, w_n)\}; \\ C_{n+1} = \bigcap_{i=1}^m C_{in}; \\ x_{n+1} = \Pi_{C_{n+1}}{}^f x_0; \end{cases} \tag{3.18}$$

where $\{\gamma_n\}$ and $\{\beta_n\} \subset (a,b)$, $0 < a < b < 1$, are sequences. Then the sequence $\{x_n\}$ converges to a point $z \in \Gamma$, where $z = \Pi_\Gamma{}^f x_0$.

Corollary 3.3 *Let E be a uniformly convex real Banach space. Let $\{S_i\}_{i=1}^m$ be a finite family of nonexpansive mappings. Assume $\Gamma = \bigcap_{i=1}^m F(S_i)\} \neq \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0, x_1 \in C, \quad C = C_1; \\ w_n = x_n + \gamma_n(x_n - x_{n-1}); \\ y_{1n} = (\beta_n w_n + (1-\beta_n)S_1 w_n); \\ y_{in} = (\beta_n S_{i-1} w_n + (1-\beta_n) S_i y_{(i-1)n}); \\ C_{in} = \{v \in C_n : \|y_{in} - v\| \leq \|w_n - v\|\}; \\ C_{n+1} = \bigcap_{i=1}^m C_{in}; \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{3.19}$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Then the sequence $\{x_n\}$ converges to a point $z \in \Gamma$, where $z = P_\Gamma x_0$.

4 Applications

4.1 Application to the equilibrium problem

Let C be a nonempty closed convex subset of a real Banach space E, and let $F : C \times C \to \mathbb{R}$ be a bifunction.

The equilibrium problem with respect to F and C is to find $z \in C$ such that

$$F(z,y) \geq 0, \quad \forall y \in C.$$

The set of solutions of the equilibrium problem above is denoted by $EP(F)$. For solving the equilibrium problem, we assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$, $\forall x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

The resolvent of a bifunction F is the operator $\mathrm{Res}_f{}^F : E \to 2^C$ defined by

$$\mathrm{Res}_f{}^F x = \bigl\{z \in C : F(z,y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C\bigr\}.$$

Lemma 4.1 ([23]) *Let E be a reflexive Banach space, and C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty)$ be a Legendre function. If the bifunction $F : C \times C \to \mathbb{R}$ satisfies conditions $(A1)$–$(A4)$, then the following holds*:

(1) $\mathrm{Res}_f{}^F$ *is single-valued*;
(2) $\mathrm{Res}_f{}^F$ *is Bregman firmly nonexpansive*;
(3) $\mathrm{Fix}(\mathrm{Res}^F) = EP(F)$;
(4) $EP(F)$ *is a closed and convex subset of* C;
(5) *For all $x \in E$ and for all $q \in \mathrm{Fix}(\mathrm{Res}^F)$*,

$$D_f\big(q, \mathrm{Res}_f{}^F x\big) + D_f\big(\mathrm{Res}_f{}^F x, x\big) \le D_f(q, x).$$

Theorem 4.2 *Let C and Q be nonempty, closed, and convex subsets of a reflexive Banach space E, and Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $F_i : C \times C \to \mathbb{R}$, $i = 1, 2, 3, \dots, m$ be bifunctions satisfying conditions $(A1)$–$(A4)$ such that $\mathrm{Res}_f{}^{F_i}$ are L_i-Lipschitz for $1 \le i \le m$. Assume $\Gamma = \bigcap_{i=1}^m EP(F_i) \ne \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0, x_1 \in C, \quad C = C_1; \\ w_n = x_n + \gamma_n(x_n - x_{n-1}); \\ y_{1n} = \nabla f^*(\beta_n \nabla f w_n + (1-\beta_n)\nabla f \mathrm{Res}_f{}^{F_1} w_n); \\ y_{in} = \nabla f^*(\beta_n \nabla f \mathrm{Res}_f{}^{F_{i-1}} w_n + (1-\beta_n)\nabla f \mathrm{Res}_f{}^{F_i} y_{(i-1)n}); \\ C_{in} = \{v \in C_n : D_f(v, y_{in}) \le D_f(v, w_n)\}; \\ C_{n+1} = \bigcap_{i=1}^m C_{in}; \\ x_{n+1} = \Pi_{C_{n+1}}{}^f x_0, \end{cases} \tag{4.1}$$

where $\{\gamma_n\}, \{\beta_n\} \subset (a, b)$, $0 < a < b < 1$, are sequences and $\mathrm{Res}_f{}^{F_i}$ are the resolvents of F_i, $i \in \{1, 2, \dots, m\}$. Then the sequence $\{x_n\}$ converges to $z = P_\Gamma{}^f x_0$.

