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Existence of solutions for a Lipschitzian vibroimpact problem with time-dependent constraints

Samir Adly[1*] and Nguyen Nang Thieu[1,2]

*Correspondence:
samir.adly@unilim.fr
[1]Laboratoire XLIM, Université de Limoges, 87060, Limoges, France
Full list of author information is available at the end of the article

Abstract

We study a mechanical system with a finite number of degrees of freedom, subjected to perfect time-dependent frictionless unilateral (possibly nonconvex) constraints with inelastic collisions on active constraints. The dynamic is described in the form of a second-order measure differential inclusion. Under some regularity assumptions on the data, we establish several properties of the set of admissible positions, which is not necessarily convex but assumed to be uniformly prox-regular. Our approach does not require any second-order information or boundedness of the Hessians of the constraints involved in the problem and are specific to moving sets represented by inequalities constraints. On that basis, we are able to discretize our problem by the time-stepping algorithm and construct a sequence of approximate solutions. It is shown that this sequence possesses a subsequence converging to a solution of the initial problem. This methodology is not only used to prove an existence result but could be also used to solve numerically the vibroimpact problem with time-dependent nonconvex constraints.

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1 Introduction

Vibroimpact systems are dynamical multibody systems subjected to perfect nonpenetration conditions that generate vibrations and impacts. Because of the impact laws, the systems involve discontinuities in the velocity and the acceleration may contain Dirac measures. Hence, vibroimpact systems cannot be modeled by ordinary differential equations, and one uses *measure differential inclusions* (see, e.g., [3, 20–24, 30, 35]).

In this paper, we consider a mechanical system with a finite number of degrees of freedom, subjected to perfect time-dependent unilateral constraints. More precisely, let $I = [0,T]$, $T > 0$, be a bounded time real interval and $d \in \mathbb{N}^* := \{1,2,\dots\}$. Let $g : I \times \mathbb{R}^d \to \mathbb{R}^d$ and $f_i : I \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,\dots,m\}$ be some functions and $m \in \mathbb{N}^*$. We denote by $q \in \mathbb{R}^d$ the representative point of the system in generalized coordinates and define the set of admissible positions at each instant $t \in I$ by

$$C(t) = \{q \in \mathbb{R}^d \mid f_i(t,q) \le 0 \ \forall i \in \{1,\dots,m\}\}$$

and the set of active constraints by $J(t,q) = \{i \in \{1,\dots,m\} \mid f_i(t,q) = 0\}$. The vibroimpact system given by g and the functions f_i is formally described by the following second-order differential inclusion in $\mathbb{R}^d$:

$$\ddot{q}(t) - g\bigl(t,q(t)\bigr) \in -\mathcal{N}_{C(t)}\bigl(q(t)\bigr), \tag{1}$$

where $\mathcal{N}_{C(t)}(q(t))$ is the Clarke normal cone [13, p. 51] to $C(t)$ at $q(t)$, $t \in I$.

Denote by $\nabla f_i(t,\cdot)(q)$ the derivative of $f_i(t,q)$ with respect to the second variable q and by $\partial f_i(\cdot,q)$ the derivative of f_i with respect to the first variable t. In what follows, given a set $\Omega \subset \mathbb{R}^d$, we denote its interior and boundary, respectively, by $\operatorname{int}(\Omega)$ and $\partial\Omega$.

Since $\mathcal{N}_{C(t)}(q) = \emptyset$ if $q(t) \notin C(t)$, if q is a solution of (1), then $q(t)$ must belong to $C(t)$ for all $t \in I$. If $q(t) \in \operatorname{int}(C(t))$ for all $t \in I$, then $\mathcal{N}_{C(t)}(q(t)) = \{0\}$ for all $t \in I$, so (1) becomes $\ddot{q} = g(t,q)$, which is an ordinary differential equation.

If $q(t) \in \operatorname{int}(C(t))$ for all $t \in (t_0,t_1) \cup (t_1,t_2)$, $q(t_1) \in \partial C(t_1)$, then

$$\dot{q}\bigl(t_1^-\bigr) \in -\mathcal{T}\bigl(t_1,q(t_1)\bigr) \quad \text{and} \quad \dot{q}\bigl(t_1^+\bigr) \in \mathcal{T}\bigl(t_1,q(t_1)\bigr), \tag{2}$$

where

$$\mathcal{T}(t,q) := \bigl\{v \in \mathbb{R}^d \mid \partial f_i(\cdot,q)(t) + \bigl\langle \nabla f_i(t,\cdot)(q), v\bigr\rangle \le 0 \ \forall i \in J(t,q)\bigr\}.$$

Observe that the set $\mathcal{T}(t,q)$ is a polyhedral convex closed set for each pair (t,q). The inclusion (2) will be proved in Sect. 4.2.

Note that the function $\dot{q}$ may be discontinuous at some $t \in I$ if $J(t,q(t))$ is nonempty. Therefore, in general, we cannot find a solution q of (1) for which there exists a differentiable derivative $\dot{q}$. Hence, we look for a solution q of (1) whose derivative $\dot{q}$ is of *bounded variation*. The latter implies that $\dot{q}$ is differentiable almost everywhere on I. Then, $\ddot{q}$ can be understood as a *Stieltjes measure*. Therefore, (1) can be extended in the distributional sense:

$$\begin{cases} \dot{q} \in BV([0,T];\mathbb{R}^d) \\ d\dot{q} - g(\cdot,q(\cdot))\,dt \in -\mathcal{N}_{C(\cdot)}(q(\cdot))\,dt, \end{cases}$$

where $BV([0,T];\mathbb{R}^d)$ stands for the space of all functions of bounded variation from $[0,T]$ to $\mathbb{R}^d$. More precisely, the second inclusion is taken in the Radon measure space $\mathcal{M}(0,T;\mathbb{R}^d)$, which is the dual space of the space of all continuous functions from $[0,T]$ to $\mathbb{R}^d$, denoted by $C([0,T],\mathbb{R}^d)$. For $\varphi \in C(I,\mathbb{R}^d)$ and for $\xi(\cdot) \in -\mathcal{N}_{C(\cdot)}(q(\cdot))$,

$$\begin{aligned} d\dot{q}: \quad & C\bigl(I,\mathbb{R}^d\bigr) \to \mathbb{R}; \\ & \langle d\dot{q},\varphi\rangle = \int_I \varphi\,d\dot{q}, \\ g\bigl(\cdot,q(\cdot)\bigr)\,dt: \quad & C\bigl(I,\mathbb{R}^d\bigr) \to \mathbb{R}; \end{aligned}$$

$$\langle g(\cdot, q(\cdot))\,dt, \varphi\rangle = \int_I \langle g(t, q(t)), \varphi(t)\rangle\,dt,$$

$$\xi(\cdot)\,dt: \quad C(I, \mathbb{R}^d) \to \mathbb{R};$$

$$\langle \xi(\cdot)\,dt, \varphi\rangle = \int_I \langle \xi(t), \varphi(t)\rangle\,dt.$$

Since the relation (2) does not uniquely define $\dot{q}(t^+)$, we will follow [21] to impose the following inealstic *impact law*

$$\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-)),$$

where $\mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-))$ is the nearest point of $\dot{q}(t^-)$ in $\mathcal{T}(t, q(t))$. In fact, J.-J. Moreau introduced the notion of inelastic shocks in 1983 in the paper [21] (see also [21, 22]).

To sum up, we are interested in investigating the following problem.

Problem $(\mathcal{P})$. Let $(q_0, p_0) \in C(0) \times \mathcal{T}(0, q_0)$. Find $q : [0, T] \to \mathbb{R}^d$, with $T > 0$, such that

(P1) q is absolutely continuous on $[0, T]$, $\dot{q} \in BV(0, T; \mathbb{R}^d)$;

(P2) $q(t) \in C(t)$ for all $t \in [0, T]$;

(P3) $d\dot{q} - g(\cdot, q(\cdot))\,dt \in -\mathcal{N}_{C(\cdot)}(q(\cdot))\,dt$;

(P4) $\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-))$ for all $t \in [0, T]$;

(P5) $q(0) = q_0$ and $\dot{q}(0) = p_0$.

Under some appropriate regularity assumptions on the data, we will prove the existence of at least one solution to problem $(\mathcal{P})$. Namely, by using a time-discretization scheme, we will construct a sequence of approximate solutions that has a subsequence converging to a solution of $(\mathcal{P})$.

There are many existence results for the vibroimpact problems with time-independent constraints (i.e., when the set of admissible positions does not depend on time: $C(t) = C$ for $t \in [0, T]$). In the single-constraint case, the results have been established by using the position-based algorithm in [32–34] and by using the velocity-based algorithm in [15, 16, 18–20]. In the multiconstraint case, several results have been obtained in [6, 25, 26, 28].

For vibroimpact problems with time-dependent constraints (i.e., when the set of admissible positions $C(t)$ depends on time), there are few solution existence theorems. Let us list some important results related to this case that are known in the literature:

Schatzman [35] established an existence result by considering a generalization of the Yosida-type approximation proposed in [31].

Assuming that the set of admissible positions at any instant is defined as a finite intersection of complements of convex sets, Bernicot and Lefebvre-Lepot [7] obtained an existence theorem.

Paoli [27, 29] proposed a time-stepping approximation scheme for the problem and proved its convergence, which gives as a byproduct a global existence result when the set of admissible positions at any instant is defined by a finite family of C^2 functions.

Attouch, Cabot and Redont [3] studied the dynamics of elastic shocks via epigraphical regularization of the nonsmooth convex potential and established an asymptotic analysis of the solutions when time $t \to +\infty$.

Cabot and Paoli [12] studied the convergence of trajectories and the exponential decay of the energy function associated to a vibroimpact problem with a linear dissipation term.

Attouch, Manigé and Redont [4] studied a nonsmooth second-order differential inclusion involving a Hessian-driven damping with applications to nonelastic shock laws.

The existence of solutions for these second-order differential problems has been studied by Bernicot and Venel [9] in a general and abstract framework. More precisely, the set $C(t)$ of admissible positions is assumed in [9] to be Lipschitz continuous in the Hausdorff distance sense and satisfies an "admissibility" property (see Sect. 2.3 [9]). The authors also considered a particular case, where the constraints are C^2 functions and have bounded second-order derivatives (see Sect. 4 in [9]). The assumptions used in this paper require less regularity on the data of the problem and could be seen as a complementary result of Theorem 3.2 and an improvement of Theorem 4.6 in [9] (see Remark 4.2 for more details).

In this paper, we give explicit conditions for the constraints without requiring any second-order differentiability information on the data involved in the constraints. We will follow the time-stepping scheme of [27] to prove the convergence of the approximate solutions. An illustrative example is given to clarify the applicability of the obtained result.

Our main result is an analog of the Peano solution existence theorem [17, Theorem 2.1, p. 10] for ordinary differential equations. Among other things, the proof relies on the Ascoli–Arzelà theorem, and the Banach–Alaoglu theorem applied to the Radon measure space $\mathcal{M}(0,T;\mathbb{R}^d)$, which is the dual space of the space of all continuous functions from $[0,T]$ to $\mathbb{R}^d$. Note that, as shown by Bounkhel [10], one can obtain existence theorems for first- and second-order nonconvex sweeping processes with perturbations by applying a fixed-point theorem.

The paper is organized as follows. In Sect. 2, we recall some preliminaries. In Sect. 3, we formulate our regularity assumptions and deduce several properties of the set of admissible positions and its Clarke's normal cone. Section 4 presents the time-discretization scheme to construct a sequence of approximate solutions and establishes the main result of the paper. The convergence of the sequence of approximate solutions is investigated in Sect. 4.1. In Sects. 4.2 and 4.3, we prove that the limit trajectory is a solution of problem $(\mathcal{P})$. To check the applicability of our result and to compare them with the existing ones, an example is presented in Sect. 5. Some concluding remarks are given in the final section.

2 Preliminaries

First, we recall some basic concepts and facts from nonsmooth analysis, which are widely used in what follows. We mainly follow the references [5, 13, 14] and [20]. Our notation is standard in variational analysis; see, e.g., [13].

Let the Euclidean space $\mathbb{R}^d$ be equipped with a standard scalar product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|\cdot\|$. The open ball (resp., closed ball) in $\mathbb{R}^d$ with center x and radius r is denoted by $\mathbb{B}(x,r)$ (resp., $\bar{\mathbb{B}}(x,r)$). The open unit ball and closed unit ball are denoted, respectively, by $\mathbb{B}$ and $\bar{\mathbb{B}}$.

The distance function $d_C(\cdot):\mathbb{R}^d\to\mathbb{R}$, where C is a nonempty subset of $\mathbb{R}^d$, is defined by setting $d_C(x)=\inf\{\|x-y\|\mid y\in C\}$. For $\rho>0$, the set $U_\rho(C)=\{x\in\mathbb{R}^d\mid d_C(x)<\rho\}$ is called the ρ-enlargement $U_\rho(C)$ of C. For x in $\mathbb{R}^d$, the set of the nearest points of x in C is called the *projection* of x onto C and is defined by $\mathbb{P}_C(x)=\{y\in C\mid \|y-x\|=d_C(x)\}$.

A function $f:Y\to\mathbb{R}$ defined on $Y\subset\mathbb{R}^d$ is said to be Lipschitz continuous with modulus $L>0$ on Y if $|f(y)-f(y')|\le L\|y-y'\|$ for all $y,y'\in Y$.

