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Convergence and stability of modified multi-step Noor iterative procedure with errors for strictly hemicontractive-type mappings in Banach spaces

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Abstract

In this paper, we introduce and study a modified multi-step Noor iterative procedure with errors for two Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces and constitute its convergence and stability. The obtained results in this paper generalize and extend the corresponding result of Hussain *et al.* (Fixed Point Theory Appl. 2012:160, 2012) and some analogous results of several authors in the literature. Finally, a numerical example is included to illustrate our analytical results and to display the efficiency of our proposed novel iterative procedure with errors.

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Keywords: Modified multi-step Noor iterative procedure with errors; Lipschitz strictly hemicontractive-type mapping; Convergence; Common stability; Almost common-stability

1 Introduction

In the last few decades, fixed-point theorem-based iterative procedures whose convergence established on the strictly hemicontractive-type mappings earn a great attention for its rigorous applications in the diverse fields of various mathematical problems; see for instance [2–5] and the references cited therein. Application of strictly hemicontractive-type mapping was initiated by Chidume and Osilike [4] for improving the consequence of Chidume [5]. After Chidume and Osilike [4], several researchers studied strictly hemicontractive-type mapping in many directions; see for instance [1–3, 6–21] and the references cited therein. Among the articles cited in [1–3, 6–21], Hussain *et al.* [1] studied Lipschitz strictly hemicontractive-type mapping in arbitrary Banach spaces to extend and improve the equivalent consequences of the monographs [4, 5, 12–15].

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, B represents a nonempty subset of an arbitrary Banach space X and X^* is a dual space of X. Let T be a single-valued map from B into itself, then $r \in B$ is called a *fixed point* of T iff $T(r) = r$. The symbols D_T, R_T and F_T denote the domain of T, the range of T and the set of fixed points of T

respectively. Let $J : X \to 2^{X^*}$ be a *normalized duality mapping* given by

$$J(r) = \{g^* \in X^* : \langle r, g^* \rangle = \|r\|^2 = \|g^*\|^2\}.$$

The mapping T is called *Lipschitzian* if there exists a $L > 0$ such that

$$\|Tq - Tr\| \leq L\|q - r\| \tag{1.1}$$

for all $q, r \in B$. If $L = 1$, then T is called a *non-expansive mapping*, and if $0 \leq L < 1$, then T is called a contraction mapping.

The mapping T is called a *strictly hemicontractive mapping* if $F_T \neq \varphi$ and if there exists a constant $t > 1$ such that

$$\|q - r\| \leq \|(1 + t')(q - r) - t't(Tq - Tr)\| \tag{1.2}$$

for all $q \in D_T$, $r \in F_T$ and $t' > 0$.

If the mapping T satisfies both inequalities (1.1) and (1.2), then it is called a *Lipschitz strictly hemicontractive mapping*.

The mapping T is called *asymptotically non-expansive* on B if there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} s_n = 0$ such that, for each $p, q \in B$,

$$\|T^n p - T^n q\| \leq (1 + s_n)\|p - q\|, \quad \forall n \geq 1.$$

T is called an *asymptotically non-expansive mapping in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{p,q\in B} \left(\|T^n p - T^n q\| - \|p - q\|\right) \leq 0.$$

The mapping T is called an *asymptotically quasi-non-expansive mapping* if there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} s_n = 0$ such that, for all $p \in B$, $q \in F_T$,

$$\|T^n p - q\| \leq (1 + s_n)\|p - q\|, \quad \forall n \geq 1.$$

According to the definitions, it is clear that an asymptotically non-expansive mapping must be an asymptotically non-expansive mapping in the intermediate sense and an asymptotically quasi-non-expansive mapping, but the converse is not always true. We may justify this concept by using the following example.

Example 1.1 (See [22]) Let $X = \mathbb{R}$ (with the usual norm), $B = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|t| < 1$. For each $u \in B$, we define

$$Tu = \begin{cases} tu \sin\frac{1}{u} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

Then T is an asymptotically non-expansive mapping in the intermediate sense and an asymptotically quasi-non-expansive mapping, but is not a Lipschitzian mapping, thus it is not an asymptotically non-expansive mapping as well as it is not a Lipschitz strictly hemicontractive mapping.

Remark 1.2 We note that an asymptotically non-expansive mapping in the intermediate sense or an asymptotically quasi-non-expansive mapping is not always a Lipschitz strictly hemicontractive mapping.

We now provide an example which shows that a Lipschitz strictly hemicontractive mapping is also an asymptotically non-expansive mapping.

Example 1.3 Let $X = \mathbb{R}$ with the usual norm and $B = [0, 2\pi]$. Define $T : B \to B$ by $Tu = \frac{u\cos u}{2}$ for each $u \in B$. Clearly $F_T = \{0\}$. For each $u \in D_T$, $r \in F_T$, $t' > 0$, choose $t = 2$. Then we have

$$\begin{aligned}\left|(1+t')(u-r) - t't(Tu - Tr)\right| &= \left|(1+t')u - 2t'Tu\right| \\ &= \left|(1+t')u - 2t' \cdot \frac{u\cos u}{2}\right| \\ &\geq (1+t')u - t'u = u = |u-r|\end{aligned}$$

and hence T is a strictly hemicontractive mapping.

And, if we consider $u = \pi$, $v = 2\pi$, then it is easy to see that $|u - v| = \pi$ and hence

$$\begin{aligned}|Tu - Tv| = \left|\frac{u\cos u}{2} - \frac{v\cos v}{2}\right| &= \frac{\pi}{2}|\cos\pi - 2\cos 2\pi| \\ &< \frac{\pi}{2}|\cos\pi - \cos 2\pi| \\ &\leq \frac{\pi}{2}|\pi - 2\pi| = \frac{\pi^2}{2} = L|u-v|,\end{aligned}$$

for all $u, v \in B$ and $L = \frac{\pi}{2} > 0$. Thus, T is a Lipschitz strictly hemicontractive mapping.

Furthermore, for a sequence $\{\frac{1}{n}\}$ we have

$$\begin{aligned}\left|T^n u - T^n v\right| = \frac{1}{2^n}|u\cos u - v\cos v| &= \frac{\pi}{2^n}|\cos\pi - 2\cos 2\pi| \\ &< \frac{\pi}{2^n}|\cos\pi - \cos 2\pi| \\ &\leq \frac{\pi}{2^n}|\pi - 2\pi| = \frac{\pi}{2^n}\cdot\pi < \left(1 + \frac{1}{n}\right)\cdot\pi = \left(1 + \frac{1}{n}\right)|u - v|,\end{aligned}$$

for all $u, v \in B$ and $n \geq 1$. Hence, T is an asymptotically non-expansive mapping. Therefore, a Lipschitz strictly hemicontractive mapping may also be an asymptotically non-expansive mapping.

The following example shows that a strictly hemicontractive mapping is neither a Lipschitzian mapping nor an asymptotically non-expansive mapping.

Example 1.4 (See [23]) Let $X = \mathbb{R}$ (with the usual norm), $B = [0, 1]$ and let φ be the Cantor ternary function. If we define $T : B \to X$ by

$$Tu = \begin{cases} \frac{u}{2} & \text{if } 0 \leq u \leq \frac{1}{2}, \\ \varphi((1-u)/2) & \text{if } \frac{1}{2} < u \leq 1, \end{cases}$$

then $T^n u \to 0$ uniformly on B and T is a strictly hemicontractive mapping. But we observe that T is neither a Lipschitzian mapping nor an asymptotically non-expansive mapping.

In 2006, Plubtieng and Wangkeeree [24] introduced and studied the following multi-step Noor iterative procedure with errors for some special type of asymptotically non-expansive mappings (asymptotically non-expansive mapping in the intermediate sense and asymptotically quasi-non-expansive mapping) in Banach spaces: For a given $u_1 \in B$, and a fixed $m \in \mathbf{N}$ (set of all positive integers), the iterative sequences $\{u_n^{(1)}\}, \{u_n^{(2)}\}, \ldots, \{u_n^{(m)}\}$ defined by

$$\left.\begin{array}{l} u_n^{(1)} = a_n^{(1)} T^n u_n + b_n^{(1)} u_n + c_n^{(1)} v_n^{(1)}, \\ u_n^{(2)} = a_n^{(2)} T^n u_n^{(1)} + b_n^{(2)} u_n + c_n^{(2)} v_n^{(2)}, \\ \ldots\ldots \\ u_n^{(m-1)} = a_n^{(m-1)} T^n u_n^{(m-2)} + b_n^{(m-1)} u_n + c_n^{(m-1)} v_n^{(m-1)}, \\ u_{n+1} = u_n^{(m)} = a_n^{(m)} T^n u_n^{(m-1)} + b_n^{(m)} u_n + c_n^{(m)} v_n^{(m)}, \quad n \geq 1, \end{array}\right\} \tag{1.3}$$

where $\{v_n^{(1)}\}, \ldots, \{v_n^{(m)}\}$ are bounded sequences in B and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$, $\{c_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for each $i \in \{1, 2, \ldots, m\}$.

The iterative procedure given by (1.3) is known as the multi-step Noor iterative procedure with errors (MNIPE). After Plubtieng and Wangkeeree [24], a numerous number of research articles have been published on different types of iterative procedures with errors for various kinds of mappings; see for instance [1, 9, 12, 25–27] and the references cited therein. Among the above-mentioned articles, Hussain *et al.* [1] studied the following special type of Ishikawa iterative procedure with errors (STIIPE) for two Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces: For a given $u_0 \in B$, the iterative sequences $\{u_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{array}{l} u_{n+1} = u_n^{(2)} = a_n^{(2)} u_n + b_n^{(2)} T u_n^{(1)} + c_n^{(2)} v_n^{(2)}, \\ u_n^{(1)} = a_n^{(1)} u_n + b_n^{(1)} S u_n + c_n^{(1)} v_n^{(1)}, \quad n \geq 0, \end{array}\right\} \tag{1.4}$$

where $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ are bounded sequences in B and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$, $\{c_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ satisfying $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for all $i \in \{1, 2\}$.

Stimulated by the work of Hussain *et al.* [1, 9], Plubtieng and Wangkeeree [24], Yu *et al.* [11], Agwu and Igbokwe [17] and Zegeye and Tufa [19] in this paper, we propose and study the following modified multi-step Noor iterative procedure with errors (MMNIPE) for two Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces: For a given $u_0 \in B$, and a fixed $m \in \mathbf{N}$, we compute the iterative sequences $\{u_n\}_{n=0}^{\infty}$ by

$$\left.\begin{array}{l} u_{n+1} = u_n^{(m)} = a_n^{(m)} u_n + b_n^{(m)} T u_n^{(m-1)} + c_n^{(m)} v_n^{(m)}, \\ u_n^{(m-1)} = a_n^{(m-1)} u_n + b_n^{(m-1)} T u_n^{(m-2)} + c_n^{(m-1)} v_n^{(m-1)}, \\ \ldots\ldots \\ u_n^{(2)} = a_n^{(2)} u_n + b_n^{(2)} T u_n^{(1)} + c_n^{(2)} v_n^{(2)}, \\ u_n^{(1)} = a_n^{(1)} u_n + b_n^{(1)} S u_n + c_n^{(1)} v_n^{(1)}, \quad n \geq 0, \end{array}\right\} \tag{1.5}$$

where $\{v_n^{(1)}\}, \ldots, \{v_n^{(m)}\}$ are bounded sequences in B and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$, $\{c_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for each $i \in \{1, 2, \ldots, m\}$.

Remark 1.5 It is clear that the iterative procedures defined by (1.4) (the STIIPE given by Hussain *et al.* [1]), the Mann iterative procedure (MIP) given by Mann [28], the Ishikawa iterative procedure (IIP) given by Ishikawa [29], the Noor iterative procedure (NIP) given by Xu and Noor [30], Mann iterative procedures with errors (MIPE) given by Liu [31] and Xu [32], the Ishikawa iterative procedure with errors (IIPE) given by Liu [31] and Xu [32] and the three-step iterative procedure with errors (TIPE) given by Cho *et al.* [33] are all special cases of the newly proposed MMNIPE given by (1.5). That is, the iterative procedure defined by (1.5) is a general iterative procedure among the above-mentioned iterative procedures.

To the best of our knowledge, there does not exist any work about the convergence and almost common-stability and common-stability of the iterative procedure given by (1.5) for Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces. From this context, here we establish the convergence, almost common-stability and common-stability of the newly proposed MMNIPE given by (1.5) for two Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces. The rest of this paper is organized as follows:

In Sect. 2, we recall some essential definitions and fundamental results. Sect. 3 is the main part of this paper. Here, we establish convergence, almost common-stability and common-stability of our proposed MMNIPE given by (1.5). In Sect. 4, we discuss a numerical example to verify the main results of this paper. Finally, in Sect. 5, we conclude this paper.

2 Preliminary notes

This section is devoted to recalling some definitions and fundamental results which are truly needed to establish the main results.

Definition 2.1 (See [4, 34]) The mapping T is called *pseudocontractive* if the inequality

$$\|q-r\| \leq \bigl\|q-r+t\bigl((I-T)q-(I-T)r\bigr)\bigr\| \tag{2.1}$$

holds for each $q,r \in B$ and for all $t > 0$. According to the result of Kato [35], it follows that T is a *pseudocontractive* if and only if there exists a $h(q-r) \in J(q-r)$ such that

$$\bigl\langle Tq-Tr, h(q-r)\bigr\rangle \leq \|q-r\|^2 \tag{2.2}$$

for all $q,r \in B$. T is called *strongly pseudocontractive* if there exists a $t > 1$ such that

$$\|q-r\| \leq \bigl\|\bigl(1+t'\bigr)(q-r)-t't(Tq-Tr)\bigr\| \tag{2.3}$$

for all $q,r \in D_T$ and $t' > 0$. T is called *local strongly pseudocontractive* if, for each $q \in D_T$, there exists a $t_q > 1$ such that

$$\|q-r\| \leq \bigl\|\bigl(1+t'\bigr)(q-r)-t't_q(Tq-Tr)\bigr\| \tag{2.4}$$

for all $q,r \in D_T$ and $t' > 0$.