Proof Putting $S_i = \mathrm{Res}_f{}^{F_i}$ in Theorem 3.1, we get the desired result. □

4.2 Application to the maximal monotone operator

A set-valued mapping $B \subset E \times E^*$ with domain $D(B) = \{x \in E : Bx \ne \emptyset\}$ and range $R(B) = \cup\{Bx : x \in D(B)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \ge 0$ whenever $(x, x^*), (y, y^*) \in B$, see, for example, [2]. A monotone mapping $B \subset E \times E^*$ is said to be maximal monotone if its graph $G(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. We know that if B is maximal monotone, then the zero of B, $B^{-1}(0) = \{x \in E : 0 \in Bx\}$ is closed and convex. Define the resolvent of B, $\mathrm{Res}_B{}^f : E \to 2^E$ by

$$\mathrm{Res}_B{}^f x = (\nabla f + B)^{-1} \circ \nabla f x.$$

We know the following (see [5]):

(1) $\mathrm{Res}_B{}^f$ is single valued;
(2) $\mathrm{Fix}(\mathrm{Res}_B{}^f) = B^{-1}0$.

Lemma 4.3 ([21]) *Let $B : E \to 2^{E*}$ be a maximal monotone mapping such that $B^{-1}(0) \neq \emptyset$. Then for all $x \in E$ and $q \in B^{-1}(0)$, we have*

$$D_f\big(q, \mathrm{Res}_B{}^f x\big) + D_f\big(\mathrm{Res}^f x, x\big) \leq D_f(q, x).$$

Theorem 4.4 *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E, and let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $B_i : E \to 2^{E*}$ $i = 1, 2, 3, \ldots, m$ be maximal monotone operators such that $\mathrm{Res}_{B_i}{}^f$ are L_i-Lipschitz for $1 \leq i \leq m$. Assume $\Gamma = \bigcap_{i=1}^m B_i{}^{-1}(0) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0, x_1 \in C, \quad C = C_1; \\ w_n = x_n + \gamma_n(x_n - x_{n-1}); \\ y_{1n} = \nabla f^*(\beta_n \nabla f w_n + (1 - \beta_n)\nabla f \mathrm{Res}_{B_1}{}^f w_n); \\ y_{in} = \nabla f^*(\beta_n \nabla f \mathrm{Res}_{B_{i-1}}{}^f w_n + (1 - \alpha_n)\nabla f \mathrm{Res}_{B_i}{}^f y_{(i-1)n}), \quad 2 \leq i \leq m; \\ C_{in} = \{v \in C_n : D_f(v, y_{in}) \leq D_f(v, w_n)\}; \\ C_{n+1} = \bigcap_{i=1}^m C_{in}; \\ x_{n+1} = \Pi_{C_{n+1}}{}^f x_0, \end{cases} \tag{4.2}$$

where $\{\gamma_n\}, \{\beta_n\} \subset (a, b)$, $0 < a < b < 1$, are sequences and $\mathrm{Res}_{B_i}{}^f$ are the resolvents of B_i. Then the sequence $\{x_n\}$ converges to a point $z \in \Gamma$, where $z = P_\Gamma{}^f x_0$.

Proof Putting $S_i = \mathrm{Res}_{B_i}{}^f$ in Theorem 3.1, we get the desired result. □

Acknowledgements
The authors appreciate the support of their institutions. They thank ACBF and AfDB for their financial support.