Definition 2.1 Let f be Lipschitz continuous near x in $\mathbb{R}^d$ and let v be any vector in $\mathbb{R}^d$. *Clarke's generalized directional derivative* of f at x in the direction v, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v) := \limsup_{y\to x, t\downarrow 0} \frac{f(y+tv)-f(y)}{t}.$$

Let C be a closed subset of $\mathbb{R}^d$ and $x \in C$.

Definition 2.2 The set $\mathcal{T}_C(x) := \{v \in \mathbb{R}^d \mid d_C^0(x;v) = 0\}$ is called *the Clarke tangent cone* to C at x. *The Clarke normal cone* to C at x is defined by polarity with $\mathcal{T}_C(x)$:

$$\mathcal{N}_C(x) = \{x^* \in \mathbb{R}^d \mid \langle x^*, v\rangle \le 0 \text{ for all } v \in \mathcal{T}_C(x)\}.$$

Definition 2.3 A vector $v \in \mathbb{R}^d$ is a *proximal subgradient* of a function $f : \mathbb{R}^d \to \mathbb{R}$ at x if there exist a real number $\sigma \ge 0$ and a neighborhood U of x such that

$$\langle v, x' - x\rangle \le f(x') - f(x) + \sigma \|x' - x\|^2,$$

for all $x' \in U$.

Definition 2.4 A vector $v \in \mathbb{R}^d$ is a *proximal normal vector* to C at $x \in C$ when it is a proximal subgradient of the indicator function of C, that is, when there exist a constant $\sigma \ge 0$ and a neighborhood U of x such that $\langle v, x' - x\rangle \le \sigma \|x' - x\|^2$ for all $x' \in U \cap C$. The set of such vectors, which is denoted by $\mathcal{N}_C^P(x)$, is said to be *the proximal normal cone* of C at x.

Definition 2.5 The set C is said to be *r-prox-regular* (or uniformly prox-regular with constant $r > 0$) whenever, for all $x \in C$, for all $\xi \in \mathcal{N}_C^P(x) \cap \mathbb{B}$, and for all $t \in (0, r)$, one has $x \in \mathbb{P}_C(x + t\xi)$.

Remark 2.1 If C is uniformly prox-regular, then $\mathcal{N}_C^P(x) = \mathcal{N}_C(x)$.

The following proposition provides a representation for the Clarke normal cone to a set, given by inequalities constraints, under some suitable assumptions.

Proposition 2.1 (See [13] Corollary 2 of Theorem 2.4.7) *Let C be given as follows*:

$$\{y \in \mathbb{R}^d \mid f_1(y) \le 0, \dots, f_m(y) \le 0\},$$

and let x be such that $f_i(x) = 0$ for $i = 1, \dots, m$. Then, if each f_i is differentiable at x and if the gradients $\nabla f_i(x), i = 1, \dots, m$, are positively linearly independent, we have

$$\mathcal{N}_C(x) = \left\{\sum_{i=1}^m \lambda_i \nabla f_i(x) \mid \lambda_i \ge 0,\ i = 1, \dots, m\right\}.$$

Lemma 2.1 (See [2, Lemma 3.2]) *Let $C \subset \mathbb{R}^d$ and $x, y \in C$ with $\|x - y\| < 2\rho$, where $\rho \in (0, +\infty]$. Then, for any $\tau \in [0, 1]$ one has $x + \tau(y - x) \in U_\rho(C)$.*

Definition 2.6 Let $f : [a,b] \to \mathbb{R}^d$ be a function. The *total variation* of f on $[a,b]$ is the nonnegative extended real number

$$\mathrm{Var}\big(f,[a,b]\big) = \sup \sum_{i=1}^{n} \big\| f(x_i) - f(x_{i-1}) \big\|,$$

where the supremum is taken over all finite partitions $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$. If $\mathrm{Var}(f,[a,b]) < +\infty$, then one says that f is a function *of bounded variation* on $[a,b]$ and writes $f \in BV([a,b],\mathbb{R}^d)$.

The next proposition is a consequence of the Ascoli–Arzelà Theorem and the Banach–Alaoglu Theorem, which gives sufficient conditions for the existence of a convergence subsequence of a sequence of absolutely continuous functions.

Proposition 2.2 (See [5, Theorem 4, p. 13]) *Let $\{x_k(\cdot)\}$ be a sequence of absolutely continuous functions from an interval $I \subset \mathbb{R}$ to a Banach space X satisfying*

(i) *For all $t \in I$, $\{x_k(t)\}_k$ is a relatively compact subset of X;*
(ii) *There exists a positive function $c(\cdot) \in L^1(I,\mathbb{R})$ such that $\|\dot{x}_k(t)\| \le c(\cdot)$ for almost all $t \in I$.*

Then, there exists a subsequence, still denoted by $\{x_k(\cdot)\}$, converging to an absolutely continuous function $x(\cdot)$ from I to X in the sense that

(a) *$x_k(\cdot)$ converges uniformly to $x(\cdot)$ over compact subsets of I;*
(b) *$\dot{x}_k(\cdot)$ converges weakly to $\dot{x}(\cdot)$ in $L^1(I,X)$.*

3 The framework

We now propose some regularity assumptions. In the notation of Sect. 1, let

$$C = \big\{(t,q) \in [0,T] \times \mathbb{R}^d \mid q \in C(t)\big\}.$$

Assumption A1 There exists an extended real $\rho \in (0,+\infty]$ such that

(i) for all $i \in \{1,\dots,m\}$, f_i is differentiable on $U_\rho(C)$ and its derivative $\nabla f_i(\cdot,\cdot) : U_\rho(C) \to \mathbb{R}$ is Lipschitz continuous with rank L;
(ii) there is $\gamma > 0$ such that for all $t \in [0,T]$ and $i \in \{1,\dots,m\}$, for all $q_1, q_2 \in U_\rho(C(t))$,

$$\big\langle \nabla f_i(t,\cdot)(q_1) - \nabla f_i(t,\cdot)(q_2), q_1 - q_2 \big\rangle \ge -\gamma \|q_1 - q_2\|^2;$$

(iii) for all $t \in [0,T]$ and for all $i \in \{1,\dots,m\}$, one has $\|\nabla f_i(t,\cdot)(q)\| \le L$ for all $q \in U_\rho(C(t))$.

Assumption A2 There is $\mu > 0$ with the property that for all $t \in [0,T]$ and $q \in C(t)$ there exists $\nu = \nu(t,q) \in \mathbb{R}^d$ with $\|\nu\| = 1$ such that for all $i \in \{1,\dots,m\}$, one has

$$\big\langle \nabla f_i(t,\cdot)(q), \nu \big\rangle \le -\mu. \tag{3}$$

Remark 3.1 From Assumption A1(i), it follows that

(i) For each $i \in \{1,\dots,m\}$, for all $t,t' \in [0,T]$ and $q,q' \in \mathbb{R}^d$,

$$\left|\partial f_i(\cdot,q)(t) - \partial f_i(\cdot,q')(t')\right| \le L\left(\left|t-t'\right| + \left\|q-q'\right\|\right);$$

(ii) for each $i \in \{1,\dots,m\}$, for all $t,t' \in [0,T]$, $q,q' \in U_\rho(C(t))$,

$$\left\|\nabla f_i(t,\cdot)(q) - \nabla f_i(t',\cdot)(q')\right\| \le L\left(\left|t-t'\right| + \left\|q-q'\right\|\right).$$

Remark 3.2 From Assumptions A1 and A2, it follows that for all $i \in \{1,\dots,m\}$, $\mu \le \|\nabla f_i(t,\cdot)(q)\| \le L$ for all $t \in [0,T]$ and $|\partial f_i(\cdot,q)(t)| \le L$ for all $q \in U_\rho(C(t))$. In particular, $\nabla f_i(t,\cdot)(q) \neq 0$ for all $i \in \{1,\dots,m\}$.

We are going to present some characterizations of the set of admissible positions $C(t)$ and the Clarke's normal cone $\mathcal{N}_{C(t)}(q)$. Thanks to Assumptions A1 and A2, the following proposition is valid.

Proposition 3.1 (See [2, Theorem 3.1]) *Suppose that Assumptions* A1(i)–(ii) *and* A2 *hold, then, for all* $t \in [0,T]$, *the set* $C(t)$ *is r-prox-regular with* $r = \min\{\rho, \frac{\mu}{\gamma}\}$.

Following the technique used in [1], we obtain the following proposition, which gives sufficient conditions to obtain Lipschitz continuity of the moving constraint set with respect to the Hausdorff distance.

Proposition 3.2 *Under Assumptions* A1(i) *and* A2, $C(\cdot)$ *is* ϑ*-Lipschitzian on* $[0,T]$, *with* $\vartheta \ge \frac{L}{\mu}$.

Proof Fix a real number ϑ such that $\vartheta \ge \mu^{-1}L$. Choose a subdivision

$$0 < T_1 < \cdots < T_p = T$$

of $[0,T]$ such that $T_k - T_{k-1} < \frac{1}{\vartheta}\rho$. Fix any k and select $s,t \in I_k := [T_{k-1},T_k]$. Then, take any $i \in \{1,\dots,m\}$. Put $u(s,t) = \vartheta|s-t|$. For any $x \in C(t)$, define $y := x + u(s,t)v$. Since $t,s \in I_k$, we have $\|y-x\| = \vartheta|s-t| < \rho$. This proves that $y \in \operatorname{int}(U_\rho(C(t)))$. By Lemma 2.1, for all $\lambda \in [0,1]$ we have

$$x(\lambda) = x + \lambda(y-x) \in \operatorname{int} U_\rho\left(C(t)\right).$$

Now, we consider the expression $f_i(t,x+u(s,t)v) - f_i(t,x)$. Since $f_i(s,\cdot)$ is differentiable on $U_\rho(C(t))$, by the mean-value theorem there exists $\lambda \in (0,1)$ such that

$$f_i\left(t,x+u(s,t)v\right) - f_i(t,x) = \left\langle \nabla f_i(t,\cdot)(x_\lambda), u(s,t)v\right\rangle,$$

with $x_\lambda = \lambda x + (1-\lambda)(x+u(s,t)v)$. Hence, by Remark 3.1, we have

$$\begin{aligned} f_i\left(s,x+u(s,t)v\right) = {} & \left[f_i\left(s,x+u(s,t)v\right) - f_i\left(t,x+u(s,t)v\right)\right] + f_i(t,x) \\ & + \left[f_i\left(t,x+u(s,t)v\right) - f_i(t,x)\right] \end{aligned}$$

$$\leq L|s-t| + f_i(t,x) + \langle \nabla f_i(t,\cdot)(x_\lambda), u(s,t)\nu \rangle.$$

By (3) and the inclusion $x \in C(t)$ we obtain

$$f_i\big(s, x + u(s,t)\nu\big) \leq L|s-t| - u(s,t)\mu = (L - \vartheta\mu)|s-t| \leq 0,$$

where the inequality is valid due to the choice of ϑ. Since $i \in \{1,\dots,m\}$ can be chosen arbitrarily, this implies that the vector $x + u(s,t)\nu = x + \vartheta|s-t|\nu$ belongs to $C(s)$. Hence, $x \in C(s) + \vartheta|s-t|(-\nu)$. It follows that

$$C(t) \subset C(s) + \vartheta|s-t|(-\nu) \subset C(s) + \vartheta|s-t|\mathbb{B}.$$

Thus, $C(\cdot)$ is ϑ-Lipschitzian on $[T_{k-1}, T_k]$. Hence, we can infer that $C(\cdot)$ is ϑ-Lipschitzian on $[0,T]$. □

4 An existence result for the vibroimpact problem

The approximate solutions will be constructed by the following time-discretization scheme. Let N be a positive natural number and $h = T/N$, we define $t_n = nh$ for all $0 \leq n \leq N$ and

1. $Q_{-1} = q_0 - hp_0$, $Q_0 = q_0$,
2. for all $n \in \{0,\dots,N\}$,

$$G_n = \int_{t_n}^{t_{n+1}} g(s, Q_n)\, ds$$

and

$$V_n = 2Q_n - Q_{n-1} + h^2 G_n, \qquad Q_{n+1} \in \operatorname*{argmin}_{x \in C(t_{n+1})} \|V_n - x\|. \tag{4}$$

Here, $\operatorname{argmin}_{x\in C(t_{n+1})} \|V_n - x\|$ denotes the solution set of the minimization problem $\min_{x\in C(t_{n+1})} \|V_n - x\|$.