Definition 2.2 (See [36–38]) Suppose $u_0 \in B$ and $u_{n+1} = f(u_n, T)$ defines an iterative procedure which yields a sequence of points $\{u_n\} \subset B$. Let $F_T \neq \varphi$ and let $\{u_n\}$ converge to a fixed point q of T. Let $\{v_n\} \subset B$ and $\{\delta_n\}$ be a sequence in $[0,\infty)$, where $\delta_n = \|v_{n+1} - f(v_n, T)\|$. Now, if $\lim_{n\to\infty} \delta_n = 0$ implies that $\lim_{n\to\infty} v_n = q$, then the iterative procedure defined by $u_{n+1} = f(u_n, T)$ is said to be *T-stable* or *stable* on B with respect to T and if $\sum_{n=0}^{\infty} \delta_n < \infty$ implies that $\lim_{n\to\infty} v_n = q$, then the iterative procedure defined by $u_{n+1} = f(u_n, T)$ is said to be an *almost T-stable* on B with respect to T.

Definition 2.3 (See [1]) Let B be a nonempty convex subset of an arbitrary Banach space X and let T and S be two self-operators on B. Suppose $u_0 \in B$ and $u_{n+1} = f(u_n, T, S)$ defines an iterative procedure which yields a sequence of points $\{u_n\} \subset B$. Let $F_T \cap F_S \neq \varphi$ and let $\{u_n\}$ converges strongly to a common fixed point q of T and S. Let $\{v_n\}$ be any bounded sequence in B and $\{\mu_n\}$ be a sequence in $[0,\infty)$, where $\mu_n = \|v_{n+1} - f(v_n, T, S)\|$. Now, if $\lim_{n\to\infty} \mu_n = 0$ implies that $\lim_{n\to\infty} v_n = r$, then the iterative procedure defined by $u_{n+1} = f(u_n, T, S)$ is said to be a *common-stable* on B and if $\sum_{n=0}^{\infty} \mu_n < \infty$ implies that $\lim_{n\to\infty} v_n = q$, then the iterative procedure defined by $u_{n+1} = f(u_n, T, S)$ is said to be an *almost common-stable* on B.

Now, we recall some lemmas which are essential to prove the main results of this paper.

Lemma 2.4 (See [39]) *Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ be nonnegative real sequences such that*

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq 0, \tag{2.5}$$

with $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty} \omega_n = \infty$, $\sum_{n=0}^{\infty} \gamma_n < \infty$ and $\lim_{n\to\infty} \beta_n = 0$. Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5 (See [40]) *Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers and $0 \leq \eta < 1$, so that*

$$\alpha_{n+1} \leq \eta\alpha_n + \beta_n, \quad \forall n \geq 0. \tag{2.6}$$

(i) *If $\lim_{n\to\infty} \beta_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.*
(ii) *If $\sum_{n=0}^{\infty} \beta_n < \infty$, then $\sum_{n=0}^{\infty} \alpha_n < \infty$.*

Lemma 2.6 (See [35]) *Let $x, y \in X$. Then $\|x\| \leq \|x + ry\|$ for every $r > 0$ if and only if there is $f \in J(x)$ such that $\operatorname{Re}(y, f) \geq 0$.*

Lemma 2.7 (See [4]) *Let $T : D_T \subseteq X \to X$ be an operator with $F_T \neq \varphi$. Then T is strictly hemicontractive if and only if there exists a $t > 1$ such that for all $x \in D_T$ and $q \in F_T$ there exists $h \in J(x - q)$ satisfying*

$$\operatorname{Re}\langle x - Tx, h(x - q)\rangle \geq \left(1 - \frac{1}{t}\right)\|x - q\|^2. \tag{2.7}$$

Lemma 2.8 (See [12]) *Let X be an arbitrary norm linear space and $T : D_T \subseteq X \to X$ be an operator.*

(a) *If T is a local strongly pseudocontractive operator and $F_T \neq \varphi$, then F_T is a singleton and T is strictly hemicontractive.*
(b) *If T is strictly hemicontractive, then F_T is a singleton.*

3 Convergence and stability of modified multi-step Noor iterative procedure with errors

In this section, we state and prove the convergence and stability of our proposed MMNIPE for two Lipschitz strictly hemicontractive-type mappings.

Let $\lambda = \frac{\sigma-1}{\sigma} \in (0,1)$, where $\sigma > 1$, L be a common Lipschitz constant of two strictly hemicontractive-type mappings T, S and I be an identity mapping on the arbitrary Banach space X. In the above-mentioned context, we state and prove the following theorems.

Theorem 3.1 *Let B be a nonempty closed convex subset of X and T and S be two Lipschitz strictly hemicontractive-type mappings from B into itself. Suppose that $\{v_n^{(1)}\}, \dots, \{v_n^{(m)}\}$ are arbitrary bounded sequences in B and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$, $\{c_n^{(i)}\}$ for each $i \in \{1,2,\dots,m\}$ are any appropriate real sequences in $[0,1]$ satisfying the following conditions*:

(1) $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$, *for each* $i \in \{1,2,3,\dots,m\}$,
(2) $c_n^{(m)} = \mathrm{o}(b_n^{(m)})$,
(3) $\lim_{n\to\infty} c_n^{(j)} = 0$, *for each* $j \in \{1,2,3,\dots,m-1\}$,
(4) $\sum_{n=0}^{\infty} b_n^{(j)} = \infty$ *for each* $j \in \{2,3,\dots,m\}$,
(5)

$$\begin{aligned} &L\Big[(1+L)b_n^{(m)} + (1+L)^2 b_n^{(m-1)} + L(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} \\ &\quad + L^2(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} + \cdots + L^{m-3}(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} \\ &\quad + L^{m-2}(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} b_n^{(1)} \\ &\quad + \big[c_n^{(m)} + (1+L)c_n^{(m-1)} + L(1+L) b_n^{(m-1)} c_n^{(m-2)} \\ &\quad + L^2(1+L) b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)} + \cdots + L^{m-3}(1+L) b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)} \\ &\quad + L^{m-2}(1+L) b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)}\big]\Big] + \frac{c_n^{(m)}}{b_n^{(m)}} \le \lambda(\lambda-\theta), \quad n \ge 0, \end{aligned}$$

where θ is a constant in $(0,\lambda)$ and $\lambda \in (0,1)$.

Assume an iterative sequence $\{u_n\}_{n=0}^{\infty}$ defined by (1.5). Let $\{w_n\}_{n=0}^{\infty}$ be any sequence in B and $\{\mu_n\}_{n=0}^{\infty}$ be a sequence defined by

$$\mu_n = \|w_{n+1} - x_n\|, \quad n \ge 0,$$

where

$$\left.\begin{aligned} &x_n = x_n^{(m)} = a_n^{(m)} w_n + b_n^{(m)} T x_n^{(m-1)} + c_n^{(m)} v_n^{(m)}, \\ &x_n^{(m-1)} = a_n^{(m-1)} w_n + b_n^{(m-1)} T x_n^{(m-2)} + c_n^{(m-1)} v_n^{(m-1)}, \\ &\dots\dots \\ &x_n^{(2)} = a_n^{(2)} w_n + b_n^{(2)} T x_n^{(1)} + c_n^{(2)} v_n^{(2)}, \\ &x_n^{(1)} = a_n^{(1)} w_n + b_n^{(1)} S w_n + c_n^{(1)} v_n^{(1)}, \quad n \ge 0. \end{aligned}\right\} \tag{3.1}$$

Then

(i) *the iterative sequence* $\{u_n\}_{n=0}^{\infty}$ *given by* (1.5) *converges strongly to the common fixed* r *of* T *and* S *and the following inequality holds*:

$$\begin{aligned}&\|u_{n+1}-r\|\\&\quad\le\big(1-\theta b_n^{(m)}\big)\|u_n-r\|+\lambda^{-1}(1+L)\big[c_n^{(m)}\big\|v_n^{(m)}-r\big\|+c_n^{(m-1)}Lb_n^{(m)}\big\|v_n^{(m-1)}-r\big\|\\&\qquad+c_n^{(m-2)}L^2b_n^{(m)}b_n^{(m-1)}\big\|v_n^{(m-2)}-r\big\|+c_n^{(m-3)}L^3b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\big\|v_n^{(m-3)}-r\big\|\\&\qquad+\cdots+c_n^{(2)}L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\big\|v_n^{(2)}-r\big\|\\&\qquad+c_n^{(1)}L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}\big\|v_n^{(1)}-r\big\|\big],\quad n\ge 0,\end{aligned}$$

(ii)
$$\begin{aligned}&\|w_{n+1}-r\|\\&\quad\le\big(1-\theta b_n^{(m)}\big)\|w_n-r\|\\&\qquad+\lambda^{-1}(1+L)\big[c_n^{(m)}\big\|v_n^{(m)}-r\big\|+L\big[b_n^{(m)}c_n^{(m-1)}\big\|v_n^{(m-1)}-r\big\|\\&\qquad+L\big[b_n^{(m)}b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)}-r\big\|+L\big[b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)}-r\big\|\\&\qquad+\cdots+L\big[b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)}-r\big\|\\&\qquad+L\big[b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big\|v_n^{(1)}-r\big\|\big]\big]\big]\big]\cdots\big]\big]+\mu_n,\quad n\ge 0,\end{aligned}$$

(iii) $\sum_{n=0}^{\infty}\mu_n<\infty$ *implies that* $\lim_{n\to\infty}w_n=r$, *so that* $\{u_n^{(m)}\}_{n=0}^{\infty}$ *is almost common-stable on* B,

(iv) $\lim_{n\to\infty}w_n=r$, *implies that* $\lim_{n\to\infty}\mu_n=0$.

Proof (i) From the condition (2), we obtain $c_n^{(m)}=\delta_n b_n^{(m)}$, and $\delta_n\to 0$ as $n\to\infty$. By an application of Lemma 2.8, we see that $F_T\cap F_S$ is singleton, and let $F_T\cap F_S=\{r\}$ for some $r\in B$. Put

$$A=\max\Big\{\sup_{n\ge0}\big\{\big\|v_n^{(1)}-r\big\|\big\},\sup_{n\ge0}\big\{\big\|v_n^{(2)}-r\big\|\big\},\sup_{n\ge0}\big\{\big\|v_n^{(3)}-r\big\|\big\},\dots,\sup_{n\ge0}\big\{\big\|v_n^{(m)}-r\big\|\big\}\Big\}.$$

Since T is strictly hemicontractive, from Lemma 2.7, we obtain

$$\begin{aligned}&\mathrm{Re}\big\langle x-Tx,h(x-r)\big\rangle\ge\lambda\|x-r\|^2\\&\Rightarrow\quad\mathrm{Re}\big\langle(I-T-\lambda I)x-(I-T-\lambda I)r,h(x-r)\big\rangle\ge 0,\quad\forall x\in B.\end{aligned}\tag{3.2}$$

Now, from (3.2) and Lemma 2.6, we have

$$\|x-r\|\le\big\|x-r+q\big[(I-T-\lambda I)x-(I-T-\lambda I)r\big]\big\|,\quad\forall x\in B,\ \text{and}\ \forall q>0.\tag{3.3}$$

Also, from the first equation of (1.5), we get

$$\begin{aligned}&u_{n+1}=a_n^{(m)}u_n+b_n^{(m)}Tu_n^{(m-1)}+c_n^{(m)}v_n^{(m)}\\&\Rightarrow\quad\big(1-b_n^{(m)}\big)u_n=\big(1-(1-\lambda)b_n^{(m)}\big)u_{n+1}+b_n^{(m)}(I-T-\lambda I)u_{n+1}\\&\qquad\qquad+b_n^{(m)}\big(Tu_{n+1}-Tu_n^{(m-1)}\big)-c_n^{(m)}\big(v_n^{(m)}-u_n\big),\end{aligned}\tag{3.4}$$

and since $r \in B$ is the fixed point of T, it follows that

$$\bigl(1 - b_n^{(m)}\bigr)r = \bigl(1 - (1 - \lambda)b_n^{(m)}\bigr)r + b_n^{(m)}(I - T - \lambda I)r. \tag{3.5}$$

