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Dislocated cone metric space over Banach algebra and α-quasi contraction mappings of Perov type

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Abstract

A dislocated cone metric space over Banach algebra is introduced as a generalisation of a cone metric space over Banach algebra as well as a dislocated metric space. Fixed point theorems for Perov-type α-quasi contraction mapping, Kannan-type contraction as well as Chatterjee-type contraction mappings are proved in a dislocated cone metric space over Banach algebra. Proper examples are provided to establish the validity of our claims.

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1 Introduction

Generalising the concept of cone metric space, Liu and Xu in [1] introduced a cone metric space over Banach algebra (in short *CMS-BA*) and proved contraction principles in such a space. They replaced the usual real contraction constant with a vector constant and scalar multiplication with vector multiplication in their results and also furnished proper examples to show that their results were different from those in a cone metric space and a metric space. While studying the applications of topology in logic programming semantics, Hitzler and Seda [2] introduced a dislocated metric space as a generalisation of a metric space and discussed the associated topologies. Later George and Khan introduced a dislocated fuzzy metric space [3], and then various fixed point results were proved in dislocated spaces. For some details, refer to [4]. On the other hand, Perov [5] generalised the Banach contraction principle by replacing the contractive factor with a matrix convergent to zero. Cvetkovic and Rakocevic [6] introduced a Perov-type quasi-contractive mapping replacing contractive factor with bounded linear operator with spectral radius less than one and obtained some interesting fixed point results in the setup of cone metric spaces.

In this work we introduce the concept of dislocated cone metric space over Banach algebra (in short *dCMS-BA*) as a generalisation of *CMS-BA* as well as a dislocated metric space and prove fixed point theorems for a Perov-type α-quasi contraction mapping in *dCMS-BA* and *CMS-BA*. Simple examples are given to illustrate the validity and superiority of our results.

2 Preliminaries

A linear space $\mathcal{A}$ over $K \in \{\mathbb{R}, \mathbb{C}\}$ is an *algebra* if for each ordered pair of elements $x, y \in \mathcal{A}$, a unique product $xy \in \mathcal{A}$ is defined such that for all $x, y, z \in \mathcal{A}$ and scalar α:

(i) $(xy)z = x(yz)$;
(iia) $x(y+z) = xy + xz$;
(iib) $(x+y)z = xz + yz$;
(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

A Banach algebra is a Banach space $\mathcal{A}$ over $K \in \{\mathbb{R}, \mathbb{C}\}$ such that, for all $x, y \in \mathcal{A}$, $\|xy\| \leq \|x\| \|y\|$.

For a given cone $P \subset \mathcal{A}$ and $x, y \in \mathcal{A}$, we say that $x \preceq y$ if and only if $y - x \in P$. Note that $\preceq$ is a partial order relation defined on $\mathcal{A}$. For more details on the basic concepts of Banach algebra, solid cone, unit element e, zero element θ, invertible elements in Banach algebra etc., the reader may refer to [1, 7].

In what follows $\mathcal{A}$ will always denote a Banach algebra, P a solid cone in $\mathcal{A}$ and e the unit element of $\mathcal{A}$.

Definition 2.1 A sequence p_n in a solid cone P of a Banach space is a c-sequence if, for each $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $p_n \ll c$ for all $n \geq n_0$.

Lemma 2.2 ([8]) *For $x \in \mathcal{A}$, $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $r(x)$ satisfies*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $r(x) < |\lambda|$, then $\lambda e - x$ is invertible in $\mathcal{A}$; moreover,

$$(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}},$$

where λ is a complex constant.

Lemma 2.3 ([9]) *Let $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than* 1, *i.e.*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1, \tag{2.1}$$

then $(e - x)$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i. \tag{2.2}$$

Lemma 2.4 ([9]) *Let $a, b \in \mathcal{A}$. If a commutes with b, then*

$$r(a+b) \leq r(a) + r(b), \qquad r(ab) \leq r(a)r(b).$$

Lemma 2.5 ([7]) *Let E be a Banach space.*

(i) *If $a, b, c \in E$ and $a \preceq b \ll c$, then $a \ll c$.*
(ii) *If $\theta \preceq a \ll c$ for each $c \gg \theta$, then $a = \theta$.*

Lemma 2.6 ([9]) *Let $\{u_n\}$ be a sequence in $\mathcal{A}$ with $\{u_n\} \to \theta$ $(n \to \infty)$. Then $\{u_n\}$ is a c-sequence.*

Lemma 2.7 ([7]) *Let $\{u_n\}$ be a c-sequence in P. If $\beta \in P$ is an arbitrarily given vector, then $\{\beta u_n\}$ is a c-sequence.*

Lemma 2.8 ([8]) *Let $\alpha \in \mathcal{A}$ and $r(\alpha) < 1$, then $\{\alpha^n\}$ is a c-sequence.*

Remark 2.9 For more on c-sequences, see [7, 8].