In this scheme, we use the approximation

$$\ddot{q}(x) \approx \frac{q(x+h) - 2q(x) + q(x-h)}{h^2}.$$

Clearly, this leads to (4). We define the discrete velocities as

$$P_n = \frac{Q_{n+1} - Q_n}{h} \quad \text{for all } n \in \{-1,\dots,N\}.$$

The sequence of approximate solutions q_N is given by

$$q_N(t) = Q_n + (t - t_n)\frac{Q_{n+1} - Q_n}{h}, \quad t \in [t_n, t_{n+1}], \forall n \in \{0,\dots,N-1\}$$

and

$$p_N(t) = P_n = \frac{Q_{n+1} - Q_n}{h}, \quad t \in [t_n, t_{n+1}], \forall n \in \{0,\dots,N-1\}.$$

For the existence of a solution to our problem we will need the following assumptions:

Assumption A3 For all $q \in \mathbb{R}^d$, $g(\cdot, q)$ is measurable on $[0, T]$ and for all $t \in [0, T]$, $g(t, \cdot)$ is continuous on $\mathbb{R}^d$. Moreover, there exist $L_g > 0$ and $F \in L^1(0, T; \mathbb{R})$ such that for almost every $t \in [0, T]$ one has

$$\|g(t,q) - g(t,\tilde{q})\| \le L_g \|q - \tilde{q}\| \quad \forall (q,\tilde{q}) \in (\mathbb{R}^d)^2 \text{ s.t. } (t,q) \in U_\rho(C), (t,\tilde{q}) \in U_\rho(C),$$
$$\|g(t,q)\| \le F(t) \quad \forall q \in \mathbb{R}^d \text{ s.t. } (t,q) \in U_\rho(C).$$

Assumption A4 For all $t \in [0, T]$, $q \in U_\rho(C(t))$, and for all $j, k \in J(t,q)$ and $j \neq k$, one has

$$\langle \nabla f_j(t,\cdot)(q), \nabla f_k(t,\cdot)(q) \rangle \ge 0.$$

Proposition 4.1 *Under Assumptions A1*(i) *and A2, for any $t \in I$ and $q \in C(t)$, the Clarke normal cone to $C(t)$ at q can be computed by the formula*

$$\mathcal{N}_{C(t)}(q) = \begin{cases} \{0\} & \text{if } q \in \operatorname{int}(C(t)), \\ \{w \in \mathbb{R}^d \mid w = \sum_{i \in J(t,q)} \lambda_i \nabla f_i(t,\cdot)(q), \lambda_i \ge 0\} & \text{if } q \in \partial C(t). \end{cases}$$

Proof If $q \in \operatorname{int}(C(t))$, then the Clarke tangent cone is equal to the whole space $\mathbb{R}^d$. Therefore, $\mathcal{N}_{C(t)}(q) = \{0\}$. Now, we consider the case when q is on the boundary $\partial C(t)$ of $C(t)$. Then, $J(t,q) \neq \emptyset$. From Assumption A2 it follows that $\{\nabla f_i(t,\cdot)(q) \mid i \in J(t,q)\}$ is positively linearly independent. Hence, by Proposition 2.1 we obtain the desired formula for $\mathcal{N}_{C(t)}(q)$. □

From Proposition 4.1 we can deduce the next formula for computing the corresponding Clarke tangent cone:

$$\mathcal{T}_{C(t)}(q) = \{v \in \mathbb{R}^d \mid \langle \nabla f_i(t,\cdot)(q), v \rangle \le 0, \forall i \in J(t,q)\}. \tag{5}$$

Lemma 4.1 *Let $t \in [0, T]$, $q \in C(t)$ and $v = v(t,q)$ be the vector that exists by Assumption A2. There exist $\rho' > 0$, $\tau \in (0, \rho']$ and $\theta \in (0, \rho']$ such that for all $t' \in I$, $|t' - t| \le \tau$, and for all q' from the open ball $\mathbb{B}(q, \theta)$ centered at q with radius θ,*

$$\langle \nabla f_i(t',\cdot)(q'), v \rangle \le -\frac{\mu}{3}, \quad \forall i \in \{1, \dots, m\}.$$

Proof Let $q \in C(t)$, v be defined in A2. For all $t' \in I$, $q' \in \mathbb{R}^d$ such that $\|q' - q\| \le \rho$, and for any $i \in \{1, \dots, m\}$, by Remark 3.1(ii) we have

$$\begin{aligned} \langle \nabla f_i(t',\cdot)(q') - \nabla f_i(t,\cdot)(q), v \rangle &\le \|\nabla f_i(t',\cdot)(q') - \nabla f_i(t,\cdot)(q)\| \|v\| \\ &\le L(|t - t'| + \|q - q'\|). \end{aligned}$$

Hence,

$$\langle \nabla f_i(t',\cdot)(q'), v \rangle \le -\mu + L(|t - t'| + \|q - q'\|).$$

Choose $\tau = \theta = \min\{\mu/3L, \rho\}$. Then, we have $\langle \nabla f_i(t',\cdot)(q'), v \rangle \le -\frac{\mu}{3}$. □

Our main result is the next theorem.

Theorem 4.1 *Suppose that Assumptions A1–A3 hold. Let $(q_0, p_0) \in C(0) \times \mathcal{T}(0, q_0)$. Then, there is a subsequence of $\{q_N\}$, still denoted by $\{q_N\}$, of the approximate solutions that converges uniformly on $[0, T]$ to a limit q satisfying* (P1)–(P3). *Furthermore, if Assumption A4 holds, then q also satisfies* (P4) *and* (P5), *and it is a solution of problem $(\mathcal{P})$ on $[0, T]$.*

To make the proof of this theorem easier to understand, we present it in the forthcoming three subsections.

4.1 Convergence of the approximate solutions

In this subsection, we shall prove that the discrete sequence $\{q_N\}$ constructed in the latter section converges to a limit, which will later be verified to be a solution of problem $(\mathcal{P})$. More precisely, we will prove that $\{p_N\}$ is uniformly bounded and it has bounded variation in Propositions 4.2 and 4.3.

Lemma 4.2 *For all $n \in \{0, \dots, N-1\}$, one has*

$$P_{n-1} - P_n + hG_n \in \mathcal{N}_{C(t_{n+1})}(Q_{n+1}). \tag{6}$$

Proof By definition of the scheme, for all $x \in C(t_{n+1})$, we have

$$\begin{aligned}\|V_n - Q_{n+1}\|^2 &\le \|V_n - x\|^2 \\ &= \|V_n - Q_{n+1}\|^2 + 2\langle V_n - Q_{n+1}, Q_{n+1} - x\rangle + \|Q_{n+1} - x\|^2.\end{aligned}$$

Hence,

$$2\langle V_n - Q_{n+1}, x - Q_{n+1}\rangle \le \|Q_{n+1} - x\|^2.$$

By definition, $V_n - Q_{n+1} = h(P_{n-1} - P_n + hG_n)$, hence

$$\langle P_{n-1} - P_n + hG_n, x - Q_{n+1}\rangle \le \frac{1}{2h}\|Q_{n+1} - x\|^2, \quad \forall x \in C(t_{n+1}). \tag{7}$$

If $Q_{n+1} \in \operatorname{int}(C(t_{n+1}))$, we can choose $\varepsilon > 0$ sufficiently small such that $x_1 = Q_{n+1} + \varepsilon E$ and $x_2 = Q_{n+1} - \varepsilon E$ belong to $C(t_{n+1})$, where $E = (1, \dots, 1) \in \mathbb{R}^d$. Then, we have

$$P_{n-1} - P_n + hG_n = 0.$$

Otherwise $J(t_{n+1}, Q_{n+1}) \neq \emptyset$. We know that by (5), the Clarke's tangent cone of $C(t_{n+1})$ at Q_{n+1} is

$$\mathcal{T}_{C(t_{n+1})}(Q_{n+1}) = \left\{w \in \mathbb{R}^d \mid \left\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), w\right\rangle \le 0, \forall i \in J(t_{n+1}, Q_{n+1})\right\}.$$

Hence, we need to show that

$$\langle P_{n-1} - P_n + hG_n, w\rangle \le 0, \quad \forall w \in \mathcal{T}_{C(t_{n+1})}(Q_{n+1}).$$

Indeed, by Assumption A2, $\operatorname{int}(\mathcal{T}_{C(t_{n+1})}(Q_{n+1})) \neq \emptyset$. Note that

$$\operatorname{int}\big(\mathcal{T}_{C(t_{n+1})}(Q_{n+1})\big) = \big\{w \in \mathbb{R}^d \mid \big\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), w\big\rangle < 0, \forall i \in J(t_{n+1}, Q_{n+1})\big\}.$$

Take any $\bar{w} \in \operatorname{int}(\mathcal{T}_{C(t_{n+1})}(Q_{n+1}))$. We will prove that $Q_{n+1} + s\bar{w} \in C(t_{n+1})$ for $s > 0$ sufficiently small. For any $s \geq 0$, there exists $q_\lambda := Q_{n+1} + \lambda s\bar{w}$ with $\lambda \in (0,1)$, such that

$$f_i(t_{n+1}, Q_{n+1} + s\bar{w}) - f_i(t_{n+1}, Q_{n+1}) = \big\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1} + \lambda s\bar{w}), s\bar{w}\big\rangle.$$

For s small enough such that $\|s\bar{w}\| \leq \rho$, we have $Q_{n+1} + s\bar{w} \in U_\rho(C(t_{n+1}))$. By Remark 3.1(ii),

$$\big\| \nabla f_i(t_{n+1},\cdot)(Q_{n+1} + \lambda s\bar{w}) - \nabla f_i(t_{n+1},\cdot)(Q_{n+1})\big\| \leq \lambda s L \|\bar{w}\|.$$

Then, $\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1} + \lambda s\bar{w}) - \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), s\bar{w}\rangle \leq \lambda L s^2 \|\bar{w}\|^2$. Hence,

$$\big\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1} + \lambda s\bar{w}), s\bar{w}\big\rangle \leq \lambda L s^2 \|\bar{w}\| + s\big\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), \bar{w}\big\rangle.$$

Since $\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), \bar{w}\rangle < 0$, we can choose s small enough such that

$$f_i(t_{n+1}, Q_{n+1} + s\bar{w}) \leq 0.$$

This implies that $Q_{n+1} + s\bar{w} \in C(t_{n+1})$. Now, we choose $x = Q_{n+1} + s\bar{w}$ satisfying $x \in C(t_{n+1})$, by (7) we obtain

$$\langle P_{n-1} - P_n + hG_n, s\bar{w}\rangle \leq \frac{1}{2h}\|s\bar{w}\|^2.$$

Letting $s \to 0$, one has

$$\langle P_{n-1} - P_n + hG_n, \bar{w}\rangle \leq 0.$$

By Assumption A2, there exits a unit vector $\nu(t_{n+1}, Q_{n+1}) \in \operatorname{int}(\mathcal{T}_{C(t_{n+1})}(Q_{n+1}))$. Therefore, for all $\nu \in \mathcal{T}_{C(t_{n+1})}(Q_{n+1})$, the sequence $\{\nu_k\}_{k\in\mathbb{N}^*}$, which is defined by

$$\nu_k = \nu + \frac{1}{k}\nu(t_{n+1}, Q_{n+1})$$

for all $k \geq 1$, converges to ν. We also see that $\nu_k \in \operatorname{int}(\mathcal{T}_{C(t_{n+1})}(Q_{n+1}))$ for all $k \geq 1$. Hence, $\operatorname{int}(\mathcal{T}_{C(t_{n+1})}(Q_{n+1}))$ is dense in $\mathcal{T}_{C(t_{n+1})}(Q_{n+1})$. This leads to

$$\langle P_{n-1} - P_n + hG_n, w\rangle \leq 0, \forall w \in \mathcal{T}_{C(t_{n+1})}(Q_{n+1}),$$

which implies that $P_{n-1} - P_n + hG_n \in \mathcal{N}_{C(t_{n+1})}(Q_{n+1})$. □

Remark 4.1 One can reformulate (6) as follows: For all $n \in \{0,\ldots,N-1\}$, there exist non-negative real numbers λ_i^n, $i = 1,\ldots,m$ such that $\lambda_i^n = 0$ for all $i \notin J(t_{n+1}, Q_{n+1})$, and

$$P_n - P_{n-1} - hG_n = -\sum_{i=1}^{m} \lambda_i^n \nabla f_i(t_{n+1},\cdot)(Q_{n+1}). \tag{8}$$

Lemma 4.3 *For each $i \in J(t_{n+1}, Q_{n+1})$ and $\|P_n\| \le \frac{\rho N}{2T}$, one has*

$$L + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \rangle \ge -\gamma h \|P_n\|^2. \tag{9}$$

Proof For all $i \in J(t_{n+1}, Q_{n+1})$, we have $f_i(t_{n+1}, Q_{n+1}) = 0 \ge f_i(t_n, Q_n)$. Thus,

$$\begin{aligned} 0 &\ge f_i(t_n, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &= f_i(t_n, Q_n) - f_i(t_{n+1}, Q_n) + f_i(t_{n+1}, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &\ge -hL - h\langle \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), P_n \rangle, \end{aligned}$$

where $q^n_{\alpha_i} = \alpha_i Q_n + (1-\alpha_i) Q_{n+1}$ for some $\alpha_i \in (0,1)$. It follows that

$$\begin{aligned} \Theta &:= L + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \rangle \\ &\ge \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), P_n \rangle \\ &= \left\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), \frac{Q_{n+1} - q^n_{\alpha_i}}{\alpha_i h} \right\rangle \\ &\ge \frac{1}{h} \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), Q_{n+1} - q^n_{\alpha_i} \rangle. \end{aligned}$$

Since $\|P_n\| \le \frac{\rho N}{2T}$, by Lemma 2.1 we know that $q^n_{\alpha_i} \in U_\rho(C(t_{n+1}))$. By Assumption A1(ii), we obtain (9). □