Now, for all $n \geq 0$ from (3.4) and (3.5), we have

$$\begin{aligned}
&\bigl(1 - b_n^{(m)}\bigr)\|u_n - r\| \\
&\quad\geq \bigl\|\bigl(1 - (1 - \lambda)b_n^{(m)}\bigr)(u_{n+1} - r) + b_n^{(m)}(I - T - \lambda I)(u_{n+1} - r)\bigr\| \\
&\qquad - b_n^{(m)}\bigl\|Tu_{n+1} - Tu_n^{(m-1)}\bigr\| - c_n^{(m)}\bigl\|v_n^{(m)} - u_n\bigr\| \\
&\quad= \bigl(1 - (1 - \lambda)b_n^{(m)}\bigr)\left\|u_{n+1} - r + \frac{b_n^{(m)}}{1 - (1 - \lambda)b_n^{(m)}}(I - T - \lambda I)(u_{n+1} - r)\right\| \\
&\qquad - b_n^{(m)}\bigl\|Tu_{n+1} - Tu_n^{(m-1)}\bigr\| - c_n^{(m)}\bigl\|v_n^{(m)} - u_n\bigr\| \\
&\quad\geq \bigl(1 - (1 - \lambda)b_n^{(m)}\bigr)\|u_{n+1} - r\| - b_n^{(m)}\bigl\|Tu_{n+1} - Tu_n^{(m-1)}\bigr\| - c_n^{(m)}\bigl\|v_n^{(m)} - u_n\bigr\|
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|u_{n+1} - r\| \\
&\quad\leq \frac{1 - b_n^{(m)}}{1 - (1 - \lambda)b_n^{(m)}}\|u_n - r\| + \frac{b_n^{(m)}}{1 - (1 - \lambda)b_n^{(m)}}\bigl\|Tu_{n+1} - Tu_n^{(m-1)}\bigr\| \\
&\qquad + \frac{c_n^{(m)}}{1 - (1 - \lambda)b_n^{(m)}}\bigl\|v_n^{(m)} - u_n\bigr\| \\
&\quad\leq \bigl(1 - \lambda b_n^{(m)}\bigr)\|u_n - r\| + \lambda^{-1}b_n^{(m)}\bigl\|Tu_{n+1} - Tu_n^{(m-1)}\bigr\| + \lambda^{-1}c_n^{(m)}\bigl\|v_n^{(m)} - u_n\bigr\| \\
&\quad\leq \bigl(1 - \lambda b_n^{(m)}\bigr)\|u_n - r\| + \lambda^{-1}Lb_n^{(m)}\bigl\|u_{n+1} - u_n^{(m-1)}\bigr\| + \lambda^{-1}c_n^{(m)}\bigl\|v_n^{(m)} - u_n\bigr\| \\
&\quad\leq \bigl(1 - \lambda b_n^{(m)}\bigr)\|u_n - r\| + \lambda^{-1}Lb_n^{(m)}\bigl\|u_{n+1} - u_n^{(m-1)}\bigr\| + \lambda^{-1}c_n^{(m)}\bigl(\bigl\|v_n^{(m)} - r\bigr\| + \|u_n - r\|\bigr) \\
&\quad= \bigl(1 - \lambda b_n^{(m)} + \lambda^{-1}c_n^{(m)}\bigr)\|u_n - r\| + \lambda^{-1}Lb_n^{(m)}\bigl\|u_{n+1} - u_n^{(m-1)}\bigr\| \\
&\qquad + \lambda^{-1}c_n^{(m)}\bigl\|v_n^{(m)} - r\bigr\|.
\end{aligned} \tag{3.6}$$

Again, from (1.5), we get

$$\begin{aligned}
&\bigl\|u_{n+1} - u_n^{(m-1)}\bigr\| \\
&\quad= \bigl\|a_n^{(m)}u_n + b_n^{(m)}Tu_n^{(m-1)} + c_n^{(m)}v_n^{(m)} - a_n^{(m-1)}u_n - b_n^{(m-1)}Tu_n^{(m-2)} - c_n^{(m-1)}v_n^{(m-1)}\bigr\| \\
&\quad= \bigl\|\bigl(1 - b_n^{(m)} - c_n^{(m)}\bigr)u_n + b_n^{(m)}Tu_n^{(m-1)} + c_n^{(m)}v_n^{(m)} - \bigl(1 - b_n^{(m-1)} - c_n^{(m-1)}\bigr)u_n \\
&\qquad - b_n^{(m-1)}Tu_n^{(m-2)} - c_n^{(m-1)}v_n^{(m-1)}\bigr\| \\
&\quad= \bigl\|b_n^{(m)}\bigl(Tu_n^{(m-1)} - u_n\bigr) + c_n^{(m)}\bigl(v_n^{(m)} - u_n\bigr) + b_n^{(m-1)}\bigl(u_n - Tu_n^{(m-2)}\bigr) - c_n^{(m-1)}\bigl(v_n^{(m-1)} - u_n\bigr)\bigr\| \\
&\quad\leq \bigl\|b_n^{(m)}\bigl(Tu_n^{(m-1)} - u_n\bigr) + c_n^{(m)}\bigl(v_n^{(m)} - u_n\bigr)\bigr\| \\
&\qquad + \bigl\|b_n^{(m-1)}\bigl(u_n - Tu_n^{(m-2)}\bigr) - c_n^{(m-1)}\bigl(v_n^{(m-1)} - u_n\bigr)\bigr\| \\
&\quad\leq b_n^{(m)}\bigl\|u_n - Tu_n^{(m-1)}\bigr\| + c_n^{(m)}\bigl\|v_n^{(m)} - u_n\bigr\| + b_n^{(m-1)}\bigl\|u_n - Tu_n^{(m-2)}\bigr\| \\
&\qquad + c_n^{(m-1)}\bigl\|v_n^{(m-1)} - u_n\bigr\|
\end{aligned}$$

$$
\begin{aligned}
&\le b_n^{(m)}\left(\|u_n - r\| + \left\|r - Tu_n^{(m-1)}\right\|\right) + c_n^{(m)}\left(\left\|v_n^{(m)} - r\right\| + \|u_n - r\|\right) \\
&\quad + b_n^{(m-1)}\left(\|u_n - r\| + \left\|r - Tu_n^{(m-2)}\right\|\right) + c_n^{(m-1)}\left(\left\|v_n^{(m-1)} - r\right\| + \|u_n - r\|\right) \\
&\le b_n^{(m)}\left(\|u_n - r\| + L\left\|u_n^{(m-1)} - r\right\|\right) + c_n^{(m)}\left(\left\|v_n^{(m)} - r\right\| + \|u_n - r\|\right) \\
&\quad + b_n^{(m-1)}\left(\|u_n - r\| + L\left\|u_n^{(m-2)} - r\right\|\right) + c_n^{(m-1)}\left(\left\|v_n^{(m-1)} - r\right\| + \|u_n - r\|\right) \\
&= \left[b_n^{(m)} + b_n^{(m-1)} + c_n^{(m)} + c_n^{(m-1)}\right]\|u_n - r\| + b_n^{(m)}L\left\|u_n^{(m-1)} - r\right\| + b_n^{(m-1)}L\left\|u_n^{(m-2)} - r\right\| \\
&\quad + c_n^{(m)}\left\|v_n^{(m)} - r\right\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\|.
\end{aligned}
\tag{3.7}
$$

But

$$
\begin{aligned}
\left\|u_n^{(m-1)} - r\right\| &= \left\|a_n^{(m-1)}u_n + b_n^{(m-1)}Tu_n^{(m-2)} + c_n^{(m-1)}v_n^{(m-1)} - r\right\| \\
&= \left\|\left(1 - b_n^{(m-1)} - c_n^{(m-1)}\right)u_n + b_n^{(m-1)}Tu_n^{(m-2)} + c_n^{(m-1)}v_n^{(m-1)} - r\right\| \\
&= \left\|u_n - r + b_n^{(m-1)}\left(Tu_n^{(m-2)} - r + r - u_n\right) + c_n^{(m-1)}\left(v_n^{(m-1)} - r + r - u_n\right)\right\| \\
&\le \|u_n - r\| + b_n^{(m-1)}\left\|Tu_n^{(m-2)} - r\right\| + b_n^{(m-1)}\|u_n - r\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\| \\
&\qquad + c_n^{(m-1)}\|u_n - r\| \\
&\le \|u_n - r\| + b_n^{(m-1)}L\left\|u_n^{(m-2)} - r\right\| + b_n^{(m-1)}\|u_n - r\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\| \\
&\qquad + c_n^{(m-1)}\|u_n - r\| \\
&= \left[1 + b_n^{(m-1)} + c_n^{(m-1)}\right]\|u_n - r\| + b_n^{(m-1)}L\left\|u_n^{(m-2)} - r\right\| \\
&\quad + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\|.
\end{aligned}
\tag{3.8}
$$

Substituting (3.8) in (3.7), we have

$$
\begin{aligned}
&\left\|u_{n+1} - u_n^{(m-1)}\right\| \\
&\le \left[b_n^{(m)} + b_n^{(m-1)} + c_n^{(m)} + c_n^{(m-1)}\right]\|u_n - r\| \\
&\quad + b_n^{(m)}L\left[\left[1 + b_n^{(m-1)} + c_n^{(m-1)}\right]\|u_n - r\| + b_n^{(m-1)}L\left\|u_n^{(m-2)} - r\right\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\|\right] \\
&\quad + b_n^{(m-1)}L\left\|u_n^{(m-2)} - r\right\| + c_n^{(m)}\left\|v_n^{(m)} - r\right\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\| \\
&= \left[b_n^{(m)} + b_n^{(m-1)} + b_n^{(m)}L\left[1 + b_n^{(m-1)} + c_n^{(m-1)}\right] + c_n^{(m)} + c_n^{(m-1)}\right]\|u_n - r\| \\
&\quad + \left[b_n^{(m)}b_n^{(m-1)}L^2 + b_n^{(m-1)}L\right]\left\|u_n^{(m-2)} - r\right\| + c_n^{(m)}\left\|v_n^{(m)} - r\right\| \\
&\quad + \left[b_n^{(m)}Lc_n^{(m-1)} + c_n^{(m-1)}\right]\left\|v_n^{(m-1)} - r\right\| \\
&= \left[(1+L)b_n^{(m)} + \left(1 + Lb_n^{(m)}\right)b_n^{(m-1)} + c_n^{(m)} + \left(1 + Lb_n^{(m)}\right)c_n^{(m-1)}\right]\|u_n - r\| \\
&\quad + Lb_n^{(m-1)}\left(1 + Lb_n^{(m)}\right)\left\|u_n^{(m-2)} - r\right\| + c_n^{(m)}\left\|v_n^{(m)} - r\right\| \\
&\quad + c_n^{(m-1)}\left(1 + Lb_n^{(m)}\right)\left\|v_n^{(m-1)} - r\right\|.
\end{aligned}
\tag{3.9}
$$

But, if we replace m by $m-1$ in (3.8), then we have

$$
\begin{aligned}
\left\|u_n^{(m-2)} - r\right\| &= \left[1 + b_n^{(m-2)} + c_n^{(m-2)}\right]\|u_n - r\| + b_n^{(m-2)}L\left\|u_n^{(m-3)} - r\right\| \\
&\quad + c_n^{(m-2)}\left\|v_n^{(m-2)} - r\right\|.
\end{aligned}
\tag{3.10}
$$

Now, substituting (3.10) in (3.9), we have

$$\begin{aligned}
&\big\|u_{n+1}-u_n^{(m-1)}\big\| \\
&\quad\le \big[(1+L)b_n^{(m)}+\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}+c_n^{(m)}+\big(1+Lb_n^{(m)}\big)c_n^{(m-1)}\big]\|u_n-r\| \\
&\qquad+L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}\big[\big[1+b_n^{(m-2)}+c_n^{(m-2)}\big]\|u_n-r\|+b_n^{(m-2)}L\big\|u_n^{(m-3)}-r\big\| \\
&\qquad+c_n^{(m-2)}\big\|v_n^{(m-2)}-r\big\|\big]+c_n^{(m)}\big\|v_n^{(m)}-r\big\|+c_n^{(m-1)}\big(1+Lb_n^{(m)}\big)\big\|v_n^{(m-1)}-r\big\| \\
&\quad\le \big[(1+L)b_n^{(m)}+(1+L)\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}+L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}+c_n^{(m)} \\
&\qquad+\big(1+Lb_n^{(m)}\big)c_n^{(m-1)}+L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)}\big]\|u_n-r\| \\
&\qquad+L^2\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\big\|u_n^{(m-3)}-r\big\|+c_n^{(m)}\big\|v_n^{(m)}-r\big\| \\
&\qquad+c_n^{(m-1)}\big(1+Lb_n^{(m)}\big)\big\|v_n^{(m-1)}-r\big\|+c_n^{(m-2)}Lb_n^{(m-1)}\big(1+Lb_n^{(m)}\big)\big\|v_n^{(m-2)}-r\big\|.
\end{aligned}\tag{3.11}$$

But, if we replace m by $m-1$ in (3.10), then we have

$$\begin{aligned}
\big\|u_n^{(m-3)}-r\big\| &= \big[1+b_n^{(m-3)}+c_n^{(m-3)}\big]\|u_n-r\|+b_n^{(m-3)}L\big\|u_n^{(m-4)}-r\big\| \\
&\quad+c_n^{(m-3)}\big\|v_n^{(m-3)}-r\big\|.
\end{aligned}\tag{3.12}$$

Substituting (3.12) in (3.11), we have

$$\begin{aligned}
&\big\|u_{n+1}-u_n^{(m-1)}\big\| \\
&\quad\le \big[(1+L)b_n^{(m)}+(1+L)\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}+L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}+c_n^{(m)} \\
&\qquad+\big(1+Lb_n^{(m)}\big)c_n^{(m-1)}+L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)}\big]\|u_n-r\| \\
&\qquad+L^2\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\big[\big[1+b_n^{(m-3)}+c_n^{(m-3)}\big]\|u_n-r\| \\
&\qquad+Lb_n^{(m-3)}\big\|u_n^{(m-4)}-r\big\|+c_n^{(m-3)}\big\|v_n^{(m-3)}-r\big\|\big]+c_n^{(m)}\big\|v_n^{(m)}-r\big\| \\
&\qquad+c_n^{(m-1)}\big(1+Lb_n^{(m)}\big)\big\|v_n^{(m-1)}-r\big\|+c_n^{(m-2)}L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}\big\|v_n^{(m-2)}-r\big\| \\
&\quad= \big[(1+L)b_n^{(m)}+(1+L)\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}+L(1+L)\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \\
&\qquad+L^2\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}+c_n^{(m)}+\big(1+Lb_n^{(m)}\big)c_n^{(m-1)} \\
&\qquad+L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)}+L^2\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big]\|u_n-q\| \\
&\qquad+L^3\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\big\|u_n^{(m-4)}-r\big\| \\
&\qquad+c_n^{(m)}\big\|v_n^{(m)}-r\big\|+c_n^{(m-1)}\big(1+Lb_n^{(m)}\big)\big\|v_n^{(m-1)}-r\big\| \\
&\qquad+c_n^{(m-2)}L\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}\big\|v_n^{(m-2)}-r\big\| \\
&\qquad+c_n^{(m-3)}L^2\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\big\|v_n^{(m-3)}-r\big\|.
\end{aligned}\tag{3.13}$$