Definition 2.10 Let X be any nonempty set and $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. Then

(i) T is an α-admissible mapping iff $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$, $x,y \in X$.

(ii) T is an α-dominated mapping iff $\alpha(x,y) \geq 1$ implies $\alpha(x,Tx) \geq 1$, $x,y \in X$.

3 Main results

In this section first we introduce the definition of a dislocated cone metric space over Banach algebra (in short dCMS-BA) and furnish examples to show that this concept is more general than that of *CMS-BA*. We then define convergence and Cauchy sequence in a dCMS-BA and then prove fixed point results in this space.

Definition 3.1 Let χ be a nonempty set and $d_{\mathrm{lc}} : \chi \times \chi \to \mathcal{A}$ be such that for all $x,y,z \in \chi$,

(dCM1) $\theta \preceq d_{\mathrm{lc}}(x,y)$ and $d_{\mathrm{lc}}(x,y) = \theta$ imply $x = y$;

(dCM2) $d_{\mathrm{lc}}(x,y) = d_{\mathrm{lc}}(y,x)$;

(dCM3) $d_{\mathrm{lc}}(x,y) \preceq d_{\mathrm{lc}}(x,z) + d_{\mathrm{lc}}(z,y)$.

Then d_{lc} is called a dislocated cone metric on χ and (χ, d_{lc}) is called a dislocated cone metric space over Banach algebra (in short *dCbMS-BA*).

Note that every metric space and *CMS-BA* is a *dCMS-BA*, but the converse is not necessarily true. Inspired by [1, 7, 10], we furnish the following examples which will establish our claim.

Example 3.2 Let $\mathcal{A} = \{a = (a_{i,j})_{3\times 3} : a_{i,j} \in \mathbb{R}, 1 \leq i,j \leq 3\}$, $\|a\| = \sum_{1\leq i,j\leq 3} |a_{i,j}|$, $P = \{a \in \mathcal{A} : a_{i,j} \geq 0, 1 \leq i,j \leq 3\}$ be a cone in $\mathcal{A}$. Let $\chi = R^+ \cup \{0\}$. Let $d_{\mathrm{lc}}\colon \chi \times \chi \to \mathcal{A}$ be given by

$$d_{\mathrm{lc}}(x,y) = \begin{pmatrix} x+y & x+y & x+y \\ 2x+2y & 2x+2y & 2x+2y \\ 3x+3y & 3x+3y & 3x+3y \end{pmatrix}.$$

Then (χ, d_{lc}) is a *dCMS-BA* over $\mathcal{A}$ but not a *CMS-BA* over Banach algebra $\mathcal{A}$.

Example 3.3 Let $\chi = \mathbb{R}$ and let $\mathcal{A} = C_{R^2}(\chi)$. For $\alpha = (f,g)$ and $\beta = (u,v)$ in $\mathcal{A}$, we define $\alpha.\beta = (f.u, g.v)$ and $\|\alpha\| = \max(\|f\|, \|g\|)$, where $\|f\| = \sup_{x\in\chi} |f(x)|$. Then $\mathcal{A}$ is a Banach algebra with unit $e = (1,1)$, zero element $\theta = (0,0)$ and $P = \{(f,g) \in \mathcal{A} : f(t) \geq 0, g(t) \geq 0, t \in \chi\}$ is a non-normal cone in $\mathcal{A}$. Consider $d_{\mathrm{lc}}\colon \chi \times \chi \to \mathcal{A}$ given by

$$d_{\mathrm{lc}}(x,y)(t) = \bigl(|x-y|\bigl(1+t^2\bigr) + |x|\bigl(t^2\bigr) + |y|\bigl(t^4\bigr), |x-y|\bigl(1+t^2\bigr)\bigr).$$

Clearly (χ, d_{lc}) is a *dCMS-BA* over $\mathcal{A}$ but not a *CMS-BA* over Banach algebra $\mathcal{A}$.

For any $a \in \chi$, the open sphere with centre a and radius $\lambda > \theta$ is given by

$$B_\lambda(a) = \left\{ b \in \chi : d_{\mathrm{lc}}(a,b) < \lambda \right\}.$$

Let $\mathcal{U} = \{Y \subseteq \chi : \forall x \in Y, \exists r > \theta$ such that $B_r(x) \subseteq Z\}$. Then $\mathcal{U}$ defines the dislocated cone metric topology for the *dCMS-BA* (χ, d_{lc}).