Lemma 4.4 *Let $N > N^0$, where $N^0 = \max\{\frac{T}{2}, \frac{6TL}{\mu\theta}\}$. Then, for all $n \in \{0, \dots, N-1\}$, we have*

$$\|P_n\| \le 2\|P_{n-1}\| + 2h\|G_n\| + \frac{6L}{\mu}.$$

Proof Let $w = \frac{6L}{\mu} v(t_n, Q_n)$, where $v(t_n, Q_n)$ is the unit vector defined in Assumption A2 for $(t,x) = (t_n, Q_n)$, i.e., for all $i \in \{1, \dots, m\}$, one has $\langle \nabla f_i(t_n, \cdot)(Q_n), v(t_n, Q_n) \rangle \le -\mu$. Then,

$$Q_n + hw \in C(t_{n+1}).$$

Indeed, by Remark 3.2 and the mean-value theorem, we have

$$f_i(t_{n+1}, Q_n + hw) \le f_i(t_n, Q_n + hw) + L|t_{n+1} - t_n|.$$

By the mean-value theorem, there exists $q^n_\alpha = \alpha Q_n + (1-\alpha)(Q_n + hw)$ with $\alpha \in (0,1)$, such that

$$f_i(t_n, Q_n + hw) - f_i(t_n, Q_n) = \langle \nabla f_i(t_n, \cdot)(q^n_\alpha), hw \rangle.$$

Since $N \ge \frac{6TL}{\mu\theta}$, $q^n_\alpha \in B(Q_n, \theta)$. By Lemma 4.1, we have

$$\langle \nabla f_i(t_n, \cdot)(q^n_\alpha), w \rangle \le \frac{-\mu}{3} \frac{6L}{\mu} = -2L.$$

Therefore, for all $i \in \{1, \dots, m\}$,

$$f_i(t_{n+1}, Q_n + hw) \leq f_i(t_n, Q_n) + \langle \nabla f_i(t_n, \cdot)(q_\alpha^n), hw \rangle + hL \leq 0.$$

We have proved that $Q_n + hw \in C(t_{n+1})$. As $Q_{n+1} \in \operatorname{argmin}_{x \in C(t_{n+1})} \|V_n - x\|$, it follows that

$$\|2Q_n - Q_{n-1} + h^2 G_n - Q_{n+1}\| \leq \|2Q_n - Q_{n-1} + h^2 G_n - Q_n - hw\|.$$

Thus, $\|P_{n-1} - P_n + hG_n\| \leq \|P_{n-1} - w + hG_n\|$. Hence, we obtain $\|P_n\| \leq 2\|P_{n-1}\| + 2h\|G_n\| + \|w\|$, which yields the conclusion. □

Proposition 4.2 *There exist $N^1 > N^0$ and $\kappa > 0$ such that*

$$\|P_n\| \leq \kappa \quad \forall n \in \{0, \dots, N-1\}, \forall N > N^1.$$

Proof We now define two real sequences $\{\kappa_k\}_{k \in \mathbb{N}}$ and $\{\tau_k\}_{k \in \mathbb{N}^*}$ by setting $\kappa_0 = \|p_0\| + 1$,

$$\begin{aligned}\kappa_k &= \kappa_{k-1} + \frac{12L}{\mu} + \|F\|_{L^1(0,T;\mathbb{R}^d)} \\ &= \kappa_0 + k\left(\frac{12L}{\mu} + \|F\|_{L^1(0,T;\mathbb{R}^d)}\right) \quad \forall k \geq 1\end{aligned}$$

and

$$\tau_k = \frac{\min\{\tau, \theta\}}{2\kappa_k} = \frac{\min\{\tau, \theta\}}{2\kappa_0 + 2k(\frac{2L}{\mu} + \|F\|_{L^1(0,T;\mathbb{R}^d)})} \quad \forall k \geq 1.$$

It is easy to see that the series $\sum_{k=1}^\infty \tau_k$ is a divergent sum, hence, there exists $k_0 \geq 1$ such that $\sum_{k=1}^{k_0} \tau_k > T$. Let $\kappa = \kappa_{k_0}$. Define

$$\bar{\kappa} = 2\kappa + 2\|F\|_{L^1(0,T;\mathbb{R}^d)} + \frac{6L}{\mu}$$

and

$$N^1 = \max\left(N^0, \frac{2T\bar{\kappa}}{\rho}, \frac{2T\bar{\kappa}}{\theta}, \frac{2T}{\tau}, \frac{2\gamma\bar{\kappa}^2 T}{L}\right).$$

We now prove that for all $N > N^1$ and we can construct a finite family of real numbers $(\tau_k^N)_{1 \leq k \leq k_0}$ such that $\tau_0^N = 0 < \tau_1^N < \dots < \tau_{k_0^N}^N = T$ with $1 \leq k_0^N \leq k_0$ and for all $k \in \{1, \dots, k_0^N\}$, in each interval $[\tau_{k-1}^N, \tau_k^N)$, one has

$$\|P_n\| \leq \kappa_k \quad \forall n \in \{0, \dots, N-1\}.$$

Consider the interval $[0, \tau_1]$ instead of $[0, T]$. From Assumption A2, we can define a vector $w_0 = \frac{6L}{\mu} v(t_0, Q_0)$. Note that $\|P_{-1}\| = \|p_0\| \leq \kappa_0 \leq \kappa$, by Lemma 4.4 we have $\|P_0\| \leq \bar{\kappa}$. Since $0 < h = \frac{T}{N} \leq \frac{\theta}{2\bar{\kappa}}$,

$$\|Q_1 - Q_0\| = h\|P_0\| \leq \frac{\theta}{2} < \theta.$$

Moreover, $|t_1 - t_0| \leq h \leq \tau/2 < \tau$, we have $(t_1, Q_1) \in \mathbb{B}(t_0,\tau) \times \mathbb{B}(Q_0,\theta)$. We will prove that $w_0 - P_0 \in \mathcal{T}_{C(t_1)}(Q_1)$. Indeed, for all $i \in J(t_1, Q_1)$, by Lemma 4.3 one has

$$\begin{aligned}\langle \nabla f_i(t_1,\cdot)(Q_1), w_0 - P_0\rangle &= \langle \nabla f_i(t_1,\cdot)(Q_1), w_0\rangle + L - \left(L + \langle \nabla f_i(t_1,\cdot)(Q_1), P_0\rangle\right)\\ &\leq \frac{-\mu\|w_0\|}{3} + L + \gamma h\|P_0\|^2\\ &\leq -2L + L + \gamma h\bar{\kappa}^2 \leq -\frac{L}{2}.\end{aligned}$$

From the latter inequality, it follows that $w_0 - P_0 \in \mathcal{T}_{C(t_1)}(Q_1)$. Since $P_{-1} - P_0 + hG_0 \in \mathcal{N}_{C(t_0)}(Q(0))$, we obtain

$$\langle (P_{-1} - w_0) - (P_0 - w_0) + hG_0, w_0 - P_0\rangle \leq 0.$$

This yields $\langle P_{-1} - w_0 + hG_0, w_0 - P_0\rangle \leq -\|P_0 - w_0\|^2$, which implies that

$$\|P_0 - w_0\| \leq \|P_{-1} - w_0\| + h\|G_0\|.$$

Hence,

$$\|P_0\| \leq \|P_{-1}\| + \frac{12L}{\mu} + h\|G_0\| \leq \kappa_1 \leq \kappa.$$

Next, we will prove by induction that

$$\|P_n - w_0\| \leq \|P_{-1} - w_0\| + h\sum_{\ell=0}^{n}\|G_\ell\| \quad \forall n \in \{0,\dots,N-1\}.$$

Indeed, let $n \in \{0,\dots,N-1\}$. Suppose that

$$\|P_k - w_0\| \leq \|P_{-1} - w_0\| + h\sum_{\ell=0}^{k}\|G_\ell\| \quad \forall k \in \{0,\dots,n-1\}.$$

Then,

$$\|P_k\| \leq 2\|w_0\| + \|P_{-1}\| + h\sum_{\ell=0}^{k}\|G_\ell\| \leq \kappa_1 \quad \text{for all } k \in \{0,\dots,n-1\}$$

and by Lemma 4.4 we infer that $\|P_n\| \leq \bar{\kappa}$. Since $0 < h \leq \frac{\theta}{2\bar{\kappa}}$,

$$\|Q_{n+1} - Q_n\| = h\|P_n\| \leq \frac{\theta}{2} < \theta.$$

Moreover, as $|t_{n+1} - t_n| \leq h < \tau$, we have $(t_{n+1}, Q_{n+1}) \in B(t_n,\tau) \times B(Q_n,\theta)$. For all $i \in J(t_{n+1}, Q_{n+1})$, by Lemma 4.3 one has

$$\begin{aligned}\Theta_0 &:= \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), w_0 - P_n\rangle\\ &= \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), w_0\rangle + L - \left(L + \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), P_n\rangle\right)\end{aligned}$$

$$\begin{aligned}&\le \frac{-\mu\|w_0\|}{3} + L + \gamma h\|P_n\|^2\\&\le -2L + L + \gamma h\bar{\kappa}^2 \le -\frac{L}{2}.\end{aligned}$$

It follows that $w_0 - P_n \in \mathcal{T}_{C(t_{n+1})}(Q_{n+1})$. Therefore,

$$\langle (P_{n-1} - w_0) - (P_n - w_0) + hG_n, w_0 - P_n\rangle \le 0.$$

This yields

$$\|P_n - w_0\| \le \|P_{n-1} - w_0\| + h\|G_n\| \le \|P_{-1} - w_0\| + h\sum_{l=0}^{n}\|G_\ell\|.$$

Hence,

$$\|P_n\| \le \|P_{-1}\| + \frac{12L}{\mu} + h\sum_{l=0}^{n}\|G_\ell\| \le \kappa_1.$$

We have shown that $\|P_n\| \le \kappa_1$ for all $n \in \{0,\dots,N\}$ on the interval $[0,\tau_1]$. Putting $\tau_0^N = 0$, we define $\tau_1^N = \min\{\tau_0^N + \tau_1, T\}$. If $\tau_0^N + \tau_1 < T$, we have $\tau_1^N - \tau_0^N = \tau_1$. If $T > \tau_1^N$, then $k_0 > 1$, $(t_{N+1}, Q_{N+1}) \in C$ and $\|P_{N+1}\| \le \kappa_1 \le \kappa$.

Assume now that $\tau_0^N + \tau_1 < T$. By Lemma 4.1 and Assumption A2, we can define a vector $w_1 = \frac{6L}{\mu}\nu(t_{N+1}, Q_{N+1})$. For the sake of simplicity, we will recount the index from 0 instead of $N+1$. By the same argument, we can prove that $\|P_n\| \le \kappa_2$ for all $n \in \{0,\dots,N-1\}$ on the interval $[\tau_1^N, \tau_1^N + \tau_2]$. We now can divide the interval $[0,T]$ into subintervals $[\tau_i^N, \tau_i^N + \tau_{i+1}]$ for $i \in \{1,\dots,k_0\}$. Repeating the same argument for finitely many steps, we obtain the desired result. □

Proposition 4.3 *There exists $\kappa' > 0$ such that, for all $N > N^1$, we have*

$$\sum_{n=0}^{N-1}\|P_n - P_{n-1}\| \le \kappa'.$$

Proof We decompose $[0,T]$ into the subintervals $[\tau_k^N, \tau_{k+1}^N]$, $k \in \{0,\dots,k_0^h - 1\}$, which were defined in the proof of Proposition 4.2. Consider the interval $[\tau_0^N, \tau_1^N]$. We have shown that

$$w_0 - P_n \in \mathcal{T}_{C(t_{n+1})}(Q_{n+1})$$

for all $n \in \{0,\dots,N-1\}$. We now prove that the closed ball $\bar{\mathbb{B}}(w_0 - P_n, \frac{1}{2}) \subset \mathcal{T}_{C(t_{n+1})}(Q_{n+1})$. Indeed, let $a \in \bar{\mathbb{B}}(w_0 - P_n, \frac{1}{2})$. Then, $\|a - (w_0 - P_n)\| \le \frac{1}{2}$. As in the proof of Proposition 4.2, one has $\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), w_0 - P_n\rangle \le -\frac{L}{2}$ for all $n \in \{0,\dots,N-1\}$. Then,

$$\begin{aligned}\Theta_1 &:= \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), a\rangle\\&= \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), a - (w_0 - P_n)\rangle + \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), w_0 - P_n\rangle\\&\le \|\nabla f_i(t_{n+1},\cdot)(Q_{n+1})\|\,\|a - (w_0 - P_n)\| - \frac{L}{2} \le 0.\end{aligned}$$

This proves that $a \in \mathcal{T}_{C(t_{n+1})}(Q_{n+1})$. Since the tangent cone $\mathcal{T}_{C(t_{n+1})}(Q_{n+1})$ is closed and convex [13, p. 51], for every $x \in \mathbb{R}^d$, by [20, Lemma 4.3, p. 22] we have

$$\left\| x - \mathbb{P}_{\mathcal{T}_{C(t_{n+1})}(Q_{n+1})}(x) \right\| \le \| x - w_0 + P_n \|^2 - \left\| \mathbb{P}_{\mathcal{T}_{C(t_{n+1})}(Q_{n+1})}(x) - w_0 + P_n \right\|^2.$$