Continuing the above procedure up to second iterative step of (1.5), we obtain

$$\begin{aligned}
&\big\|u_{n+1}-u_n^{(m-1)}\big\| \\
&\quad\le \big[(1+L)b_n^{(m)}+(1+L)\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}+L(1+L)\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \\
&\qquad+L^2\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}+\cdots+L^{m-3}\big(1+Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}
\end{aligned}$$

$$\begin{aligned}
&+ \big[c_n^{(m)} + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)} + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)} \\
&+ L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} \\
&+ \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \cdots b_n^{(2)}c_n^{(3)}\big]\big]\|u_n - r\| \\
&+ L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \cdots b_n^{(2)}\big\|u_n^{(1)} - r\big\| + \big[c_n^{(m)}\big\|v_n^{(m)} - r\big\| \\
&+ c_n^{(m-1)}\big(1 + Lb_n^{(m)}\big)\big\|v_n^{(m-1)} - r\big\| + c_n^{(m-2)}L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}\big\|v_n^{(m-2)} - r\big\| \\
&+ c_n^{(m-3)}L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\big\|v_n^{(m-3)} - r\big\| \\
&+ \cdots + c_n^{(2)}L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \cdots b_n^{(3)}\big\|v_n^{(2)} - r\big\|\big].
\end{aligned} \tag{3.14}$$

Now, from the last equation of (1.5), we have

$$\begin{aligned}
\big\|u_n^{(1)} - r\big\| &= \big\|a_n^{(1)}u_n + b_n^{(1)}Su_n + c_n^{(1)}v_n^{(1)} - r\big\| \\
&\le \|u_n - r\| + b_n^{(1)}\|Su_n - r\| + b_n^{(1)}\|u_n - r\| + c_n^{(1)}\|u_n - r\| + c_n^{(1)}\big\|v_n^{(1)} - r\big\| \\
&\le \big(1 + b_n^{(1)} + c_n^{(1)}\big)\|u_n - r\| + b_n^{(1)}L\|u_n - r\| + c_n^{(1)}\big\|v_n^{(1)} - r\big\| \\
&= \big[1 + (1 + L)b_n^{(1)} + c_n^{(1)}\big]\|u_n - r\| + c_n^{(1)}\big\|v_n^{(1)} - r\big\|.
\end{aligned} \tag{3.15}$$

Substituting (3.15) in (3.14), we have

$$\begin{aligned}
&\big\|u_{n+1} - u_n^{(m-1)}\big\| \\
&\le \big[(1 + L)b_n^{(m)} + (1 + L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)} + L(1 + L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \\
&\quad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \cdots b_n^{(2)} \\
&\quad + \big[c_n^{(m)} + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)} + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)} \\
&\quad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} \\
&\quad + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \cdots b_n^{(2)}c_n^{(3)}\big]\big]\|u_n - r\| \\
&\quad + L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \cdots b_n^{(2)}\big[\big(1 + (1 + L)b_n^{(1)} + c_n^{(1)}\big)\|u_n - r\| \\
&\quad + c_n^{(1)}\big\|v_n^{(1)} - r\big\|\big] + c_n^{(m)}\big\|v_n^{(m)} - r\big\| + c_n^{(m-1)}\big(1 + Lb_n^{(m)}\big)\big\|v_n^{(m-1)} - r\big\| \\
&\quad + c_n^{(m-2)}L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}\big\|v_n^{(m-2)} - r\big\| \\
&\quad + c_n^{(m-3)}L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\big\|v_n^{(m-3)} - r\big\| \\
&\quad + \cdots + c_n^{(2)}L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \cdots b_n^{(3)}\big\|v_n^{(2)} - r\big\| \\
&= \big[(1 + L)b_n^{(m)} + (1 + L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)} + L(1 + L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \\
&\quad + L^2(1 + L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\quad + \cdots + L^{m-3}(1 + L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \cdots b_n^{(2)} \\
&\quad + L^{m-2}(1 + L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \cdots b_n^{(2)}b_n^{(1)} \\
&\quad + \big[c_n^{(m)} + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)} + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)} \\
&\quad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} \\
&\quad + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \cdots b_n^{(2)}c_n^{(2)}
\end{aligned}$$

$$
\begin{aligned}
&+ L^{m-2}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}c_n^{(1)}\bigr]\bigr]\|u_n - r\| \\
&+ c_n^{(m)}\bigl\|v_n^{(m)} - r\bigr\| + c_n^{(m-1)}\bigl(1 + Lb_n^{(m)}\bigr)\bigl\|v_n^{(m-1)} - r\bigr\| \\
&+ c_n^{(m-2)}L\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}\bigl\|v_n^{(m-2)} - r\bigr\| \\
&+ c_n^{(m-3)}L^2\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\bigl\|v_n^{(m-3)} - r\bigr\| \\
&+ \cdots + c_n^{(2)}L^{m-3}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\bigl\|v_n^{(2)} - r\bigr\| \\
&+ c_n^{(1)}L^{m-2}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}\bigl\|v_n^{(1)} - r\bigr\|.
\end{aligned}
\tag{3.16}
$$

Substituting (3.16) in (3.6), we have

$$
\begin{aligned}
&\|u_{n+1} - r\| \\
&\quad\le \bigl(1 - \lambda b_n^{(m)} + \lambda^{-1}c_n^{(m)}\bigr)\|u_n - r\| \\
&\qquad + \lambda^{-1}Lb_n^{(m)}\bigl[\bigl[(1 + L)b_n^{(m)} + (1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)} \\
&\qquad + L(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)} \\
&\qquad + L^2(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\qquad + \cdots + L^{m-3}(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)} \\
&\qquad + L^{m-2}(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}b_n^{(1)} \\
&\qquad + \bigl[c_n^{(m)} + \bigl(1 + Lb_n^{(m)}\bigr)c_n^{(m-1)} + L\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}c_n^{(m-2)} \\
&\qquad + L^2\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} + \cdots + L^{m-3}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}c_n^{(2)} \\
&\qquad + L^{m-2}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}c_n^{(1)}\bigr]\bigr]\|u_n - r\| \\
&\qquad + c_n^{(m)}\bigl\|v_n^{(m)} - r\bigr\| + c_n^{(m-1)}\bigl(1 + Lb_n^{(m)}\bigr)\bigl\|v_n^{(m-1)} - r\bigr\| \\
&\qquad + c_n^{(m-2)}L\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}\bigl\|v_n^{(m-2)} - r\bigr\| \\
&\qquad + c_n^{(m-3)}L^2\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\bigl\|v_n^{(m-3)} - r\bigr\| \\
&\qquad + \cdots + c_n^{(2)}L^{m-3}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\bigl\|v_n^{(2)} - r\bigr\| \\
&\qquad + c_n^{(1)}L^{m-2}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}\bigl\|v_n^{(1)} - r\bigr\|\bigr] + \lambda^{-1}c_n^{(m)}\bigl\|v_n^{(m)} - r\bigr\| \\
&\quad= \bigl(1 - \lambda b_n^{(m)} + \lambda^{-1}c_n^{(m)}\bigr)\|u_n - r\| \\
&\qquad + \lambda^{-1}Lb_n^{(m)}\bigl[(1 + L)b_n^{(m)} + (1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)} \\
&\qquad + L(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)} \\
&\qquad + L^2(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\qquad + \cdots + L^{m-3}(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)} \\
&\qquad + L^{m-2}(1 + L)\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}b_n^{(1)} \\
&\qquad + \bigl[c_n^{(m)} + \bigl(1 + Lb_n^{(m)}\bigr)c_n^{(m-1)} + L\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}c_n^{(m-2)} \\
&\qquad + L^2\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} \\
&\qquad + \cdots + L^{m-3}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}c_n^{(2)} \\
&\qquad + L^{m-2}\bigl(1 + Lb_n^{(m)}\bigr)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}c_n^{(1)}\bigr]\bigr]\|u_n - r\|
\end{aligned}
$$

$$
\begin{aligned}
&+ c_n^{(m)}\lambda^{-1}\bigl(1+Lb_n^{(m)}\bigr)\bigl\|v_n^{(m)}-r\bigr\| + c_n^{(m-1)}\lambda^{-1}L\bigl(1+Lb_n^{(m)}\bigr)b_n^{(m)}\bigl\|v_n^{(m-1)}-r\bigr\| \\
&+ c_n^{(m-2)}\lambda^{-1}L^2\bigl(1+Lb_n^{(m)}\bigr)b_n^{(m)}b_n^{(m-1)}\bigl\|v_n^{(m-2)}-r\bigr\| \\
&+ \cdots + c_n^{(2)}\lambda^{-1}L^{m-2}\bigl(1+Lb_n^{(m)}\bigr)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\bigl\|v_n^{(2)}-r\bigr\| \\
&+ c_n^{(1)}\lambda^{-1}L^{m-1}\bigl(1+Lb_n^{(m)}\bigr)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}\bigl\|v_n^{(1)}-r\bigr\| \\
\le{}& \bigl[1-b_n^{(m)}\bigl[\lambda-\lambda^{-1}L\bigl[(1+L)b_n^{(m)}+(1+L)^2b_n^{(m-1)}+L(1+L)^2b_n^{(m-1)}b_n^{(m-2)} \\
&+ L^2(1+L)^2b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}+\cdots+L^{m-3}(1+L)^2b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)} \\
&+ L^{m-2}(1+L)^2b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}b_n^{(1)} \\
&+ c_n^{(m)}+(1+L)c_n^{(m-1)}+L(1+L)b_n^{(m-1)}c_n^{(m-2)} \\
&+ L^2(1+L)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}+\cdots+L^{m-3}(1+L)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)} \\
&+ L^{m-2}(1+L)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\bigr]-\lambda^{-1}\delta_n\bigr]\bigr]\|u_n-r\| \\
&+ \lambda^{-1}\bigl(1+Lb_n^{(m)}\bigr)\bigl[c_n^{(m)}\bigl\|v_n^{(m)}-q\bigr\|+c_n^{(m-1)}Lb_n^{(m)}\bigl\|v_n^{(m-1)}-r\bigr\| \\
&+ c_n^{(m-2)}L^2b_n^{(m)}b_n^{(m-1)}\bigl\|v_n^{(m-2)}-r\bigr\|+c_n^{(m-3)}L^3b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\bigl\|v_n^{(m-3)}-r\bigr\| \\
&+ \cdots+c_n^{(2)}L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\bigl\|v_n^{(2)}-r\bigr\| \\
&+ c_n^{(1)}L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}\bigl\|v_n^{(1)}-r\bigr\|\bigr] \\
\le{}& \bigl(1-\theta b_n^{(m)}\bigr)\|u_n-r\|+\lambda^{-1}(1+L)\bigl[c_n^{(m)}\bigl\|v_n^{(m)}-q\bigr\|+c_n^{(m-1)}Lb_n^{(m)}\bigl\|v_n^{(m-1)}-r\bigr\| \\
&+ c_n^{(m-2)}L^2b_n^{(m)}b_n^{(m-1)}\bigl\|v_n^{(m-2)}-r\bigr\|+c_n^{(m-3)}L^3b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\bigl\|v_n^{(m-3)}-r\bigr\| \\
&+ \cdots+c_n^{(2)}L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\bigl\|v_n^{(2)}-r\bigr\| \\
&+ c_n^{(1)}L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}\bigl\|v_n^{(1)}-r\bigr\|\bigr] \\
\le{}& \bigl(1-\theta b_n^{(m)}\bigr)\|u_n-r\|+\delta_n\lambda^{-1}(1+L)b_n^{(m)}A\bigl[1+Lb_n^{(m)}b_n^{(m-1)}+L^2b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)} \\
&+ L^3b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}+\cdots+L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\bigr] \\
&+ L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}].
\end{aligned}
\tag{3.17}
$$

Now, if we put

$$
\begin{aligned}
&\alpha_n = \|u_n-r\|, \\
&\omega_n = \theta b_n^{(m)}, \\
&\beta_n = \theta^{-1}\delta_n\lambda^{-1}(1+L)b_n^{(m)}A\bigl[1+Lb_n^{(m)}b_n^{(m-1)} \\
&\qquad + L^2b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}+L^3b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\qquad + \cdots+L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}+L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}\bigr], \\
&\gamma_n = 0,
\end{aligned}
$$

in (3.17), then, by condition (3), we observe that

$$\alpha_{n+1} \le (1-\omega_n)\alpha_n+\omega_n\beta_n+\gamma_n, \quad n\ge 0,$$

with $\{\omega_n\}_{n=0}^{\infty}\subset[0,1]$, $\sum_{n=0}^{\infty}\omega_n=\infty$, $\sum_{n=0}^{\infty}\gamma_n<\infty$ and $\lim_{n\to\infty}\beta_n=0$.

Hence, from Lemma 2.4, we have

$$\lim_{n\to\infty}\alpha_n = 0.$$

That is $\lim_{n\to\infty}\|u_n - r\| = 0$.