Definition 3.4 Let (χ, d_{lc}) be a *dCMS-BA* over $\mathcal{A}$, $p \in \chi$ and $\{p_n\}$ be a sequence in χ.

(i) $\{p_n\}$ converges to p if, for each $c \in \mathcal{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d_{\mathrm{lc}}(p_n, p) \ll c$ for all $n \geq n_0$. We write it as $\mathrm{Lim}_{n \to \infty} p_n = p$.

(ii) $\{p_n\}$ is a Cauchy sequence if and only if for each $c \in \mathcal{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d_{\mathrm{lc}}(p_n, p_m) \ll c$ for all $n, m \geq n_0$.

(iii) (χ, d_{lc}) is a complete *dCMS* if and only if every Cauchy sequence in (χ, d_{lc}) is convergent.

Proposition 3.5 *Let (χ, d_{lc}) be a dCMS-BA over $\mathcal{A}$, P be a soloid cone and $\{p_n\}$ be a sequence in χ. If $\{p_n\}$ converges to $p \in \chi$, then*

(i) *$d_{\mathrm{lc}}(p_n, p)$ is a c-sequence;*

(ii) *$d_{\mathrm{lc}}(p_n, p_{n+r})$ is a c-sequence.*

Proof Follows from Definitions 2.1,3.1 and 3.4(i). □

In [11] Samet et al. introduced the concept of α-admissible mappings and proved fixed point theorems for *alpha-psi* contractive-type mappings, which paved the way for proving new and existing results in fixed point theory. As in [11] and others, we give the following definitions.

Definition 3.6 Let (χ, d_{lc}) be a *dCMS-BA*, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. Then

(i) T is an α-admissible mapping iff $\alpha(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, $x, y \in X$.

(ii) T is an α-dominated mapping iff $\alpha(x,y) \geq 1$ implies $\alpha(x, Tx) \geq 1$, $x, y \in X$.

(iii) α is a triangular function iff $\alpha(x,y) \geq 1$, $\alpha(y,z) \geq 1$ imply $\alpha(x,z) \geq 1$, $x, y, z \in X$.

(iv) (χ, d_{lc}) is α-regular iff for any sequence $\{x_p\}$ in χ with $\alpha(x_p, x_{p+1}) \geq 1$ and $x_p \to x_*$ as $p \to \infty$, then $\alpha(x_p, x_*) \geq 1$

However, for proving the uniqueness of the fixed point, different hypotheses were used by different authors. In the sequel Popescu [12] considered the following condition:

(**K**) For all $x \neq y \in \chi$, there exists $w \in \chi$ such that $\alpha(x,w) \geq 1$, $\alpha(y,w) \geq 1$ and $\alpha(w, Tw) \geq 1$.

We now introduce the following definitions.

Definition 3.7 Let (χ, d_{lc}) be a *dCMS-BA*, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. Then

(i) T is an α-identical function iff $\alpha(Tx, Tx) \geq 1$ for all $x \in \chi$.

(ii) T is weak semi α-admissible iff $\alpha(x,y) \geq 1$ implies $\alpha(x, T^2 y) \geq 1$ for any $x, y \in \chi$.

(iii) T satisfies condition (**G**) iff $\alpha(x, Tx) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x,y) \geq 1$ or $\alpha(Tx, Ty) \geq 1$ for any $x, y \in \chi$.

(iv) T satisfies condition ($\mathbf{G}'$) iff for all $x \neq y \in \chi$ with $\alpha(x, Tx) \geq 1$ and $\alpha(y, Ty) \geq 1$, there exists $w \in \chi$ such that $\alpha(x, w) \geq 1$, $\alpha(y, w) \geq 1$, $\alpha(w, w) \geq 1$ and $\alpha(w, Tw) \geq 1$.

Example 3.8 Let $\chi = [0, \infty]$, $Tx = x^2$ for all $x \in X$. Let

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1] \text{ or } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then T is an α-identical function and T satisfies conditions ($\mathbf{G}$) and ($\mathbf{G}'$), but T does not satisfy condition ($\mathbf{K}$) and T is not α-dominated.

Example 3.9 Let $\chi = [-n, n]$ for some $n \in \mathbb{N}$, $Tx = -x$ for all $x \in \chi$. Let

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [-n,0] \text{ or } x, y \in (0,n], \\ 0 & \text{otherwise.} \end{cases}$$

Then T is an α-identical function and T satisfies conditions ($\mathbf{G}$) and ($\mathbf{G}'$), but T is not α-dominated and does not satisfy condition ($\mathbf{K}$).