Applying this with $x = P_{n-1} - P_n + hG_n$, we obtain

$$\| P_{n-1} - P_n + hG_n - \bar{P} \| \le \| P_{n-1} - P_n + hG_n - w_0 + P_n \|^2 - \| \tilde{P} - w_0 + P_n \|^2,$$

where $\bar{P} = \mathbb{P}_{\mathcal{T}_{C(t_{n+1})}(Q_{n+1})}(P_{n-1} - P_n + hG_n)$. It follows that

$$\begin{aligned} &\| P_{n-1} - P_n + hG_n - \bar{P} \| \\ &\quad \le \| P_{n-1} + hG_n - w_0 \|^2 - \| \bar{P} - w_0 + P_n \|^2. \end{aligned}$$

Recall that $P_{n-1} - P_n + hG_n \in \mathcal{N}_{C(t_{n+1})}(Q_{n+1})$ (see Lemma 4.2). Since $\mathcal{N}_{C(t_{n+1})}(Q_{n+1})$ is the dual cone of $\mathcal{T}_{C(t_{n+1})}(Q_{n+1})$, $\bar{P} = 0$. We obtain

$$\begin{aligned} \Theta_2 &:= \| P_{n-1} - P_n \| \\ &= \| P_{n-1} - P_n + hG_n - hG_n \| \\ &\le h\|G_n\| + \| P_{n-1} - P_n + hG_n \| \\ &= h\|G_n\| + \left\| (P_{n-1} - P_n + hG_n) - \bar{P} \right\| \\ &\le h\|G_n\| + \| P_{n-1} - P_n + hG_n \|^2 - \| P_n - w_0 \|^2 \\ &\le h\|G_n\| + \| P_{n-1} - w_0 \|^2 - \| P_n - w_0 \|^2 + h^2 \|G_n\|^2 + 2h \langle G_n, P_{n-1} - w_0 \rangle \\ &\le h\|G_n\| + \| P_{n-1} - w_0 \|^2 - \| P_n - w_0 \|^2 + h^2 \|G_n\|^2 + 2h \|G_n\| \| P_{n-1} - w_0 \| \\ &= \bigl(1 + h\|G_n\| + 2\| P_{n-1} - w_0 \|\bigr) h\|G_n\| + \| P_{n-1} - w_0 \|^2 - \| P_n - w_0 \|^2 \\ &\le \bigl(1 + h\|G_n\| + 2\| P_{n-1} \| + 2\|w_0\|\bigr) h\|G_n\| + \| P_{n-1} - w_0 \|^2 - \| P_n - w_0 \|^2. \end{aligned}$$

It follows that $\| P_{n-1} - P_n \| \le h(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2\kappa + \frac{12L}{\mu}) \|G_n\| + \| P_{n-1} - w_0 \|^2 - \| P_n - w_0 \|^2$ for $n = 0, \dots, N-1$. Adding these inequalities, we obtain

$$\begin{aligned} \Theta_3 &:= \sum_{n=0}^{N-1} \| P_{n-1} - P_n \| \\ &\le \left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2\kappa + \frac{12L}{\mu} \right) \sum_{n=0}^{N-1} h\|G_n\| + \| P_0 - w_0 \|^2 - \| P_N - w_0 \|^2 \\ &\le T \left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2\kappa + \frac{12L}{\mu} \right) \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2 \left(\kappa + \frac{6L}{\mu} \right)^2. \end{aligned}$$

Similarly, we can obtain the same result for all the subintervals $[\tau_i^N, \tau_{i+1}^N]$, where $i \in \{1, \dots, k_0\}$. Since the number of the subintervals $[\tau_i^N, \tau_{i+1}^N]$ is finite, the proof is complete. □

From Propositions 4.2 and 4.3 we can infer that the sequence $\{q_N\}$ is uniformly Lipschitz continuous and that the sequence $\{p_N\}$ is uniformly bounded in $L^\infty(0,T;\mathbb{R}^d)$ and in $BV([0,T];\mathbb{R}^d)$. For any $t \in [0,T]$, it is clear that $q_N(t)$ is bounded for all N. Moreover, since p_N is the derivative of q_N, by Proposition 2.2, there exists a subsequence of $\{q_N\}$, still denoted by $\{q_N\}$, converging uniformly to an absolutely continuous function q over $[0,T]$. In addition, by [20, Theorem 2.1], we can extract subsequences of $\{p_N\}$, still denoted by $\{p_N\}$ and find $p \in BV([0,T];\mathbb{R}^d)$ such that

$$p_N \to p \quad \text{pointwise in } [0,T],$$
$$dp_N \rightharpoonup dp \quad \text{weakly* in } \mathcal{M}\big(0,T;\mathbb{R}^d\big).$$

4.2 Properties of the limit trajectory

In this subsection, we will prove that the limit trajectory q satisfies the properties (P1)–(P3).

The definitions of q_N and p_N imply that

$$q_N(t) = q_0 + \int_0^t p_N(s)\,ds \quad \forall t \in [0,T]\ \forall n > N^1.$$

Passing to the limit as $N \to +\infty$, by the dominated convergence theorem [11, Theorem 4.2, p. 90] we obtain

$$q(t) = q_0 + \int_0^t p(s)\,ds \quad \forall t \in [0,T]. \tag{10}$$

Hence, $\dot{q} = p \in BV([0,T];\mathbb{R}^d)$, which implies that q is Lipschitz continuous with rank κ on $[0,T]$.

Proposition 4.4 *For all $t \in [0,T]$, $q(t) \in C(t)$.*

Proof Indeed, for all $t \in [0,T]$ and for all $N > N^1$, there exists $n \in \{0,\dots N-1\}$ such that $t \in [t_n, t_{n+1}]$. Then, for all $i \in \{1,\dots,m\}$,

$$\begin{aligned} f_i\big(t,q(t)\big) - f_i\big(t_n,q_N(t_n)\big) &= f_i\big(t,q(t)\big) - f_i\big(t,q_N(t_n)\big) + f_i\big(t,q_N(t_n)\big) - f_i\big(t_n,q_N(t_n)\big) \\ &\le L\big\|q(t) - q_N(t_n)\big\| + L|t_n - t| \\ &\le L\big\|q(t) - q_N(t_n)\big\| + hL \\ &\le L\big(\big\|q(t) - q(t_n)\big\| + \big\|q(t_n) - q_N(t_n)\big\|\big) + hL. \end{aligned}$$

Since q is Lipschitz continuous with modulus κ, we have

$$\begin{aligned} & f_i\big(t,q(t)\big) - f_i\big(t_n,q_N(t_n)\big) \\ &\quad \le L\big(\kappa(t-t_n) + \sup\big\{\big\|q(s) - q_N(s)\big\|_{\mathbb{R}^d} \mid s \in [0,T]\big\}\big) + hL \\ &\quad \le L\big(\kappa h + \|q - q_N\|_{C([0,T];\mathbb{R}^d)}\big) + hL. \end{aligned} \tag{11}$$

Since $\{q_N\}$ converges uniformly to q on $[0,T]$, $f_i(t_n,q_N(t_n)) = f_i(t_n,Q_n) \le 0$, and (11) holds for all $N > N^1$, we can conclude that $f_i(t,q(t)) \le 0$.

The proof is complete. □

We are now going to show that the limit trajectory satisfies property (P3). By the definition of p_N, the Stieltjes measure $d\dot{q}_N = dp_N$ is a sum of Dirac's measures

$$dp_N(t) = \sum_{n=0}^{N-1}(P_n - P_{n-1})\delta(t-t_n).$$

Define

$$g_N(t) = \sum_{n=0}^{N-1} hG_n\delta(t-t_n) - \sum_{n=0}^{N-1}\sum_{i=1}^{m}\lambda_i^n(\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) - \nabla f_i(t_n,\cdot)\big(q(t_n)\big)\delta(t-t_n),$$

and

$$U_N(t) = \sum_{n=0}^{N-1}\sum_{i=1}^{m}\delta(t-t_n)\lambda_i^n\nabla f_i(t,\cdot)\big(q(t)\big),$$

where the constants λ_i^n are given in Remark 4.1. Then, (8) can be rewritten as

$$dp_N(t) = -U_N(t) + g_N(t). \tag{12}$$

Lemma 4.5 *For all $i \in \{1,\dots,m\}$ and for all $N > N^1$ we have*

$$\sum_{n=0}^{N-1}\big|\lambda_i^n\big| \le \frac{1}{\mu}\big(\kappa' + \|F\|_{L^1(0,T;\mathbb{R})}\big).$$

Proof Let $i \in \{1,\dots,m\}$, $n \in \{0,\dots,N-1\}$. By (8) we have

$$\left\|\sum_{i=1}^{m}\lambda_i^n\nabla f_i(t_{n+1},\cdot)(Q_{n+1})\right\| \le \|P_n - P_{n-1}\| + h\|G_n\|.$$

By Assumption A1(ii), for fixed n, there exists v such that $\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}), v\rangle \le -\mu$. Hence,

$$\begin{aligned}\langle P_n - P_{n-1} + hG_n, v\rangle &= \left\langle\sum_{i=1}^{m}\lambda_i^n\nabla f_i(t_{n+1},\cdot)(Q_{n+1}), v\right\rangle \\ &= \sum_{i=1}^{m}\lambda_i^n\big\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}), v\big\rangle \\ &\le \sum_{i=1}^{m}\lambda_i^n(-\mu).\end{aligned}$$

For every fixed i, we have

$$\lambda_i^n \le \sum_{i=1}^{m}\lambda_i^n \le \frac{1}{\mu}\big(\|P_n - P_{n-1}\| + \|hG_n\|\big).$$

Hence,

$$\sum_{n=0}^{N-1}\left|\lambda_i^n\right| = \sum_{n=0}^{N-1}\lambda_i^n \le \frac{1}{\mu}\sum_{n=0}^{N-1}\bigl(\|P_n - P_{n-1}\| + h\|G_n\|\bigr) \le \frac{1}{\mu}\bigl(\kappa' + \|F\|_{L^1(0,T;\mathbb{R})}\bigr).$$

The proof is complete. □

Let $\Lambda_i^N(t) = \sum_{n=0}^{N-1}\lambda_i^n\delta(t-t_n)$. By the above lemma, Λ_i^N is uniformly bounded, then there exists a subsequence of $\{\Lambda_i^N\}$ converging weakly* to nonnegative measure Λ_i in $\mathcal{M}(0,T;\mathbb{R})$. Therefore, U_N has a subsequence that converges weakly* to U in $\mathcal{M}(0,T;\mathbb{R}^d)$ with $U(t) = \sum_{i=1}^m \Lambda_i(t)\nabla f_i(t,\cdot)(q(t))$. Since $\nabla f_i(t,\cdot)(q(t)) \in \mathcal{N}_{C(t)}(q(t))$, we obtain $U \in \mathcal{N}_{C(\cdot)}(q(\cdot))\,dt$.

Lemma 4.6 *The sequence $\{g_N\}$ converges weakly* to $g(\cdot,q)\,dt$ in $\mathcal{M}(0,T;\mathbb{R}^d)$, where $g(\cdot,q)\,dt$ is the measure of density $g(\cdot,q)$ with respect to Lebesgue's measure on* $[0,T]$.

Proof Let $\varphi \in C([0,T];\mathbb{R}^d)$. By the definition of g_N, we have

$$\begin{aligned}
\langle g_N,\varphi\rangle &= \sum_{n=0}^{N-1} h\bigl\langle G_n,\varphi(t_n)\bigr\rangle + \sum_{n=0}^{N-1}\sum_{i=1}^m \lambda_i^n\bigl\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) \\
&\quad - \nabla f_i(t_n,\cdot)\bigl(q(t_n)\bigr),\varphi(t_n)\bigr\rangle \\
&= \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\bigl\langle g(s,Q_n),\varphi(t_n)\bigr\rangle\,ds + \sum_{n=0}^{N-1}\sum_{i=1}^m\lambda_i^n\bigl\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) \\
&\quad - \nabla f_i(t_n,\cdot)\bigl(q(t_n)\bigr),\varphi(t_n)\bigr\rangle \qquad (13)\\
&= \int_0^T\bigl\langle g\bigl(s,q(s)\bigr),\varphi(s)\bigr\rangle\,ds + \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\bigl\langle g(s,Q_n) - g\bigl(s,q(s)\bigr),\varphi(s)\bigr\rangle\,ds \\
&\quad + \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\bigl\langle g(s,Q_n),\varphi(t_n)-\varphi(s)\bigr\rangle\,ds + \sum_{n=0}^{N-1}\sum_{i=1}^m\lambda_i^n\bigl\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) \\
&\quad - \nabla f_i(t_n,\cdot)\bigl(q(t_n)\bigr),\varphi(t_n)\bigr\rangle.
\end{aligned}$$

Moreover, for all $n \in \{0,\dots,N-1\}$, we have $(t_n,q(t_n)) \in C$ and

$$\begin{aligned}
\bigl\|Q_{n+1} - q(t_n)\bigr\| &\le \|Q_{n+1} - Q_n\| + \bigl\|q_N(t_n) - q(t_n)\bigr\| \\
&\le \kappa h + \|q - q_N\|_{C([0,T];\mathbb{R}^d)}.
\end{aligned}$$