This ensures that the sequence $\{u_n\}_{n=0}^{\infty}$ of the MMNIPE given by (1.5) converges strongly to the common fixed r of T and S.

(ii) From the first equation of (3.1), we get

$$\begin{aligned}
&x_n = a_n^{(m)} w_n + b_n^{(m)} T x_n^{(m-1)} + c_n^{(m)} v_n^{(m)} \\
\Rightarrow\quad &\bigl(1 - b_n^{(m)}\bigr) w_n = \bigl(1 - (1-\lambda) b_n^{(m)}\bigr) x_n + b_n^{(m)} (I - T - \lambda I) x_n \\
&\qquad + b_n^{(m)}\bigl(T x_n - T x_n^{(m-1)}\bigr) - c_n^{(m)}\bigl(v_n^{(m)} - w_n\bigr).
\end{aligned}\tag{3.18}$$

Now, for all $n \geq 0$ combining (3.5) and (3.18), we obtain

$$\begin{aligned}
\bigl(1 - b_n^{(m)}\bigr)\|w_n - r\| &\geq \bigl\|\bigl(1 - (1-\lambda) b_n^{(m)}\bigr)(x_n - r) + b_n^{(m)}(I - T - \lambda I)(x_n - r)\bigr\| \\
&\quad - b_n^{(m)}\bigl\|T x_n - T x_n^{(m-1)}\bigr\| - c_n^{(m)}\bigl\|v_n^{(m)} - w_n\bigr\| \\
&= \bigl(1 - (1-\lambda) b_n^{(m)}\bigr)\left\| x_n - r + \frac{b_n^{(m)}}{1 - (1-\lambda) b_n^{(m)}}(I - T - \lambda I)(x_n - r)\right\| \\
&\quad - b_n^{(m)}\bigl\|T x_n - T x_n^{(m-1)}\bigr\| - c_n^{(m)}\bigl\|v_n^{(m)} - w_n\bigr\| \\
&\geq \bigl(1 - (1-\lambda) b_n^{(m)}\bigr)\|x_n - r\| - b_n^{(m)}\bigl\|T x_n - T x_n^{(m-1)}\bigr\| - c_n^{(m)}\bigl\|v_n^{(m)} - w_n\bigr\| \\
\Rightarrow\quad \|x_n - r\| &\leq \frac{(1 - b_n^{(m)})}{(1 - (1-\lambda) b_n^{(m)})}\|w_n - r\| + \frac{b_n^{(m)}}{(1 - (1-\lambda) b_n^{(m)})}\bigl\|T x_n - T x_n^{(m-1)}\bigr\| \\
&\quad + \frac{c_n^{(m)}}{(1 - (1-\lambda) b_n^{(m)})}\bigl\|v_n^{(m)} - w_n\bigr\| \\
&\leq \bigl(1 - \lambda b_n^{(m)}\bigr)\|w_n - r\| + \lambda^{-1} L b_n^{(m)}\bigl\|x_n - x_n^{(m-1)}\bigr\| + \lambda^{-1} c_n^{(m)}\bigl\|v_n^{(m)} - w_n\bigr\| \\
&\leq \bigl(1 - \lambda b_n^{(m)}\bigr)\|w_n - r\| + \lambda^{-1} L b_n^{(m)}\bigl\|x_n - x_n^{(m-1)}\bigr\| \\
&\quad + \lambda^{-1} c_n^{(m)}\bigl(\bigl\|v_n^{(m)} - r\bigr\| + \|w_n - r\|\bigr) \\
&= \bigl(1 - \lambda b_n^{(m)} + \lambda^{-1} c_n^{(m)}\bigr)\|w_n - r\| + \lambda^{-1} L b_n^{(m)}\bigl\|x_n - x_n^{(m-1)}\bigr\| \\
&\quad + \lambda^{-1} c_n^{(m)}\bigl\|v_n^{(m)} - r\bigr\|.
\end{aligned}\tag{3.19}$$

But, applying the second equation of (3.1), we have

$$\begin{aligned}
&\bigl\|x_n - x_n^{(m-1)}\bigr\| \\
&\quad\leq \bigl\|b_n^{(m)}\bigl(T x_n^{(m-1)} - w_n\bigr) + c_n^{(m)}\bigl(v_n^{(m)} - w_n\bigr)\bigr\| \\
&\qquad + \bigl\|b_n^{(m-1)}\bigl(w_n - T x_n^{(m-2)}\bigr) - c_n^{(m-1)}\bigl(v_n^{(m-1)} - w_n\bigr)\bigr\| \\
&\quad\leq b_n^{(m)}\bigl\|w_n - T x_n^{(m-1)}\bigr\| + c_n^{(m)}\bigl\|v_n^{(m)} - w_n\bigr\| + b_n^{(m-1)}\bigl\|w_n - T x_n^{(m-2)}\bigr\| \\
&\qquad + c_n^{(m-1)}\bigl\|v_n^{(m-1)} - w_n\bigr\| \\
&\quad\leq b_n^{(m)}\bigl(\|w_n - r\| + \bigl\|r - T x_n^{(m-1)}\bigr\|\bigr) + c_n^{(m)}\bigl(\bigl\|v_n^{(m)} - r\bigr\| + \|w_n - r\|\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\quad + b_n^{(m-1)}\left(\|w_n - r\| + \left\|r - Tx_n^{(m-2)}\right\|\right) + c_n^{(m-1)}\left(\left\|v_n^{(m-1)} - r\right\| + \|w_n - r\|\right) \\
&\le b_n^{(m)}\left(\|w_n - r\| + L\left\|x_n^{(m-1)} - r\right\|\right) + c_n^{(m)}\left(\left\|v_n^{(m)} - r\right\| + \|w_n - r\|\right) \\
&\quad + b_n^{(m-1)}\left(\|w_n - r\| + L\left\|x_n^{(m-2)} - r\right\|\right) + c_n^{(m-1)}\left(\left\|v_n^{(m-1)} - r\right\| + \|w_n - r\|\right) \\
&\le \left[b_n^{(m)} + b_n^{(m-1)} + c_n^{(m)} + c_n^{(m-1)}\right]\|w_n - r\| + Lb_n^{(m)}\left\|x_n^{(m-1)} - r\right\| \\
&\quad + Lb_n^{(m-1)}\left\|x_n^{(m-2)} - r\right\| + c_n^{(m)}\left\|v_n^{(m)} - r\right\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\|.
\end{aligned}
\tag{3.20}
$$

But after a simple calculation we get

$$
\begin{aligned}
&\left\|x_n^{(m-1)} - r\right\| \\
&\quad \le \|w_n - r\| + Lb_n^{(m-1)}\left\|x_n^{(m-2)} - r\right\| + b_n^{(m-1)}\|w_n - r\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\| \\
&\qquad + c_n^{(m-1)}\|w_n - r\| \\
&\quad = \left[1 + b_n^{(m-1)} + c_n^{(m-1)}\right]\|w_n - r\| + Lb_n^{(m-1)}\left\|x_n^{(m-2)} - r\right\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\|.
\end{aligned}
\tag{3.21}
$$

Substituting (3.21) in (3.20), we have

$$
\begin{aligned}
&\left\|x_n - x_n^{(m-1)}\right\| \\
&\quad \le \left[b_n^{(m)} + b_n^{(m-1)} + c_n^{(m)} + c_n^{(m-1)}\right]\|w_n - r\| \\
&\qquad + Lb_n^{(m)}\left[\left[1 + b_n^{(m-1)} + c_n^{(m-1)}\right]\|w_n - r\| + Lb_n^{(m-1)}\left\|x_n^{(m-2)} - r\right\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\|\right] \\
&\qquad + Lb_n^{(m-1)}\left\|x_n^{(m-2)} - r\right\| + c_n^{(m)}\left\|v_n^{(m)} - r\right\| + c_n^{(m-1)}\left\|v_n^{(m-1)} - r\right\| \\
&\quad = \left[(1 + L)b_n^{(m)} + \left(1 + Lb_n^{(m)}\right)b_n^{(m-1)} + c_n^{(m)} + \left(1 + Lb_n^{(m)}\right)c_n^{(m-1)}\right]\|w_n - r\| \\
&\qquad + Lb_n^{(m-1)}\left(1 + Lb_n^{(m)}\right)\left\|x_n^{(m-2)} - r\right\| + c_n^{(m)}\left\|v_n^{(m)} - r\right\| \\
&\qquad + c_n^{(m-1)}\left(1 + Lb_n^{(m)}\right)\left\|v_n^{(m-1)} - r\right\|.
\end{aligned}
\tag{3.22}
$$

Now, from the third equation of (3.1), we get

$$
\begin{aligned}
&\left\|x_n^{(m-2)} - r\right\| \\
&\quad \le \|w_n - r\| + Lb_n^{(m-2)}\left\|x_n^{(m-3)} - r\right\| + b_n^{(m-2)}\|w_n - r\| + c_n^{(m-2)}\left\|v_n^{(m-2)} - r\right\| \\
&\qquad + c_n^{(m-2)}\|w_n - r\| \\
&\quad = \left[1 + b_n^{(m-2)} + c_n^{(m-2)}\right]\|w_n - r\| + Lb_n^{(m-2)}\left\|x_n^{(m-3)} - r\right\| + c_n^{(m-2)}\left\|v_n^{(m-2)} - r\right\|.
\end{aligned}
\tag{3.23}
$$

Substituting (3.23) in (3.22), we have

$$
\begin{aligned}
&\left\|x_n - x_n^{(m-1)}\right\| \\
&\quad \le \left[(1 + L)b_n^{(m)} + \left(1 + Lb_n^{(m)}\right)b_n^{(m-1)} + c_n^{(m)} + \left(1 + Lb_n^{(m)}\right)c_n^{(m-1)}\right]\|w_n - r\| \\
&\qquad + Lb_n^{(m-1)}\left(1 + Lb_n^{(m)}\right)\left[\left[1 + b_n^{(m-2)} + c_n^{(m-2)}\right]\|w_n - r\| + Lb_n^{(m-2)}\left\|x_n^{(m-3)} - r\right\|\right. \\
&\qquad \left. + c_n^{(m-2)}\left\|v_n^{(m-2)} - r\right\|\right] + c_n^{(m)}\left\|v_n^{(m)} - r\right\| + c_n^{(m-1)}\left(1 + Lb_n^{(m)}\right)\left\|v_n^{(m-1)} - r\right\| \\
&\quad = \left[(1 + L)b_n^{(m)} + (1 + L)\left(1 + Lb_n^{(m)}\right)b_n^{(m-1)} + L\left(1 + Lb_n^{(m)}\right)b_n^{(m-1)}b_n^{(m-2)} + c_n^{(m)}\right. \\
&\qquad \left. + \left(1 + Lb_n^{(m)}\right)c_n^{(m-1)} + L\left(1 + Lb_n^{(m)}\right)b_n^{(m-1)}c_n^{(m-2)}\right]\|w_n - r\|
\end{aligned}
$$

$$+ L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\big\|x_n^{(m-3)} - r\big\| + c_n^{(m)}\big\|v_n^{(m)} - r\big\|$$
$$+ \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)}\big\|v_n^{(m-1)} - r\big\| + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)} - r\big\|.$$

Continuing the above procedure up to second iterative step of (3.1), we obtain

$$\begin{aligned}
&\big\|x_n - x_n^{(m-1)}\big\| \\
&\quad\le \Big[(1+L)b_n^{(m)} + (1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)} + L(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \\
&\qquad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)} \\
&\qquad + \Big[c_n^{(m)} + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)} + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)} \\
&\qquad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} \\
&\qquad + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(2)}c_n^{(3)}\Big]\Big]\|w_n - r\| \\
&\qquad + L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}\big\|x_n^{(1)} - r\big\| + \Big[c_n^{(m)}\big\|v_n^{(m)} - r\big\| \\
&\qquad + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)}\big\|v_n^{(m-1)} - r\big\| + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)} - r\big\| \\
&\qquad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)} - r\big\| \\
&\qquad + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)} - r\big\|\Big].
\end{aligned} \tag{3.24}$$

Now, from the last equation of (3.1), we have

$$\begin{aligned}
\big\|x_n^{(1)} - r\big\| &= \big\|a_n^{(1)}w_n + b_n^{(1)}Sw_n + c_n^{(1)}v_n^{(1)} - r\big\| \\
&\le \|w_n - r\| + b_n^{(1)}\|Sw_n - r\| + b_n^{(1)}\|w_n - r\| + c_n^{(1)}\|w_n - r\| + c_n^{(1)}\big\|v_n^{(1)} - r\big\| \\
&= \big(1 + b_n^{(1)} + c_n^{(1)}\big)\|w_n - r\| + b_n^{(1)}\|Sw_n - r\| + c_n^{(1)}\big\|v_n^{(1)} - r\big\| \qquad (3.25) \\
&\le \big(1 + b_n^{(1)} + c_n^{(1)}\big)\|w_n - r\| + b_n^{(1)}L\|w_n - r\| + c_n^{(1)}\big\|v_n^{(1)} - r\big\| \\
&= \big(1 + (1+L)b_n^{(1)} + c_n^{(1)}\big)\|w_n - r\| + c_n^{(1)}\big\|v_n^{(1)} - r\big\|.
\end{aligned}$$