Funding
This work is supported from ACBF and AfDB Research Grant Funds to AUST.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematical Sciences, Bayero University, Kano, Nigeria. [2]Department of Mathematical Sciences, African University of Science and Technology, Abuja, Nigeria.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 November 2020 Accepted: 17 January 2022 Published online: 14 March 2022

References

1. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Iterative construction of fixed points of nearly asymptotically nonexansive mappings. J. Nonlinear Convex Anal. **8**, 61–79 (2007)
2. Alber, Y., Ryazantseva, I.: Nonlinear Ill-Posed Problems of Monotone Type. Springer, Dordrecht (2006)

3. Asplund, E., Rockafellar, R.T.: Gradients of convex functions. Trans. Am. Math. Soc. **139**, 443–467 (1969)
4. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun. Contemp. Math. **3**, 615–647 (2001)
5. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Bregman monotone optimization algorithms. SIAM J. Control Optim. **42**, 596–636 (2003)
6. Bonnans, J.F., Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer, New York (2000)
7. Bregman, L.M.: The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. **7**, 200–217 (1967)
8. Butnariu, D., Resmerita, E.: Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. **2006**, Article ID 84919 (2006)
9. Butnariu, D., Iusem, A.N.: Totally Convex Functions for Fixed Point Computation and Infinite Dimensional Optimization. Kluwer Academic, Dordrecht (2000)
10. Chidume, C.E., Ali, B.: Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **326**, 960–973 (2007)
11. Chidume, C.E., Ali, B.: Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **330**, 377–387 (2007)
12. Hiriart-Urruty, J.B., Lemarchal, C.: Convex Analysis and Minimization Algorithms II. Grundlehren der Mathematischen Wissenschaften, vol. 306. Springer, Berlin (1993)
13. Kitkuan, D., Kumam, P., Moreno, J.M., Sitthithakerngkiet, K.: Inertial viscosity forward backward splitting algorithm for monotone inclusions and its application to image restoration problems. Int. J. Comput. Math. https://doi.org/10.1080/00207160.2019.1649661
14. Kohsaka, F., Takahashi, W.: Proximal point algorithms with Bregman functions in Banach spaces. J. Nonlinear Convex Anal. **6**, 505–523 (2005)
15. Kumam, P., Saluja, G.S., Nashine, H.K.: Convergence of modified S-iteration process for two asymptotically nonexpansive mappings in the intermediate sense in *CAT*(0) spaces. J. Inequal. Appl. **2014**, 368 (2014)
16. Naraghirad, E., Yao, J.C.: Bregman weak relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2013**, 141 (2013)
17. Phon-on, A., Makaje, N., Sama-Ae, A., Khongraphan, K.: An inertial S-iteration process. Fixed Point Theory Appl. (2019). https://doi.org/10.1186/s13663-019-0654-7
18. Rehman, H., Kumam, P., Abubakar, A.B., Cho, Y.J.: The extragradient algorithm with inertial effects extended to equilibrium problems. Comput. Appl. Math. **39**, Article ID 100 (2020). https://doi.org/10.1007/s40314-020-1093-0
19. Rehman, H., Kumam, P., Kumam, W., Shutaywi, M., Jirakitpuwapat, W.: The inertial sub-gradient extra-gradient method for a class of pseudo-monotone equilibrium problems. Symmetry **12**, 463 (2020). https://doi.org/10.3390/sym12030463
20. Rehman, H., Kumam, P., Argyros, I.K., Deebani, W., Kumam, W.: Inertial extra-gradient method for solving a family of strongly pseudomonotone equilibrium problems in real Hilbert spaces with application in variational inequality problem. Symmetry **12**, 503 (2020). https://doi.org/10.3390/sym12040503
21. Reich, S., Sabach, S.: Two convergence theorems for a proximal methods in reflexive Banach spaces. Numer. Funct. Anal. Optim. **31**, 22–44 (2010)
22. Reich, S., Sabach, S.: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. **10**, 471–485 (2009)
23. Reich, S., Sabach, S.: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. **73**, 122–135 (2010)
24. Suparatulatorn, R., Cholamjiak, W., Suantai, S.: A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs. Numer. Algorithms **77**, 479–490 (2018)
25. Zalinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, River Edge (2002)