Let $\varepsilon_n := \|Q_{n+1} - q(t_n)\|$. From Remark 3.1 and Lemma 4.5 it follows that

$$\begin{aligned}
\Theta_4 &:= \left\|\sum_{n=0}^{N-1}\sum_{i=1}^m\lambda_i^n\bigl\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) - \nabla f_i(t_n,\cdot)\bigl(q(t_n)\bigr),\varphi(t_n)\bigr\rangle\right\| \\
&\le \sum_{n=0}^{N-1}\sum_{i=1}^m\lambda_i^n L(h+\varepsilon_n)\bigl\|\varphi(t_n)\bigr\|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{N-1}\sum_{i=1}^{m}\lambda_i^n L\big((\kappa+1)h+\|q-q_N\|_{C([0,T];\mathbb{R}^d)}\big)\|\varphi\|_{C([0,T];\mathbb{R}^d)}\\
&\leq \frac{mL}{\mu}\big((\kappa+1)h+\|q-q_N\|_{C([0,T];\mathbb{R}^d)}\big)\|\varphi\|_{C([0,T];\mathbb{R}^d)}\\
&\quad\times\big(\mathrm{Var}\big(p_N,[0,T]\big)+\|F\|_{L^1(0,T;\mathbb{R}^d)}\big).
\end{aligned}$$

In addition,

$$\begin{aligned}
\left|\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\big\langle g(s,Q_n)-g\big(s,q(s)\big),\varphi(s)\big\rangle\,ds\right| &\leq \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} L_g\big\|Q_n-q(s)\big\|\,\big\|\varphi(s)\big\|\,ds\\
&\leq L_g\big(\kappa h+\|q-q_N\|_{C([0,T],\mathbb{R}^d)}\big)\int_0^T\big\|\varphi(s)\big\|\,ds.
\end{aligned}$$

We also have

$$\begin{aligned}
\left|\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\big\langle g(s,Q_n),\varphi(t_n)-\varphi(s)\big\rangle\,ds\right| &\leq \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\big\|g(s,Q_n)\big\|\,\big\|\varphi(t_n)-\varphi(s)\big\|\,ds\\
&\leq \omega_\varphi(h)\|F\|_{L^1([0,T];\mathbb{R}^d)},
\end{aligned}$$

where ω_φ denotes the modulus of continuity of φ. Therefore, letting N go to ∞ in (13) we obtain

$$\langle g_N,\varphi\rangle\to\int_0^T\big\langle g\big(s,q(s)\big),\varphi(s)\big\rangle\,ds.$$

The proof is complete. □

Passing (12) to the limit yields $dp-g(\cdot,q)\,dt\in-\mathcal{N}_{C(\cdot)}(q(\cdot))\,dt$.

4.3 Checking the impact law and the initial data

In this subsection, we will prove that the limit trajectory satisfies the impact law (P4) and the initial data (P5).

Lemma 4.7 *If $J(t,q)\neq\emptyset$, then $\dot{q}(t^+)\in\mathcal{T}(t,q(t))$.*

Proof Let $t\in I$ be chosen arbitrarily. Consider an index i such that $f_i(t,q(t))=0$. We have

$$\begin{aligned}
0&\geq f_i\big(t+\varepsilon,q(t+\varepsilon)\big)-f_i\big(t,q(t)\big)\\
&=\varepsilon\partial f_i\big(\cdot,q(t)\big)(t)+\big\langle\nabla f_i(t,\cdot)\big(q(t)\big),q(t+\varepsilon)-q(t)\big\rangle+o(\varepsilon).
\end{aligned}$$

Dividing both sides by ε and letting $\varepsilon\to0$, we obtain

$$\partial f_i\big(\cdot,q(t)\big)(t)+\big\langle\nabla f_i(t,\cdot)\big(q(t)\big),\dot{q}\big(t^+\big)\big\rangle\leq0.$$

We have shown that $\dot{q}(t^+)\in\mathcal{T}(t,q(t))$.

Similarly, we can prove that $\dot{q}(t^-)\in-\mathcal{T}(t,q(t))$. □

Lemma 4.8 *For each $i \in J(t_{n+1}, Q_{n+1})$ and $\|P_n\| \le \frac{\rho N}{2T}$, one has*

$$\partial f_i(\cdot, Q_{n+1})(t_{n+1}) + \big\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \big\rangle \ge -h\big(L + L\|P_n\| + \gamma \|P_n\|^2\big).$$

Proof For all $i \in J(t_{n+1}, Q_{n+1}), f_i(t_{n+1}, Q_{n+1}) = 0 \ge f_i(t_n, Q_n)$. Thus,

$$\begin{aligned} 0 &\ge f_i(t_n, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &= f_i(t_n, Q_n) - f_i(t_{n+1}, Q_n) + f_i(t_{n+1}, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &= -h\partial f_i(\cdot, Q_n)\big(t^n_\alpha\big) - h\big\langle \nabla f_i(t_{n+1}, \cdot)\big(q^n_\beta\big), P_n \big\rangle, \end{aligned}$$

where $t^n_\alpha = \alpha t_n + (1-\alpha)t_{n+1}$ and $q^n_\beta = \beta Q_n + (1-\beta)Q_{n+1}$ for some $\alpha, \beta \in (0,1)$, satisfying

$$\big\langle \partial f_i(\cdot, Q_n)\big(t^n_\alpha\big), t_n - t_{n+1} \big\rangle = f_i(t_n, Q_n) - f_i(t_{n+1}, Q_n),$$

and

$$\big\langle \nabla f_i(t_{n+1}, \cdot)\big(q^n_\beta\big), Q_n - Q_{n+1} \big\rangle = f_i(t_{n+1}, Q_n) - f_i(t_{n+1}, Q_{n+1}).$$

Hence,

$$\begin{aligned} \big\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \big\rangle &\ge -\partial f_i(\cdot, Q_n)\big(t^n_\alpha\big) + \big\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) \\ &\quad - \nabla f_i(t_{n+1}, \cdot)\big(q^n_\beta\big), P_n \big\rangle \\ &\ge -\partial f_i(\cdot, Q_n)\big(t^n_\alpha\big) + \frac{1}{\beta h}\big\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) \\ &\quad - \nabla f_i(t_{n+1}, \cdot)\big(q^n_\beta\big), Q_{n+1} - q^n_\beta \big\rangle. \end{aligned}$$

Since $h\|P_n\| \le \frac{\rho}{2}$, by Lemma 2.1 we know that $q^n_\beta \in U_\rho(C(t_{n+1}))$. Therefore, by Remark 3.1(i),

$$\begin{aligned} \big\| \partial f_i(\cdot, Q_{n+1})(t_{n+1}) - \partial f_i(\cdot, Q_n)\big(t^n_\alpha\big) \big\| &\ge -L\big(|t_{n+1} - t_\alpha| + \|Q_{n+1} - Q_n\|\big) \\ &= -Lh\big(\alpha + \|P_n\|\big) \\ &\ge -Lh\big(1 + \|P_n\|\big). \end{aligned}$$

Then, by Assumption A1(ii), one has

$$\begin{aligned} \frac{1}{\beta h}\big\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)\big(q^n_\beta\big), Q_{n+1} - q^n_\beta \big\rangle &\ge -\frac{\gamma}{\beta h}\big\| Q_{n+1} - q^n_\beta \big\|^2 \\ &= -\gamma \beta h \|P_n\|^2 \\ &\ge -\gamma h \|P_n\|^2. \end{aligned}$$

Hence,

$$\partial f_i(\cdot, Q_{n+1})(t_{n+1}) + \big\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \big\rangle \ge -h\big(L + L\|P_n\| + \gamma \|P_n\|^2\big).$$

The proof is complete. □

Proposition 4.5 *For all $t \in (0,T)$, one has $\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q)}(\dot{q}(t^-))$.*

Proof Step 1: We consider the case that $J(t,q(t)) = \emptyset$. Since f_i are continuous for all $i \in \{1,\dots,m\}$, we may define $\rho_t \in (0, \min(\rho, t, T-t))$ such that, for all $i \in \{1,\dots,m\}$ we have

$$f_i(s,y) \le \frac{1}{2} f_i\big(t,q(t)\big) < 0 \quad \forall s \in [t-\rho_t, t+\rho_t], y \in \bar{\mathbb{B}}\big(q(t),\rho_t\big)$$

and we define $N_t > \max\{N^1, \frac{4T(\kappa+1)}{\rho_t}\}$ such that $\|q - q_N\|_{C([0,T];\mathbb{R}^d)} \le \frac{\rho_t}{4}$ for all $N > N_t$. Then, for all $\tilde{\rho} \in (0,\rho_t]$ and for all $N > N_t$, we define

$$n_- = \left\lfloor \frac{t - \frac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor + 1, \qquad n_+ = \left\lfloor \frac{t + \frac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor.$$

It follows that

$$\begin{aligned} 2h < (n_- - 1)h \le t - \frac{\tilde{\rho}}{4(\kappa+1)} < hn_- < \cdots < hn_+ \\ \le t + \frac{\tilde{\rho}}{4(\kappa+1)} < (n_+ + 1)h < T - 2h \end{aligned}$$

and

$$P_{n_- -1} = p_N\left(t - \frac{\tilde{\rho}}{4(\kappa+1)}\right), \qquad P_{n_+} = p_N\left(t + \frac{\tilde{\rho}}{4(\kappa+1)}\right).$$

By relation (8) we have

$$P_{n_+} - P_{n_- -1} = \sum_{n=n_-}^{n_+} hG_n - \sum_{n=n_-}^{n_+} \sum_{i=1}^{m} \lambda_i^n \nabla f_i(t_{n+1},\cdot)(Q_{n+1}).$$

Moreover, for all $n \in \{n_-,\dots,n_+\}$ we have $t_n = nh \in [t - \frac{\tilde{\rho}}{4(\kappa+1)}, t + \frac{\tilde{\rho}}{4(\kappa+1)}]$ and

$$\begin{aligned} |t_{n+1} - t| &\le \frac{\tilde{\rho}}{4(\kappa+1)} + h \le \frac{\rho_t}{2(\kappa+1)} < \rho_t, \\ \big\|Q_{n+1} - q(t)\big\| &\le \big\|Q_{n+1} - q_N(t)\big\| + \big\|q_N(t) - q(t)\big\| \\ &\le \kappa |t_{n+1} - t| + \|q - q_N\|_{C([0,T];\mathbb{R}^d)} < \rho_t. \end{aligned}$$

It follows that $f_i(t_{n+1},Q_{n+1}) < 0$ and $\lambda_i^n = 0$ for all $i \in \{1,\dots,m\}$ and for all $n \in \{n_-,\dots,n_+\}$. Thus,

$$\begin{aligned} \left\| p_N\left(t + \frac{\tilde{\rho}}{4(\kappa+1)}\right) - p_N\left(t - \frac{\tilde{\rho}}{4(\kappa+1)}\right)\right\| &= \left\| \sum_{n=n_-}^{n_+} hG_n \right\| \\ &\le \int_{t_{n_-}}^{t_{n_+ +1}} F(s)\,ds \\ &\le \int_{t - \frac{\tilde{\rho}}{4(\kappa+1)}}^{t + \frac{\tilde{\rho}}{4(\kappa+1)} + h} F(s)\,ds. \end{aligned}$$

Letting N go to infinity, we obtain that $\|p(t^+) - p(t^-)\| = 0$. This means that

$$\dot{q}(t^-) = p(t^-) = p(t^+) = \dot{q}(t^+).$$

Step 2: Now, let $t \in (0,T)$ be such that $J(t,q(t)) \neq \emptyset$. Consider the case if $J(t,q(t)) = \{1,\ldots,m\}$, we let $\rho_t = \frac{1}{2}\min(\rho,t,T-t)$. Otherwise, using the continuity of the mappings $f_i, i \in \{1,\ldots,m\}$ we may define ρ_t in $(0,\min(\rho,t,T-t))$ such that, for all $i \in \{1,\ldots,m\} \setminus J(t,q(t))$ we have

$$f_i(s,y) \le \frac{1}{2} f_i\big(t,q(t)\big) < 0 \quad \forall s \in [t-\rho_t, t+\rho_t], y \in \bar{B}\big(q(t),\rho_t\big).$$

Then, by the uniform convergence of (q_N) to q on $[0,T]$, we can define

$$N_t > \max\left(N^1, \frac{4T(\kappa+1)}{\rho_t}\right)$$

such that $\|q - q_N\|_{C([0,T];\mathbb{R}^d)} \le \frac{\rho_t}{4}$ for all $N > N_t$. We will show that for all $N > N_t$ and for all $t_n \in [t - \frac{\rho_t}{4(\kappa+1)}, t + \frac{\rho_t}{4(\kappa+1)}]$, $J(t_{n+1}, Q_{n+1}) \subset J(t,q(t))$. Indeed, let $N > N_t$ and $t_n \in [t - \frac{\rho_t}{4(\kappa+1)}, t + \frac{\rho_t}{4(\kappa+1)}]$. We have

$$\begin{aligned}
|t_{n+1} - t| &\le \frac{\rho_t}{4(\kappa+1)} + h \le \frac{\rho_t}{2(\kappa+1)} < \rho_t, \\
\big\|Q_{n+1} - q(t)\big\| &\le \big\|Q_{n+1} - q_N(t)\big\| + \big\|q_N(t) - q(t)\big\| \\
&\le \kappa|t_{n+1} - t| + \|q - q_N\|_{C([0,T];\mathbb{R}^d)} < \rho_t.
\end{aligned}$$