Combining (3.24) and (3.25), we obtain

$$\begin{aligned}
&\big\|x_n - x_n^{(m-1)}\big\| \\
&\quad\le \Big[(1+L)b_n^{(m)} + (1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)} + L(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} \\
&\qquad + L^2(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\qquad + \cdots + L^{m-3}(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)} \\
&\qquad + L^{m-2}(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}b_n^{(1)} \\
&\qquad + \Big[c_n^{(m)} + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)} + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)} \\
&\qquad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)} \\
&\qquad + L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\Big]\Big]\|w_n - r\| \\
&\qquad + \Big[c_n^{(m)}\big\|v_n^{(m)} - r\big\| + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)}\big\|v_n^{(m-1)} - r\big\| \\
&\qquad + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)} - r\big\|
\end{aligned}$$

$$\begin{aligned}
&+ L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)} - r\big\| \\
&+ \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)} - r\big\| \\
&+ L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big\|v_n^{(1)} - r\big\|\big].
\end{aligned} \tag{3.26}$$

Inserting (3.26) in (3.19), we find

$$\begin{aligned}
&\|x_n - r\| \\
&\le \big[1 - \lambda b_n^{(m)} + \lambda^{-1}c_n^{(m)} + \lambda^{-1}Lb_n^{(m)}\big[(1+L)b_n^{(m)} + (1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)} \\
&\quad + L(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} + L^2(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\quad + \cdots + L^{m-3}(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)} \\
&\quad + L^{m-2}(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}b_n^{(1)} \\
&\quad + \big[c_n^{(m)} + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)} + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)} \\
&\quad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)} \\
&\quad + L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big]\big]\big]\|w_n - r\| \\
&\quad + \lambda^{-1}Lb_n^{(m)}\big[c_n^{(m)}\big\|v_n^{(m)} - r\big\| + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)}\big\|v_n^{(m-1)} - r\big\| \\
&\quad + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)} - r\big\| \\
&\quad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)} - r\big\| \\
&\quad + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)} - r\big\| \\
&\quad + L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big\|v_n^{(1)} - r\big\|\big] + \lambda^{-1}c_n^{(m)}\big\|v_n^{(m)} - r\big\| \\
&= \big[1 - b_n^{(m)}\big[\lambda - \lambda^{-1}L\big[(1+L)b_n^{(m)} + (1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)} \\
&\quad + L(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)} + L^2(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\quad + \cdots + L^{m-3}(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)} \\
&\quad + L^{m-2}(1+L)\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}b_n^{(1)} \\
&\quad + \big[c_n^{(m)} + \big(1 + Lb_n^{(m)}\big)c_n^{(m-1)} + L\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}c_n^{(m-2)} \\
&\quad + L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} + \cdots + L^{m-3}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)} \\
&\quad + L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big]\big] - \lambda^{-1}\delta_n\big]\big]\|w_n - r\| \\
&\quad + \lambda^{-1}\big(1 + Lb_n^{(m)}\big)c_n^{(m)}\big\|v_n^{(m)} - r\big\| + \lambda^{-1}L\big(1 + Lb_n^{(m)}\big)b_n^{(m)}c_n^{(m-1)}\big\|v_n^{(m-1)} - r\big\| \\
&\quad + \lambda^{-1}L^2\big(1 + Lb_n^{(m)}\big)b_n^{(m)}b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)} - r\big\| \\
&\quad + \lambda^{-1}L^3\big(1 + Lb_n^{(m)}\big)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)} - r\big\| \\
&\quad + \cdots + \lambda^{-1}L^{m-2}\big(1 + Lb_n^{(m)}\big)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)} - r\big\| \\
&\quad + \lambda^{-1}L^{m-1}\big(1 + Lb_n^{(m)}\big)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big\|v_n^{(1)} - r\big\| \\
&\le \big[1 - b_n^{(m)}\big[\lambda - \lambda^{-1}L\big[(1+L)b_n^{(m)} + (1+L)^2b_n^{(m-1)} \\
&\quad + L(1+L)^2b_n^{(m-1)}b_n^{(m-2)} + L^2(1+L)^3b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)} \\
&\quad + \cdots + L^{m-3}(1+L)^2b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}
\end{aligned}$$

$$
\begin{aligned}
&+ L^{m-2}(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} b_n^{(1)} \\
&+ \big[c_n^{(m)} + (1+L)c_n^{(m-1)} + L(1+L)b_n^{(m-1)} c_n^{(m-2)} \\
&+ L^2(1+L)b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)} + \cdots + L^{m-3}(1+L)b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)} \\
&+ L^{m-2}(1+L)b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)}\big]\big] - \lambda^{-1}\delta_n\big]\big]\|w_n - r\| \\
&+ \lambda^{-1}(1+L)c_n^{(m)} \big\|v_n^{(m)} - r\big\| + \lambda^{-1}L(1+L)b_n^{(m)} c_n^{(m-1)} \big\|v_n^{(m-1)} - r\big\| \\
&+ \lambda^{-1}L^2(1+L)b_n^{(m)} b_n^{(m-1)} c_n^{(m-2)} \big\|v_n^{(m-2)} - r\big\| \\
&+ \lambda^{-1}L^3(1+L)b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)} \big\|v_n^{(m-3)} - r\big\| \\
&+ \cdots + \lambda^{-1}L^{m-2}(1+L)b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)} \big\|v_n^{(2)} - r\big\| \\
&+ \lambda^{-1}L^{m-1}(1+L)b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)} \big\|v_n^{(1)} - r\big\| \\
\le{}& \big(1 - \theta b_n^{(m)}\big)\|w_n - r\| \\
&+ \lambda^{-1}(1+L)\big[c_n^{(m)} \big\|v_n^{(m)} - r\big\| + L\big[b_n^{(m)} c_n^{(m-1)} \big\|v_n^{(m-1)} - r\big\| \\
&+ L\big[b_n^{(m)} b_n^{(m-1)} c_n^{(m-2)} \big\|v_n^{(m-2)} - r\big\| + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)} \big\|v_n^{(m-3)} - r\big\| \\
&+ \cdots + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)} \big\|v_n^{(2)} - r\big\| \\
&+ L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)} \big\|v_n^{(1)} - r\big\|\big]\big]\big]\big] \cdots \big]\big].
\end{aligned}
\tag{3.27}
$$

Now, using (3.27), we obtain

$$
\begin{aligned}
&\|w_{n+1} - r\| \\
&\quad\le \|w_{n+1} - x_n\| + \|x_n - r\| \\
&\quad\le \big(1 - \theta b_n^{(m)}\big)\|w_n - r\| \\
&\qquad + \lambda^{-1}(1+L)[c_n^{(m)} \big\|v_n^{(m)} - r\big\| + L[b_n^{(m)} c_n^{(m-1)} \big\|v_n^{(m-1)} - r\big\| \\
&\quad\le \big(1 - \theta b_n^{(m)}\big)\|w_n - r\| + \lambda^{-1}(1+L)\big[c_n^{(m)} A + L\big[b_n^{(m)} c_n^{(m-1)} A \\
&\qquad + L\big[b_n^{(m)} b_n^{(m-1)} c_n^{(m-2)} A + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)} A \\
&\qquad + \cdots + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)} A \\
&\qquad + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)} A\big]\big]\big]\big] \cdots \big]\big] + \mu_n \\
&\quad= \big(1 - \theta b_n^{(m)}\big)\|w_n - r\| + \lambda^{-1}(1+L)A b_n^{(m)}\big[\delta_n + L\big[c_n^{(m-1)} \\
&\qquad + L\big[b_n^{(m-1)} c_n^{(m-2)} + L\big[b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)} + \cdots + L\big[b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)} \\
&\qquad + L\big[b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)}\big]\big]\big]\big] \cdots \big]\big] + \mu_n, \quad n \ge 0.
\end{aligned}
\tag{3.28}
$$

(iii) If we put

$$
\begin{aligned}
&\alpha_n = \|w_n - r\|, \\
&\omega_n = \theta b_n^{(m)},
\end{aligned}
$$

$$\begin{aligned}\beta_n &= \theta^{-1}\lambda^{-1}(1+L)A\big[\delta_n + L\big[c_n^{(m-1)} + L\big[b_n^{(m-1)}c_n^{(m-2)} + L\big[b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} \\ &\quad + \cdots + L\big[b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)} + L\big[b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big]\big]\big]\big]\cdots\big]\big], \\ \gamma_n &= \mu_n, \quad \forall n \geq 0,\end{aligned}$$

in (3.28), then we obtain

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty}\omega_n = \infty$, $\sum_{n=0}^{\infty}\gamma_n < \infty$ and $\lim_{n\to\infty}\beta_n = 0$.

Hence, from Lemma 2.4, we have $\lim_{n\to\infty}\alpha_n = 0$.

That is $\lim_{n\to\infty}\|w_n - r\| = 0$. Hence, $\lim_{n\to\infty} w_n = r$ and this ensures that $\{u_n^{(m)}\}_{n=0}^{\infty}$ is almost common-stable on B.

(iv) Considering $\lim_{n\to\infty} w_n = r$, we have $\lim_{n\to\infty} w_{n+1} = r$.

Now, using (3.27), we get

$$\begin{aligned}\mu_n &= \|w_{n+1} - x_n\| \leq \|w_{n+1} - r\| + \|x_n - r\| \\ &\leq \|w_{n+1} - r\| + \big(1 - \theta b_n^{(m)}\big)\|w_n - r\| \\ &\quad + \lambda^{-1}(1+L)Ab_n^{(m)}\big[\delta_n + L\big[c_n^{(m-1)} + L\big[b_n^{(m-1)}c_n^{(m-2)} + L\big[b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)} \\ &\quad + \cdots + L\big[b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)} + Lb_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big]\big]\big]\cdots\big]\big] \to 0,\end{aligned}$$

as $n \to \infty$, this means that $\lim_{n\to\infty}\mu_n = 0$.

This completes the proof. □

Theorem 3.2 *Let B, X, T, S, θ, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{v_n^{(1)}\},\ldots,\{v_n^{(m)}\}$, and $\{\mu_n\}_{n=0}^{\infty}$ be as in Theorem 3.1 and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$, $\{c_n^{(i)}\}$ for each $i \in \{1,2,\ldots,m\}$ be any appropriate real sequences in $[0,1]$ satisfying the conditions (1), (3), (4), (5) of Theorem 3.1 with the following property*:

$$\sum_{n=0}^{\infty} c_n^{(m)} < \infty.$$

Then the results of Theorem 3.1 hold.

Proof The proof of this theorem is similar to the proof of Theorem 3.1, so here we omit it. □

Theorem 3.3 *Let B, X, T, S, θ, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{v_n^{(1)}\},\ldots,\{v_n^{(m)}\}$, and $\{\mu_n\}_{n=0}^{\infty}$ be as in Theorem 3.1 and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$, $\{c_n^{(i)}\}$ for each $i \in \{1,2,\ldots,m\}$ be any appropriate real sequences in $[0,1]$ satisfying the conditions (1), (3) and (5) of Theorem 3.1 with the following property*:

$$\lim_{n\to\infty} c_n^{(m)} = 0, \quad \textit{and} \quad b_n^{(m)} \geq h > 0, \quad \forall n \geq 0,$$

where h is a constant.

Then

(i) *the iterative sequence* $\{u_n\}_{n=0}^{\infty}$ *given by* (1.5) *converges strongly to the common fixed r of T and S and the following inequality holds*:

$$\|u_{n+1}-r\| \leq (1-\theta h)\|u_n-r\| + D, \quad \forall n \geq 0,$$

where

$$\begin{aligned} D = \lambda^{-1}(1+L)\Big[&\sup_{n\geq 0}\big\{c_n^{(m)}\big\|v_n^{(m)}-r\big\|\big\} + Lb_n^{(m)}\sup_{n\geq 0}\big\{c_n^{(m-1)}\big\|v_n^{(m-1)}-r\big\|\big\} \\ &+ L^2 b_n^{(m)}b_n^{(m-1)}\sup_{n\geq 0}\big\{c_n^{(m-2)}\big\|v_n^{(m-2)}-r\big\|\big\} \\ &+ L^3 b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\sup_{n\geq 0}\big\{c_n^{(m-3)}\big\|v_n^{(m-3)}-r\big\|\big\} \\ &+ \cdots + L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\sup_{n\geq 0}\big\{c_n^{(2)}\big\|v_n^{(2)}-r\big\|\big\} \\ &+ L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}\sup_{n\geq 0}\big\{c_n^{(1)}\big\|v_n^{(1)}-r\big\|\big\}\Big], \end{aligned}$$

(ii)

$$\begin{aligned} &\|w_{n+1}-r\| \\ &\quad\leq (1-\theta h)\|w_n-r\| + \lambda^{-1}(1+L)\big[c_n^{(m)}\big\|v_n^{(m)}-r\big\| + L\big[c_n^{(m-1)}\big\|v_n^{(m-1)}-r\big\| \\ &\qquad + L\big[c_n^{(m-2)}\big\|v_n^{(m-2)}-r\big\| + L\big[c_n^{(m-3)}\big\|v_n^{(m-3)}-r\big\| \\ &\qquad + \cdots + L\big[c_n^{(2)}\big\|v_n^{(2)}-r\big\| + L\big[c_n^{(1)}\big\|v_n^{(1)}-r\big\|\big]\big]\big]\big]\cdots\big]\big] + \mu_n, \quad \forall n \geq 0, \end{aligned}$$

(iii) $\lim_{n\to\infty} w_n = r$, *if and only if* $\lim_{n\to\infty}\mu_n = 0$.

Proof (i) Following the proof of Theorem 3.1, we have

$$\begin{aligned} &\|u_{n+1}-r\| \\ &\quad\leq \big(1-\theta b_n^{(m)}\big)\|u_n-r\| + \lambda^{-1}(1+L)c_n^{(m)}\big\|v_n^{(m)}-r\big\| + \lambda^{-1}L(1+L)b_n^{(m)}c_n^{(m-1)}\big\|v_n^{(m-1)}-r\big\| \\ &\qquad + \lambda^{-1}L^2(1+L)b_n^{(m)}b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)}-r\big\| \\ &\qquad + \lambda^{-1}L^3(1+L)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)}-r\big\| \\ &\qquad + \cdots + \lambda^{-1}L^{m-2}(1+L)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)}-r\big\| \\ &\qquad + \lambda^{-1}L^{m-1}(1+L)b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big\|v_n^{(1)}-r\big\| \\ &\quad\leq (1-\theta h)\|u_n-r\| + \lambda^{-1}(1+L)\big[c_n^{(m)}\big\|v_n^{(m)}-r\big\| + Lb_n^{(m)}c_n^{(m-1)}\big\|v_n^{(m-1)}-r\big\| \\ &\qquad + L^2 b_n^{(m)}b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)}-r\big\| + L^3 b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)}-r\big\| \\ &\qquad + \cdots + L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)}-r\big\| \\ &\qquad + L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big\|v_n^{(1)}-r\big\|\big] \\ &\quad\leq (1-\theta h)\|u_n-r\| + \lambda^{-1}(1+L)\Big[\sup_{n\geq 0}\big\{c_n^{(m)}\big\|v_n^{(m)}-r\big\|\big\} + Lb_n^{(m)}\sup_{n\geq 0}\big\{c_n^{(m-1)}\big\|v_n^{(m-1)}-r\big\|\big\} \\ &\qquad + L^2 b_n^{(m)}b_n^{(m-1)}\sup_{n\geq 0}\big\{c_n^{(m-2)}\big\|v_n^{(m-2)}-r\big\|\big\} \end{aligned}$$

$$
\begin{aligned}
&\quad + L^3 b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \sup_{n\geq 0}\{c_n^{(m-3)}\|v_n^{(m-3)} - r\|\} \\
&\quad + \cdots + L^{m-2} b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} \sup_{n\geq 0}\{c_n^{(2)}\|v_n^{(2)} - r\|\} \\
&\quad + L^{m-1} b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} \sup_{n\geq 0}\{c_n^{(1)}\|v_n^{(1)} - r\|\}\Big] \\
&= (1-\theta h)\|u_n - r\| + D, \quad n \geq 0.
\end{aligned}
$$

Now, putting

$$
\begin{aligned}
&\alpha_n = \|u_n - r\|, \qquad \eta = 1-\theta h \quad \text{and} \\
&\beta_n = \lambda^{-1}(1+L)\Big[\sup_{n\geq 0}\{c_n^{(m)}\|v_n^{(m)} - r^*\|\} + L b_n^{(m)} \sup_{n\geq 0}\{c_n^{(m-1)}\|v_n^{(m-1)} - r^*\|\} \\
&\quad + L^2 b_n^{(m)} b_n^{(m-1)} \sup_{n\geq 0}\{c_n^{(m-2)}\|v_n^{(m-2)} - r^*\|\} \\
&\quad + L^3 b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \sup_{n\geq 0}\{c_n^{(m-3)}\|v_n^{(m-3)} - r^*\|\} \\
&\quad + \cdots + L^{m-2} b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} \sup_{n\geq 0}\{c_n^{(2)}\|v_n^{(2)} - r^*\|\} \\
&\quad + L^{m-1} b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} \sup_{n\geq 0}\{c_n^{(1)}\|v_n^{(1)} - r^*\|\}\Big], \quad \forall n \geq 0,
\end{aligned}
$$

we get $0 \leq \eta < 1$ and $\lim_{n\to\infty} \beta_n = 0$. Hence, by an application of Lemma 2.5, we obtain

$$
\lim_{n\to\infty} \|u_n - r\| = 0.
$$

This ensures that the sequence $\{u_n\}_{n=0}^{\infty}$ converges strongly to the common fixed r of T and S.

(ii) Again, from (3.27), we find

$$
\begin{aligned}
&\|w_{n+1} - r\| \\
&\leq \big(1-\theta b_n^{(m)}\big)\|w_n - r\| + \lambda^{-1}(1+L)\big[c_n^{(m)}\|v_n^{(m)} - r\| + L\big[b_n^{(m)} c_n^{(m-1)}\|v_n^{(m-1)} - r\| \\
&\quad + L\big[b_n^{(m)} b_n^{(m-1)} c_n^{(m-2)}\|v_n^{(m-2)} - r\| + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)}\|v_n^{(m-3)} - r\| \\
&\quad + \cdots + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)}\|v_n^{(2)} - r\| \\
&\quad + L\big[b_n^{(m)} b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)}\|v_n^{(1)} - r\|\big]\big]\big]\big] \cdots\big]\big] + \mu_n \\
&\leq (1-\theta h)\|w_n - r\| + \lambda^{-1}(1+L)\big[c_n^{(m)}\|v_n^{(m)} - r\| + L\big[c_n^{(m-1)}\|v_n^{(m-1)} - r\| \\
&\quad + L\big[c_n^{(m-2)}\|v_n^{(m-2)} - r\| + L\big[c_n^{(m-3)}\|v_n^{(m-3)} - r\| \\
&\quad + \cdots + L\big[c_n^{(2)}\|v_n^{(2)} - r\| + L\big[c_n^{(1)}\|v_n^{(1)} - r\|\big]\big]\big]\big] \cdots\big]\big] + \mu_n.
\end{aligned}
$$

(iii) Considering $\lim_{n\to\infty} w_n = r$, we have $\lim_{n\to\infty} w_{n+1} = r$.

Now, using (3.27), we get

$$
\mu_n = \|w_{n+1} - x_n\|
$$

$$
\begin{aligned}
&\le \|w_{n+1}-r\| + \|x_n - r\| \\
&\le \|w_{n+1}-r\| + (1-\theta h)\|w_n - r\| \\
&\quad + \lambda^{-1}(1+L)b_n^{(m)}\big[\delta_n \big\| v_n^{(m)} - r\big\| + L\big[c_n^{(m-1)}\big\|v_n^{(m-1)} - r\big\| \\
&\quad + L\big[b_n^{(m-1)}c_n^{(m-2)}\big\|v_n^{(m-2)} - r\big\| + L\big[b_n^{(m-1)}b_n^{(m-2)}c_n^{(m-3)}\big\|v_n^{(m-3)} - r\big\| \\
&\quad + \cdots + L\big[b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}c_n^{(2)}\big\|v_n^{(2)} - r\big\| \\
&\quad + L\big[b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}c_n^{(1)}\big\|v_n^{(1)} - r\big\|\big]\big]\big]\big]\cdots\big]\big] \to 0,
\end{aligned}
$$

as $n \to \infty$, this means that $\lim_{n\to\infty} \mu_n = 0$.

Conversely, suppose that $\lim_{n\to\infty} \mu_n = 0$. Now, by setting

$$
\begin{aligned}
&\alpha_n = \|w_n - r\|, \qquad \eta = 1-\theta h \quad \text{and} \\
&\beta_n = \lambda^{-1}(1+L)\Big[\sup_{n\ge 0}\big\{c_n^{(m)}\big\|v_n^{(m)} - r^*\big\|\big\} + Lb_n^{(m)}\sup_{n\ge 0}\big\{c_n^{(m-1)}\big\|v_n^{(m-1)} - r^*\big\|\big\} \\
&\quad + L^2 b_n^{(m)}b_n^{(m-1)}\sup_{n\ge 0}\big\{c_n^{(m-2)}\big\|v_n^{(m-2)} - r^*\big\|\big\} \\
&\quad + L^3 b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\sup_{n\ge 0}\big\{c_n^{(m-3)}\big\|v_n^{(m-3)} - r^*\big\|\big\} \\
&\quad + \cdots + L^{m-2}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}\cdots b_n^{(3)}\sup_{n\ge 0}\big\{c_n^{(2)}\big\|v_n^{(2)} - r^*\big\|\big\} \\
&\quad + L^{m-1}b_n^{(m)}b_n^{(m-1)}b_n^{(m-2)}b_n^{(m-3)}\cdots b_n^{(2)}\sup_{n\ge 0}\big\{c_n^{(1)}\big\|v_n^{(1)} - r^*\big\|\big\}\Big] + \mu_n, \quad \forall n \ge 0,
\end{aligned}
$$

we get $0 \le \eta < 1$ and $\lim_{n\to\infty} \beta_n = 0$. Hence, by an application of Lemma 2.5, we obtain

$$
\lim_{n\to\infty} \|w_n - r\| = 0.
$$

This completes the proof. □

The following corollaries show the convergence, almost common-stability and common-stability for the corresponding modified multi-step Noor iterative procedure without errors for two Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces.

Corollary 3.4 *Let B be a nonempty closed convex subset of X and T and S be two Lipschitz strictly hemicontractive-type mappings from B into itself. Suppose that $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $\{\beta_n^{(i)}\}_{n=0}^{\infty}$ for each $i \in \{1,2,\dots,m\}$ are any appropriate real sequences in $[0,1]$ satisfying the following conditions*:

(6) $\alpha_n^{(i)} + \beta_n^{(i)} = 1$, *for each* $i \in \{1,2,3,\dots,m\}$,
(7) $\sum_{n=0}^{\infty} \alpha_n^{(j)} = \infty$ *for each* $j \in \{2,3,\dots,m\}$,
(8)

$$
\begin{aligned}
&L\big[(1+L)\beta_n^{(m)} + (1+L)^2\big[\beta_n^{(m-1)} + L(1+L)^2\beta_n^{(m-1)}\beta_n^{(m-2)} + L^2\beta_n^{(m-1)}\beta_n^{(m-2)}\beta_n^{(m-3)} \\
&\quad + \cdots + L^{m-3}\beta_n^{(m-1)}\beta_n^{(m-2)}\cdots\beta_n^{(2)} + L^{m-2}\beta_n^{(m-1)}\beta_n^{(m-2)}\cdots\beta_n^{(2)}\beta_n^{(1)}\big]\big] \\
&\le \lambda(\lambda-\theta), \quad n \ge 0,
\end{aligned}
$$

where θ is a constant in $(0,\lambda)$ and $\lambda \in (0,1)$.

For $u_0 \in B$, we assume an iterative sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{aligned} &u_{n+1} = u_n^{(m)} = \alpha_n^{(m)} u_n + \beta_n^{(m)} T u_n^{(m-1)}, \\ &u_n^{(m-1)} = \alpha_n^{(m-1)} u_n + \beta_n^{(m-1)} T u_n^{(m-2)}, \\ &\ldots\ldots \\ &u_n^{(2)} = \alpha_n^{(2)} u_n + \beta_n^{(2)} T u_n^{(1)}, \\ &u_n^{(1)} = \alpha_n^{(1)} u_n + \beta_n^{(1)} S u_n, \quad n \geq 0 \end{aligned}\right\} \tag{3.29}$$

and let $\{w_n\}_{n=0}^{\infty}$ be any sequence in B and $\{\mu_n\}_{n=0}^{\infty}$ be a sequence defined by

$$\mu_n = \|w_{n+1} - x_n\|, \quad n \geq 0,$$

where

$$\left.\begin{aligned} &x_n = x_n^{(m)} = \alpha_n^{(m)} w_n + \beta_n^{(m)} T x_n^{(m-1)}, \\ &x_n^{(m-1)} = \alpha_n^{(m-1)} w_n + \beta_n^{(m-1)} T x_n^{(m-2)}, \\ &\ldots\ldots \\ &x_n^{(2)} = \alpha_n^{(2)} w_n + \beta_n^{(2)} T x_n^{(1)}, \\ &x_n^{(1)} = \alpha_n^{(1)} w_n + \beta_n^{(1)} S w_n, \quad n \geq 0. \end{aligned}\right\} \tag{3.30}$$

Then

(i) *the iterative sequence $\{u_n\}_{n=0}^{\infty}$ given by (3.29) converges strongly to the common fixed r of T and S,*

(ii) *$\sum_{n=0}^{\infty} \mu_n < \infty$ implies that $\lim_{n\to\infty} w_n = r$, so that $\{u_n^{(m)}\}_{n=0}^{\infty}$ is almost common-stable on B,*

(iii) *$\lim_{n\to\infty} w_n = r$, implies that $\lim_{n\to\infty} \mu_n = 0$.*

Proof The proof follows from the proof of Theorem 3.1 and, for brevity, here we omit it. □

Corollary 3.5 *Let B, X, T, S, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ be as in Corollary 3.4 and θ be as in Theorem 3.1. Suppose that $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $\{\beta_n^{(i)}\}_{n=0}^{\infty}$ for each $i \in \{1, 2, \ldots, m\}$ are any appropriate real sequences in $[0, 1]$ satisfying the similar condition of condition (3) of Theorem 3.1 and the conditions (6), (7) and (8) of Corollary 3.4 along with the following property:*

$$\beta_n^{(m)} \geq h > 0, \quad \forall n \geq 0,$$

where h is a constant.

Then

(i) *the iterative sequence $\{u_n\}_{n=0}^{\infty}$ given by (3.29) converges strongly to the common fixed r of T and S and the following inequality holds:*

$$\|u_{n+1} - r\| \leq (1 - \theta h)\|u_n - r\|, \quad \forall n \geq 0,$$

(ii) *$\|w_{n+1} - r\| \leq (1 - \theta h)\|w_n - r\| + \mu_n$, $\forall n \geq 0$,*

(iii) *$\lim_{n\to\infty} w_n = r$, if and only if $\lim_{n\to\infty} \mu_n = 0$.*

Proof The proof follows from the proof of Theorem 3.3 and, for brevity, here we omit it. □

Remark 3.6

(a) If we put $m = 2$ in our Theorem 3.1, Theorem 3.2 and Theorem 3.3, then we can easily establish Theorem 9, Theorem 10 and Theorem 11 of Hussain *et al.* [1], respectively. Therefore, we can comment that the results of Hussain *et al.* [1] are special case of our results.