In addition, we have

$$f_i(t_{n+1}, Q_{n+1}) < 0 \quad \forall i \notin J\big(t,q(t)\big).$$

Therefore, $J(t_{n+1},Q_{n+1}) \subset J(t,q(t))$. Represent $J(t,q(t))$ as $J(t,q(t)) = J_1(t,q(t)) \cup J_2(t,q(t))$ with

$$\begin{aligned}
J_1\big(t,q(t)\big) = \bigg\{ &i \in J\big(t,q(t)\big) \mid \rho_i \in (0,\rho_t], \exists N_i > N_t, \forall N > N_i, \\
&\forall t_n \in \left[t - \frac{\rho_i}{4(\kappa+1)}, t + \frac{\rho_i}{4(\kappa+1)}\right] \cap [0,T], f_i(t_{n+1},Q_{n+1}) < 0 \bigg\}
\end{aligned}$$

and

$$\begin{aligned}
J_2\big(t,q(t)\big) = \bigg\{ &i \in J\big(t,q(t)\big) \mid \forall \rho_i \in (0,\rho_t], \forall N_i > N_t, \exists N > N_i, \\
&\exists t_n \in \left[t - \frac{\rho_i}{4(\kappa+1)}, t + \frac{\rho_i}{4(\kappa+1)}\right] \cap [0,T], f_i(t_{n+1},Q_{n+1}) = 0 \bigg\}.
\end{aligned}$$

Since $J_1(t,q(t))$ is a finite set, we may define

$$\begin{cases}
\tilde{\rho}_t = \min\{\rho_i \mid i \in J_1(t,q(t))\}, \tilde{N}_t = \max\{N_i \mid i \in J_1(t,q(t))\} & \text{if } J_1(t,q(t)) \neq \emptyset, \\
\tilde{\rho}_t = \rho_t, \tilde{N}_t = N_t & \text{if } J_1(t,q(t)) = \emptyset.
\end{cases}$$

Now, let $\tilde{\rho} \in (0, \tilde{\rho}_t]$ and $N > \tilde{N}_t$. As before, we define

$$n_- = \left\lfloor \frac{t - \frac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor + 1, \qquad n_+ = \left\lfloor \frac{t + \frac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor,$$

which implies that

$$\begin{aligned} 2h < (n_- - 1)h &\le t - \frac{\tilde{\rho}}{4(\kappa+1)} < n_- h < \cdots < n_+ h \\ &\le t + \frac{\tilde{\rho}}{4(\kappa+1)} < (n_+ + 1)h < T - 2h \end{aligned}$$

and

$$P_{n_- - 1} = p_N\left(t - \frac{\tilde{\rho}}{4(\kappa+1)}\right), \qquad P_{n_+} = p_N\left(t + \frac{\tilde{\rho}}{4(\kappa+1)}\right).$$

Thus,

$$P_{n_+} - P_{n_- - 1} = \sum_{n=n_-}^{n_+} hG_n - \sum_{n=n_-}^{n_+} \sum_{i=1}^{m} \lambda_i^n \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}).$$

Since $J(t_{n+1}, Q_{n+1}) \subset J(t, q(t))$, $i \notin J(t_{n+1}, Q_{n+1})$ implies that $i \in J_1(t, q(t))$. Thus,

$$P_{n_+} - P_{n_- - 1} = \sum_{n=n_-}^{n_+} hG_n - \sum_{i \in J_2(t,q(t))} \sum_{n=n_-}^{n_+} \lambda_i^n \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}). \tag{14}$$

If $J_2(t, q(t)) = \emptyset$ using the same arguments as in Step 1, we can obtain that $\dot{q}(t^+) = \dot{q}(t^-)$. Moreover, since $q(s) \in C(s)$ for all $s \in [0, T]$, $\dot{q}(t^+) \in \mathcal{T}(t, q(t))$. It follows that $\dot{q}(t^+) = \dot{q}(t^-) \in \mathcal{T}(t, q(t))$ and therefore we have $\dot{q}(t^-) = \dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-))$. For the case where $J_2(t, q(t)) \neq \emptyset$, we rewrite (14) as follows

$$\begin{aligned} \Theta_5 &:= p_N\left(t + \frac{\tilde{\rho}}{4(\kappa+1)}\right) - p_N\left(t - \frac{\tilde{\rho}}{4(\kappa+1)}\right) \\ &= -\sum_{i \in J_2(t,q(t))} \sum_{n=n_-}^{n_+} \lambda_i^n \nabla f_i(t, \cdot)\big(q(t)\big) + \sum_{n=n_-}^{n_+} hG_n \\ &\quad - \sum_{i \in J_2(t,q(t))} \sum_{n=n_-}^{n_+} \lambda_i^n \big(\nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t, \cdot)\big(q(t)\big)\big). \end{aligned} \tag{15}$$

Before continuing the proof, we prove the following two technical lemmas.

Lemma 4.9 *We have*

$$p\big(t^+\big) - p\big(t^-\big) \in -\sum_{i \in J_2(t,q(t))} \mathbb{R}_+ \nabla f_i(t, \cdot)\big(q(t)\big).$$

Proof We can estimate the last two terms of (15) as follows

$$\left\|\sum_{n=n_-}^{n_+} hG_n\right\| \le \int_{t_{n_-}}^{t_{n_+ +1}} F(s)\,ds \le \int_{t-\frac{\tilde\rho}{4(\kappa+1)}}^{t+\frac{\tilde\rho}{4(\kappa+1)}+h} F(s)\,ds$$

and, let $\Delta_i^n(t) = \lambda_i^n(\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) - \nabla f_i(t,\cdot)(q(t)))$, using Lemma 4.5 and Remark 3.1(ii) we have

$$\begin{aligned}
\Theta_6 &:= \left\|\sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+} \Delta_i^n(t)\right\| \\
&\le \sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+} \left\|\Delta_i^n(t)\right\| \\
&\le \sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+} \lambda_i^n L\bigl(|t_{n+1}-t| + \left\|Q_{n+1}-q(t)\right\|\bigr) \\
&\le \sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+} \lambda_i^n L\left(\left(h+\frac{\tilde\rho}{4(\kappa+1)}\right) + \|q-q_N\|_{C([0,T];\mathbb{R}^d)}\right) \\
&\le L\left(\left(h+\frac{\tilde\rho}{4(\kappa+1)}\right) + \|q-q_N\|_{C([0,T];\mathbb{R}^d)}\right) \\
&\quad \times \frac{m}{\mu}\bigl(\operatorname{Var}\bigl(p_N,[0,T]\bigr) + \|F\|_{L^1(0,T;\mathbb{R}^d)}\bigr).
\end{aligned}$$

From (15), it follows that

$$\begin{aligned}
&\lim_{\tilde\rho\to 0^+}\lim_{N\to\infty}\left\| p_N\left(t+\frac{\tilde\rho}{4(\kappa+1)}\right) - p_N\left(t-\frac{\tilde\rho}{4(\kappa+1)}\right)\right. \\
&\quad \left. + \sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+} \lambda_i^n \nabla f_i(t,\cdot)\bigl(q(t)\bigr)\right\| = 0.
\end{aligned} \tag{16}$$

We now will prove that the set $S := \sum_{i\in J_2(t,q(t))} \mathbb{R}_+ \nabla f_i(t,\cdot)(q(t))$ is a closed subset of $\mathbb{R}$. Indeed, let $\{x_n\}$, with $x_n = \sum_{i\in J_2(t,q(t))} x_{i,n}\nabla f_i(t,\cdot)(q(t))$, be a sequence in S converging to some x^*. By Assumption A2, there exists $\nu = \nu(t,q(t))$ such that $\|\nu\| = 1$ and

$$\begin{aligned}
\langle x_n,\nu\rangle = \left\langle \sum_{i\in J_2(t,q(t))} x_{i,n}\nabla f_i(t,\cdot)\bigl(q(t)\bigr),\nu\right\rangle &= \sum_{i\in J_2(t,q(t))} x_{i,n}\bigl\langle \nabla f_i(t,\cdot)\bigl(q(t)\bigr),\nu\bigr\rangle \\
&\le (-\mu)\sum_{i\in J_2(t,q(t))} x_{i,n}.
\end{aligned}$$

From this it follows that

$$0 \le x_{i,n} \le \sum_{i\in J_2(t,q(t))} x_{i,n} \le \frac{1}{\mu}\langle x_n,-\nu\rangle \le \frac{1}{\mu}\|x_n\|.$$

Since $\{x_n\}$ is a convergent sequence, there exists $l > 0$ such that for each $i \in J_2(t,q(t))$ we have $0 \le x_{i,n} < l$ for all n. Hence, there exists a subsequence of $\{x_{i,n}\}$, denoted by $\{x_{i,n'}\}$ and a nonnegative number x_i^* such that for all $i \in J_2(t,q(t))$

$$x_{i,n'} \overset{n' \to +\infty}{\longrightarrow} x_i^*.$$

Since the sequence $\{x_n\}$ converges to x^*, the sequence $\{x_{n'}\}$ also converges to x^*. We have

$$\left\| x_{n'} - \sum_{i \in J_2(t,q(t))} x_i^* \nabla f_i(t,\cdot)\big(q(t)\big) \right\| \le \sum_{i \in J_2(t,q(t))} \left| x_{i,n'} - x_i^* \right| \left\| \nabla f_i(t,\cdot)\big(q(t)\big) \right\|.$$

From this we obtain the limit

$$x^* = \sum_{i \in J_2(t,q(t))} x_i^* \nabla f_i(t,\cdot)\big(q(t)\big) \in S.$$

We have shown that $\sum_{k \in J(t,q)} \mathbb{R}_+ \nabla f_i(t,\cdot)(q)$ is closed. Hence, by (16) we obtain the desired result. □

Lemma 4.10 *For all $i \in J_2(t,q(t))$, one has*

$$\partial f_i\big(\cdot,q(t)\big)(t) + \big\langle \nabla f_i(t,\cdot)\big(q(t)\big), \dot{q}\big(t^+\big)\big\rangle = 0.$$

Proof By Lemma 4.7, $\dot{q}(t^+) \in \mathcal{T}(t,q(t))$. Hence, for each $i \in J_2(t,q(t))$,

$$\partial f_i\big(\cdot,q(t)\big)(t) + \big\langle \nabla f_i(t,\cdot)\big(q(t)\big), \dot{q}\big(t^+\big)\big\rangle \le 0.$$

We only need to prove that

$$\partial f_i\big(\cdot,q(t)\big)(t) + \big\langle \nabla f_i(t,\cdot)\big(q(t)\big), \dot{q}\big(t^+\big)\big\rangle \ge 0; \quad \forall i \in J_2\big(t,q(t)\big).$$

Let $i \in J_2(t,q(t))$ and $\tilde{\rho} \in (0,\tilde{\rho}_t]$. By the definition of $J_2(t,q(t))$, there exists a subsequence $\{N_\alpha\}_{\alpha \in \mathbb{N}}$ strictly increasing to infinity such that, for all $\alpha \in \mathbb{N}$ we have $N_\alpha > \tilde{N}_t$. Let $h_\alpha = T/N_\alpha$, then there exists $nh_\alpha \in [t - \frac{\tilde{\rho}}{4(\kappa+1)}, t + \frac{\tilde{\rho}}{4(\kappa+1)}]$ such that $f_i(t_{n+1},Q_{n+1}) = 0$, i.e., $i \in J(t_{n+1},Q_{n+1})$. We define

$$n_\alpha = \max\left\{ n \in \mathbb{N} \mid nh_\alpha \in \left[t - \frac{\tilde{\rho}}{4(\kappa+1)}, t + \frac{\tilde{\rho}}{4(\kappa+1)} \right] \text{ and } i \in J(t_{n+1},Q_{n+1}) \right\}.$$

By Lemma 4.8 we have

$$\partial f_i(\cdot,Q_{n+1})(t_{n+1}) + \big\langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), P_n \big\rangle \ge -h\big(L + L\|P_n\| + \gamma \|P_n\|^2\big).$$

It follows that

$$\begin{aligned} \Theta_7 &:= \partial f_i\big(\cdot,q(t)\big)(t) + \big\langle \nabla f_i(t,\cdot)\big(q(t)\big), P_{n_+} \big\rangle \\ &\ge -h_\alpha\big(1 + \kappa + \gamma\kappa^2\big) + \big(\partial f_i\big(\cdot,q(t)\big)(t) - \nabla f_i(t_{n_\alpha+1},\cdot)(Q_{n_\alpha+1})\big) \\ &\quad + \big\langle \nabla f_i(t,\cdot)\big(q(t)\big), P_{n_+} - P_{n_\alpha} \big\rangle \\ &\quad + \big\langle \nabla f_i(t,\cdot)\big(q(t)\big) - \nabla f_i(t_{n_\alpha+1},\cdot)(Q_{n_\alpha+1}), P_{n_\alpha} \big\rangle. \end{aligned} \tag{17}$$

We can estimate the second and fourth terms of the right-hand side of (17) as follows

$$\begin{aligned}\Theta_8 &:= \partial f_i\big(\cdot, q(t)\big)(t) - \nabla f_i(t_{n_\alpha+1}, \cdot)(Q_{n_\alpha+1}) \\ &\geq -L\big(|t - t_{n_\alpha+1}| + \|Q_{n_\alpha+1} - q(t)\|\big) \\ &\geq -L\left(\frac{\tilde{\rho}}{4(\kappa+1)} + h_\alpha + \|q - q_{N_\alpha}\|_{C([0,T];\mathbb{R}^d)}\right)\end{aligned}$$

and

$$\begin{aligned}\Theta_9 &:= \big\langle \nabla f_i(t, \cdot)\big(q(t)\big) - \nabla f_i(t_{n_\alpha+1}, \cdot)(Q_{n_\alpha+1}), P_{n_\alpha}\big\rangle \\ &\geq -L\big(|t - t_{n_\alpha+1}| + \|Q_{n_\alpha+1} - q(t)\|\big)\|P_{n_\alpha}\| \\ &\geq -L\kappa\left(\frac{\tilde{\rho}}{4(\kappa+1)} + h_\alpha + \|q - q_{N_\alpha}\|_{C([0,T];\mathbb{R}^d)}\right).\end{aligned}$$

If $n_\alpha = n_+$, the third term of the right-hand side of (17) vanishes. Otherwise, we rewrite it as follows

$$\begin{aligned}\Psi &:= \big\langle \nabla f_i(t, \cdot)\big(q(t)\big), P_{n_+} - P_{n_\alpha}\big\rangle \\ &= \left\langle \nabla f_i(t, \cdot)\big(q(t)\big), \sum_{n=n_\alpha+1}^{n_+} hG_n\right\rangle + \big\langle \nabla f_i(t, \cdot)\big(q(t)\big), e_1\big\rangle \\ &\geq -L\int_{t-\frac{\tilde{\rho}}{4(\kappa+1)}}^{t+\frac{\tilde{\rho}}{4(\kappa+1)}} F(s)\,ds + \big\langle \nabla f_i(t, \cdot)\big(q(t)\big), e_2\big\rangle + \big\langle \nabla f_i(t, \cdot)\big(q(t)\big), e_1 - e_2\big\rangle,\end{aligned}$$

where

$$\begin{aligned}e_1 &= \sum_{n=n_\alpha+1}^{n_+} \sum_{j\in J(t_{n+1},Q_{n+1})} \lambda_j^n \nabla f_j(t_{n+1}, \cdot)(Q_{n+1}), \\ e_2 &= \sum_{n=n_\alpha+1}^{n_+} \sum_{j\in J(t_{n+1},Q_{n+1})} \lambda_j^n \nabla f_j(t, \cdot)\big(q(t)\big).\end{aligned}$$

Since $i \notin J(t_{n+1}, Q_{n+1})$ for all $n \in \{n_\alpha + 1, \ldots, n_+\}$ by the definition of n_α and the inclusion $J(t_{n+1}, Q_{n+1}) \subset J(t, q(t))$, Assumption A4 implies that the second term of the right-hand side of this last inequality is nonnegative. Furthermore, the last term can be estimated as

$$\begin{aligned}\big\langle \nabla f_i(t, \cdot)\big(q(t)\big), e_1 - e_2\big\rangle &\geq -\sum_{n=n_\alpha+1}^{n_+} \sum_{j\in J(t_{n+1},Q_{n+1})} \lambda_j^n L^2\big(|t - t_{n+1}| + \|Q_{n+1} - q(t)\|\big) \\ &\geq -L^2 m\left(\frac{\tilde{r}}{4(\kappa+1)} + h_\alpha + \|q - q_{N_\alpha}\|_{C([0,T];\mathbb{R}^d)}\right) \\ &\quad \times \big(\mathrm{Var}\big(p_N, [0, T]\big) + \|F\|_{L^1(0,T;\mathbb{R}^d)}\big).\end{aligned}$$

Then, passing the right-hand side of (17) to the limit and recalling that $P_{n_+} = p_N(t + \frac{\tilde{\rho}}{4(\kappa+1)})$, we obtain

$$\Psi_0 := \lim_{\tilde{\rho}\to 0^+} \lim_{N_\alpha\to\infty} \partial f_i\big(\cdot, q(t)\big)(t) + \big\langle \nabla f_i(t, \cdot)\big(q(t)\big), P_{n_+}\big\rangle$$

$$
\begin{aligned}
&= \partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), p(t^+)\rangle \\
&\geq 0.
\end{aligned}
$$

This means that $\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), \dot{q}(t^+)\rangle \geq 0$. □

We now continue the proof of Proposition 4.5. We have $\dot{q}(t^+) \in \mathcal{T}(t, q(t))$ and

$$
\dot{q}(t^+) - \dot{q}(t^-) \in - \sum_{i \in J_2(t,q(t))} \mathbb{R}_+ \nabla f_i(t,\cdot)(q(t)).
$$

Hence, there exist nonnegative real numbers $\bar{\lambda}_i$, for $i \in J_2(t, q(t))$, such that

$$
\dot{q}(t^+) - \dot{q}(t^-) = - \sum_{i \in J_2(t,q(t))} \bar{\lambda}_i \nabla f_i(t,\cdot)(q(t))
$$

for all $w \in \mathcal{T}(t, q(t))$

$$
\langle \dot{q}(t^-) - \dot{q}(t^+), w - \dot{q}(t^+)\rangle = \sum_{i \in J_2(t,q(t))} \bar{\lambda}_i \langle \nabla f_i(t,\cdot)(q(t)), w - \dot{q}(t^+)\rangle.
$$

However, using the previous proposition, for all $w \in \mathcal{T}(t, q(t))$ and for all $i \in J_2(t, q(t))$, we have

$$
\begin{aligned}
\langle \nabla f_i(t,\cdot)(q(t)), w - \dot{q}(t^+)\rangle &= \big(\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), w\rangle\big) - \big(\partial f_i(\cdot, q(t))(t) \\
&\quad + \langle \nabla f_i(t,\cdot)(q(t)), \dot{q}(t^+)\rangle\big) \\
&= \partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), w\rangle\big) \\
&\leq 0.
\end{aligned}
$$

Hence,

$$
\langle \dot{q}(t^-) - \dot{q}(t^+), w - \dot{q}(t^+)\rangle \leq 0 \quad \forall w \in \mathcal{T}(t, q(t)).
$$

As $\mathcal{T}(t, q(t))$ is a closed convex subset of $\mathbb{R}^d$, the above is equivalent to

$$
\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-)).
$$

The proof is complete. □

Finally, we observe that the limit trajectory satisfies the initial data. Indeed, with (10) we have immediately $q(0) = q_0$. Moreover, $p_0 \in \mathcal{T}(0, q_0)$ we can prove that $\dot{q}(0^+) = p_0 = \mathbb{P}_{\mathcal{T}(0,q_0)}(p_0)$ by the same kind of computations. Indeed, if $t = t_0 = 0$, we may define $\rho_{t_0} \in (0, \min(\rho, T))$ such that

$$
J(s,y) \subset J(t_0, q(t_0)) \forall s \in [t_0 - \rho_{t_0}, t_0 + \rho_{t_0}] \cap [0, T] \quad \forall y \in \bar{\mathbb{B}}(q(t_0), \rho_{t_0})
$$

and we define N_{t_0} (respectively, $\tilde{\rho}_{t_0}$ and $\tilde{N}_{t_0}$ if $J(t_0, q(t_0)) \neq \emptyset$). in the same way as previously. Then, for all $\tilde{\rho} \in (0, \rho_{t_0}]$ and for all $N > h_{t_0}$ (respectively, for all $\tilde{\rho} \in (0, \tilde{\rho}_{t_0}]$ and for all

$N > \tilde{N}_{t_0}$ if $J(t_0, q(t_0)) \neq \emptyset$) we define

$$n_- = 0, \qquad n_+ = \left[\frac{t_0 + \frac{\tilde{\rho}}{4(\kappa+1)}}{h} \right].$$

We obtain

$$P_{n_- -1} = P_{-1} = p_0, \qquad P_{n_+} = p_N\left(t_0 + \frac{\tilde{\rho}}{4(\kappa + 1)} h \right).$$

Using the same computation as above, we obtain $\dot{q}(0^+) = p_0$.

Remark 4.2 A similar existence result was proved in [9, Theorem 4.6]. Let us mention that our proof does not require any second-order information or boundedness on the constraints f_i such as $(A3)$ and $(A4)$ used in [9]. In fact, the boundedness conditions on $|\nabla^2 f_i(t,\cdot)(q)|$ and $|\partial^2 f_i(\cdot,q)(t)| + |\partial(\nabla f_i(\cdot,\cdot)(q))(t)|$ used in [9] are not necessary in our analysis. Moreover, the condition (R_q) used in [9] is replaced here by the weak uniform Slater condition A2. Our existence result is more specific to constraints inequalities, uses less regularity assumptions on the constraints f_i and could be seen as complementary to [9, Theorem 3.2]. In fact, Theorem 3.2 in [9] gives a global existence result for second-order differential inclusions involving a general abstract prox-regular and Lipschitz continuous set $C(t)$. When applying this result to the particular case of finite inequality constraints

$$C(t) = \left\{ q \in \mathbb{R}^d \mid f_i(t,q) \leq 0 \; \forall i \in \{1, \dots, m\} \right\}, \tag{18}$$

two main questions arise: under which conditions on the data f_i the set $C(t)$ is Lipschitz continuous? and is prox-regular? It is well known that the sublevel of prox-regular functions may fail to be prox-regular and also the prox-regularity of sets is not stable under intersection (see [2] for more details). Our aim here is to give some verifiable and practical conditions on the data f_i to satisfy both the prox-regularity and Lipschitz continuity properties of the set $C(t)$ in (18). Another way to obtain Theorem 4.1 is to assume A1–A3 to prove via Propositions 3.1 and 3.2 the prox-regularity and the Lipschitz continuity of the set $C(t)$ given in (18) and then apply the general Theorem 3.2 in [9]. For the convenience of the reader, we prefer to give a direct and self-contained proof specific to constraints inequalities based on the time-stepping algorithm. We mention that this technique for proving the existence result for nonsmooth second-order differential inclusion problems was also used in [7, 8, 27]. The following example shows that the Assumptions $(A3)$ and $(A4)$ in [9] could not be satisfied.

5 Example

Let $t \in [0,1]$ and for $i \in \{1,2\}$, $f_i : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_1\big(t,(x,y)\big) = \begin{cases} -y - t & \text{if } x \leq 0, \\ -\frac{1}{4}x^2 - y - t & \text{if } 0 \leq x \leq 1, \\ -\frac{1}{2}x + \frac{1}{4} - y - t & \text{if } x \geq 1, \end{cases}$$

and

$$f_2\big(t,(x,y)\big)=\begin{cases}-y-t & \text{if } x\geq 4,\\ -\frac{1}{4}(4-x)^2-y-t & \text{if } 3\leq x\leq 4,\\ \frac{1}{2}(x-4)+\frac{1}{4}-y-t & \text{if } x\leq 3.\end{cases}$$

Consider the problem $\mathcal{P}$ with the set $C(t)=\{q=(x,y)\in\mathbb{R}^2 \mid f_i(t,q)\leq 0, i\in\{1,2\}\}$ and $g(t,q)=0$.

Observe that $f_i(\cdot,\cdot)$, $i\in\{1,2\}$ are differentiable and their derivatives are Lipschitz continuous with rank $L=\frac{\sqrt{5}}{2}$. This shows that the Assumption A1(i) holds. Note that $\partial f_1(\cdot,q)(t)=\partial f_2(\cdot,q)(t)=-1$ and

$$\nabla f_1(t,\cdot)(x,y)=\begin{cases}(0,-1) & \text{if } x\leq 0,\\ (-\frac{1}{2}x,-1) & \text{if } 0\leq x\leq 1,\\ (-\frac{1}{2},-1) & \text{if } x\geq 1,\end{cases}$$

and

$$\nabla f_2(t,\cdot)\big((x,y)\big)=\begin{cases}(0,-1) & \text{if } x\geq 4,\\ (\frac{1}{2}(4-x),-1) & \text{if } 3\leq x\leq 4,\\ (\frac{1}{2},-1) & \text{if } x\leq 3.\end{cases}$$

Assumption A1(ii) is always true for $v=(0,1)$ and $\mu=1$. We also have $\|f_i(t,\cdot)(x,y)\|\leq L$ and therefore, Assumption A1(iii) holds. Assumption A2 is satisfied with the choice of $\gamma=\frac{1}{2}$. If $J(t,q)=\{1,2\}$ we have

$$\big\langle\nabla f_1(t,\cdot)(q),\nabla f_2(t,\cdot)(q)\big\rangle=-\frac{1}{2}\frac{1}{2}+(-1)(-1)=\frac{3}{4}\geq 0.$$

Hence, Assumption A4 holds. We have shown that Assumptions A1–A4 are satisfied for the above problem. By Theorem 4.1, the problem has a solution.

Note that the second-order derivative with respect to the second variable q of f_1 (of f_2) does not exist at $q=(0,y)$ (at $q=(4,y)$, respectively) for any $y\in\mathbb{R}$. Hence, $f_1,f_2\notin C^2([0,1]\times\mathbb{R}^2;\mathbb{R})$. This shows that the assumptions proposed in [7, 9, 27] cannot be applied to ensure the existence solution for this example.

6 Conclusions

In this paper, we have presented some regularity conditions for the data to ensure the existence of solutions for a class of vibroimpact problems. These conditions require neither the second-order differentiability nor convexity of the constraint functions. Some assumptions relate to the uniformly prox-regularity of the set of admissible positions. We also give an example to illustrate the applicability of the provided assumptions.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this article. SA and NNT investigated the problem, proposed regularity assumptions, proved the results, and gave an illustrative example. All authors read and approved the final manuscript.

Author details
[1]Laboratoire XLIM, Université de Limoges, 87060, Limoges, France. [2]Institute of Mathematics, VietNam Academy Of Science And Technology, Hanoi, Vietnam.

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