(b) Since the MIP given by Mann [28], the IIP given by Ishikawa [29], the NIP given by Xu and Noor [30], the MIPE given in Liu [31] and Xu [32], the IIPE given by Liu [31] and Xu [32] and the TIPE given by Cho *et al.* [33] are all special cases of our newly proposed MMNIPE given by (1.5), by setting the appropriate values of m and $c_n^{(m)}$ in our Theorem 3.1, Theorem 3.2 and Theorem 3.3, we can easily obtain the convergence, almost common-stability and common-stability criteria of the above-mentioned iterative procedures for two Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces.

4 Examples

In this section, we provide a numerical example to verify our analytical results and to show a numerical comparison between our newly proposed MMNIPE given by (1.5) and some other most analogous iterative procedures with errors.

Example 4.1 Consider B is a nonempty subset of an arbitrary Banach space X with the usual norm and let $B = \mathbb{R}$. Suppose that T and S are two self-maps on B which are defined as follows:

$$Tu = \frac{1}{3}\sin^2 u, \quad \text{and} \quad Su = \frac{2}{3}u^2.$$

Now, if we let $L = \frac{2}{3}$, $\sigma = \frac{3}{2}$, $\theta = \frac{1}{300}$, then it is obvious that $F_T \cap F_S = \{0\}$, $\lambda = \frac{\sigma-1}{\sigma} = \frac{3/2-1}{3/2} = \frac{1}{3} \in (0,1)$ and

$$|Tu_1 - Tu_2| = \frac{1}{3}\left|\sin^2 u_1 - \sin^2 u_2\right| \le \frac{2}{3}|\sin u_1 - \sin u_2||\sin u_1 + \sin u_2| \le L|u_1 - u_2|,$$

$$|Su_1 - Su_2| = \frac{2}{3}\left|u_1^2 - u_2^2\right| = \frac{2}{3}|u_1 - u_2||u_1 + u_2| \le L|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R}.$$

Hence, both T and S are Lipschitzian mappings on B.

Likewise, using (1.1) we have

$$\begin{aligned}
&\left|\left(1+t'\right)(q-r) - t't(Tq - Tr)\right| \\
&\quad \ge \left(1+t'\right)|q-r| - t't|Tq - Tr| \\
&\quad = |q-r| + t'\left(|q-r| - t|Tq - Tr|\right) \\
&\quad \ge |q-r| + t'\left(|q-r| - tL|q-r|\right) = \left(1 + t'(1-tL)\right)|q-r| \\
&\quad \ge \|q-r\|
\end{aligned}$$

for any $q, r \in \mathbb{R}$ and $t' > 0$. Therefore, T is strongly pseudocontractive and hence Lemma 2.8 confirms that T is strictly hemicontractive on B. Also, the similar arguments hold for the mapping S. Hence both T and S are Lipschitz strictly hemicontractive mappings on B.

Now, if we consider

$$b_n^{(m)} = \frac{5}{9}\cdot\frac{1}{\sqrt{n}+100}, \qquad c_n^{(m)} = \frac{1}{(\sqrt{n}+100)^2}, \qquad a_n^{(m)} = 1-\left(b_n^{(m)}+c_n^{(m)}\right),$$
$$b_n^{(j)} = c_n^{(j)} = \frac{3}{5}\cdot\frac{1}{n+100}, \qquad a_n^{(j)} = 1-\left(b_n^{(j)}+c_n^{(j)}\right),$$
$$\text{where } j=1,2,\ldots,m-1, \text{ and } \forall n\geq 0.$$

Then, for $n=0$, $m=10$, we obtain

$$\begin{aligned}
&L\Big[(1+L)b_n^{(m)} + (1+L)^2 b_n^{(m-1)} + L(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} \\
&\quad + L^2(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} + \cdots + L^{m-3}(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} \\
&\quad + L^{m-2}(1+L)^2 b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} b_n^{(1)} \\
&\quad + \Big[c_n^{(m)} + (1+L)c_n^{(m-1)} + L(1+L) b_n^{(m-1)} c_n^{(m-2)} \\
&\quad + L^2(1+L) b_n^{(m-1)} b_n^{(m-2)} c_n^{(m-3)} + \cdots + L^{m-3}(1+L) b_n^{(m-1)} b_n^{(m-2)} \cdots b_n^{(3)} c_n^{(2)} \\
&\quad + L^{m-2}(1+L) b_n^{(m-1)} b_n^{(m-2)} b_n^{(m-3)} \cdots b_n^{(2)} c_n^{(1)}\Big]\Big] + \frac{c_n^{(m)}}{b_n^{(m)}} \leq 0.042231081 \leq 0.11.
\end{aligned}$$

Therefore, by an application of Theorem 3.1, we can say that the iterative sequence $\{u_n\}_{n=0}^{\infty}$ defined by (1.5) converges strongly to the common fixed 0 of T and S in B and the corresponding MMNIPE given by (1.5) is common-stable as well as almost common-stable on B.

Analogously, by applying Theorem 3.2 and Theorem 3.3, we can easily prove that the iterative sequence $\{u_n\}_{n=0}^{\infty}$ defined by (1.5) converges strongly to the common fixed 0 of T and S in B and the corresponding MMNIPE given by (1.5) is common-stable as well as almost common-stable on B.

For the numerical experiment, here we consider our newly proposed MMNIPE given by (1.5) for $m=10$, and compared it with the STIIPE given by Hussain *et al.* [1], the IIPE given

Table 1 Numerical results corresponding to $u_0 = 0.5$ for 1000 iterative steps

Step no.	Our proposed MMNIPE	STIIPE of Hussain *et al.*	IIPE of Liu	TIPE of Cho *et al.*
1	0.5	0.5	0.5	0.5
2	0.497667276003391	0.497668105410462	0.998544685005923	0.497667277380927
3	0.495335888378632	0.495337523105695	1.328130014700319	0.495335891074855
4	0.493012550083953	0.493014966750906	1.572697385396054	0.493012554043365
5	0.490699600582596	0.490702776576019	1.765873713917432	0.490699605752504
…	…	…	…	…
25	0.446858824895112	0.446873361526477	3.033120278084475	0.446858846100032
…	…	…	…	…
50	0.397841478361139	0.397863341508779	3.297911825793324	0.397841507128528
…	…	…	…	…
100	0.315560781936281	0.315586416536455	3.153669674069195	0.315560811436780
…	…	…	…	…
500	0.049930372833121	0.049936108842272	1.278420609396875	0.049930378403359
…	…	…	…	…
1000	0.005520028104577	0.005520671212411	0.526921124813466	0.005520028727839

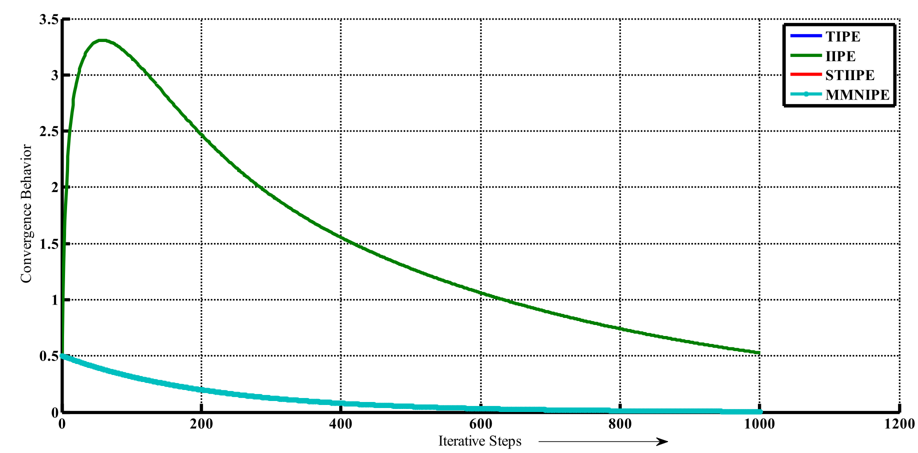

Figure 1 Convergence behavior corresponding to $u_0 = 0.5$ for 1000 iterative steps

by Liu [31], and the TIPE given by Cho *et al.* [33]. By using MATLAB programming language, we computed the different iterative steps and the numerical comparison is shown in Table 1. Furthermore, the convergence behaviors of these iterative procedures with errors are shown in Fig. 1. For all iterative procedure, we take the initial approximation $u_0 = 0.5$. For our proposed MMNIPE given by (1.5), we consider $v_n^{(i)} = \frac{1}{n+1}$, where $i = 1, 2, 3, \dots, 10$ and

$$b_n^{(10)} = \frac{5}{9} \cdot \frac{1}{\sqrt{n} + 100}, \qquad c_n^{(10)} = \frac{1}{(\sqrt{n} + 100)^2},$$

$$a_n^{(10)} = 1 - \left(b_n^{(10)} + c_n^{(10)}\right) = \frac{9n + 1795\sqrt{n} + 89{,}491}{9(\sqrt{n} + 100)^2},$$

$$b_n^{(j)} = c_n^{(j)} = \frac{3}{5} \cdot \frac{1}{n + 100}, \qquad a_n^{(j)} = 1 - \left(b_n^{(j)} + c_n^{(j)}\right) = \frac{5n + 494}{5(n + 100)},$$

where $j = 1, 2, 3, \dots, 9$ and $\forall n \geq 0$.

For the STIIPE given by Hussain *et al.* [1], we consider $v_n^{(i)} = \frac{1}{n+1}$, where $i = 1, 2$ and

$$b_n^{(2)} = \frac{5}{9} \cdot \frac{1}{\sqrt{n} + 100}, \qquad c_n^{(2)} = \frac{1}{(\sqrt{n} + 100)^2},$$

$$a_n^{(2)} = 1 - \left(b_n^{(2)} + c_n^{(2)}\right) = \frac{9n + 1795\sqrt{n} + 89{,}491}{9(\sqrt{n} + 100)^2},$$

$$b_n^{(1)} = c_n^{(1)} = \frac{3}{5} \cdot \frac{1}{n + 100}, \qquad a_n^{(1)} = 1 - \left(b_n^{(1)} + c_n^{(1)}\right) = \frac{5n + 494}{5(n + 100)} \quad \text{and} \quad \forall n \geq 0.$$

For the IIPE given by Liu [31], we consider $v_n^{(i)} = \frac{1}{n+1}$, where $i = 1, 2$ and

$$b_n^{(2)} = \frac{5}{9} \cdot \frac{1}{\sqrt{n} + 100}, \qquad a_n^{(2)} = 1 - b_n^{(2)} = \frac{9\sqrt{n} + 895}{9(\sqrt{n} + 100)},$$

$$b_n^{(1)} = \frac{3}{5} \cdot \frac{1}{n + 100}, \qquad a_n^{(1)} = 1 - b_n^{(1)} = \frac{5n + 497}{5(n + 100)} \quad \text{and} \quad \forall n \geq 0.$$

For the TIPE given by Cho *et al.* [33], we consider $v_n^{(i)} = \frac{1}{n+1}$, where $i = 1, 2, 3$ and

$$b_n^{(3)} = \frac{5}{9} \cdot \frac{1}{\sqrt{n} + 100}, \qquad c_n^{(3)} = \frac{1}{(\sqrt{n} + 100)^2},$$

$$a_n^{(3)} = 1 - \left(b_n^{(3)} + c_n^{(3)}\right) = \frac{9n + 1795\sqrt{n} + 89{,}491}{9(\sqrt{n} + 100)^2},$$

$$b_n^{(j)} = c_n^{(j)} = \frac{3}{5} \cdot \frac{1}{n + 100}, \qquad a_n^{(j)} = 1 - \left(b_n^{(j)} + c_n^{(j)}\right) = \frac{5n + 494}{5(n + 100)},$$

where $j = 1, 2$ and $\forall n \geq 0$.

The comparison table (Table 1) confirms that the rate of convergence of our proposed MMNIPE given by (1.5) is better than that of the STIIPE given by Hussain *et al.* [1], the IIPE given by Liu [31] and the TIPE given by Cho *et al.* [33].

5 Conclusion

In this study, we established the convergence, almost common-stability and common-stability criteria of our proposed MMNIPE given by (1.5) for two Lipschitz strictly hemicontractive-type mappings in arbitrary Banach spaces. The obtained results of this paper provided easy and straightforward techniques for proving the convergence, almost common-stability and common-stability criteria of the proposed MMNIPE given by (1.5). Furthermore, the results of this paper extended the corresponding results of Hussain *et al.* [1, 7–9], Zegeye *et al.* [2], Meche *et al.* [3], Chidume and Osilike [4], Chidume [5], Liu *et al.* [12], Zeng [13], Yu *et al.* [11], Yang [25], Chidume [36], Deng [37, 38] and Liu [39]. According to the Remark 3.6, our results generalized and unify the corresponding results of Hussain *et al.* [1], Mann [28], Ishikawa [29], Xu and Noor [30], Liu [31] and Xu [32] and Cho *et al.* [33] in the case of establishing the fixed-point theorem-based iterative procedures for two Lipschitz strictly hemicontractive-type mappings. At the end of this work, we discussed a computational numerical example which verify our main results and compare the performance of our proposed MMNIPE given by (1.5) with other most analogous iterative procedures with errors. From the comparison table (Table 1), we conclude that our proposed MMNIPE given by (1.5) superior over the STIIPE given by Hussain *et al.* [1] and the TIPE given by Cho *et al.* [33] in the case of convergence at the common fixed point of two Lipschitz strictly hemicontractive-type mappings.

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