Example 3.10 Let $A = [-n, 0]$, $B = [0, n]$ and $\chi = A \cup B$ for some $n \in \mathbb{N}$. Let $Tx = -x$ for all $x \in \chi$ and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x, y) \in \{A \times B, B \times A\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then α is not triangular and T is not α-identical, but T is weak semi α-admissible and α-dominated. T does not satisfy condition ($\mathbf{G}$) but satisfies condition ($\mathbf{G}'$).

Example 3.11 Let $A = [-n, 0)$, $B = (0, n]$ and $\chi = A \cup \{0\} \cup B$ for some $n \in \mathbb{N}$. Let $Tx = \frac{x^2}{n}$ for all $x \in \chi$ and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x, y) \in \{A \times A, B \times B\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then α is triangular and T is α-identical, but T is not weak semi α-admissible and not α-dominated. T satisfies conditions ($\mathbf{G}$) and ($\mathbf{G}'$) but does not satisfy condition ($\mathbf{K}$).

Lemma 3.12 *Let X be a nonempty set and $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. Let $\{x_n\}$ be the Picard sequence starting with x_0. If $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$, and if α is a triangular function and T is α-admissible, then for all $n \geq 1$ and $0 \leq p \leq q \leq n$, $\alpha(x_p, x_q) \geq 1$.*

Proof For the proof, we will make use of the principle of mathematical induction.

As $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, x_1) \geq 1$ and T is α-admissible, $\alpha(x_1, x_1) = \alpha(Tx_0, Tx_0) \geq 1$, and so the result holds good for $n = 1$. Again, by α-admissibility of T, we get $\alpha(x_1, x_2) \geq 1$, $\alpha(x_2, x_2) \geq 1$, and then, since α is triangular, we get $\alpha(x_0, x_2) \geq 1$. Thus the result holds good for $n = 2$. Suppose the result is true for $n = r$, i.e. $\alpha(x_p, x_q) \geq 1$ for all $0 \leq p \leq q \leq r$. We will show that it is true for $n = r + 1$. It is enough to consider the case $\alpha(x_p, x_{r+1})$, $0 \leq p \leq r + 1$. By induction hypothesis and α admissibility of T, we have $\alpha(x_p, x_{r+1}) \geq 1$ for all $1 \leq p \leq r + 1$. Since $\alpha(x_0, x_1) \geq 1$, by α admissibility of T and triangularity of function α, we get $\alpha(x_0, x_{r+1}) \geq 1$, and thus the result is true for $n = r + 1$. Hence, by the principle of mathematical induction, the result is true for all n. □

Lemma 3.13 *Let X be a nonempty set and $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. Let $\{x_n\}$ be the Picard sequence starting with x_0 such that $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. If T is α-admissible and weak semi α-admissible, then for all $n \geq 1$ and $0 \leq p \leq q \leq n$, $\alpha(x_p, x_q) \geq 1$.*

Proof As $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, x_1) \geq 1$ and T is α-admissible, $\alpha(x_1, x_1) = \alpha(Tx_0, Tx_0) \geq 1$, and so the result holds good for $n = 1$. Again, by α-admissibility of T, we get $\alpha(x_1, x_2) \geq 1$, $\alpha(x_2, x_2) \geq 1$. Since T is weak semi α-admissible and $\alpha(x_0, x_0) \geq 1$, we get $\alpha(x_0, x_2) \geq 1$. Thus the result holds good for $n = 2$. Suppose the result is true for $n = r$, i.e. $\alpha(x_p, x_q) \geq 1$ for all $0 \leq p \leq q \leq r$. We will show that it is true for $n = r + 1$. It is enough to consider the case $\alpha(x_p, x_{r+1})$, $0 \leq p \leq r + 1$. By induction hypothesis and α admissibility of T, we have $\alpha(x_p, x_{r+1}) \geq 1$ for all $1 \leq p \leq r + 1$. If r is even, then using $\alpha(x_0, x_1) \geq 1$ and repeatedly using weak semi α admissibility of T, we get α, we get $\alpha(x_0, x_{r+1}) \geq 1$. If r is odd, then using $\alpha(x_0, x_0) \geq 1$ and repeatedly using weak semi α admissibility of T, we get α, we get $\alpha(x_0, x_{r+1}) \geq 1$. Thus the result is true for $n = r + 1$. Hence, by the principle of mathematical induction, the result is true for all n. □

Definition 3.14 Let (χ, d_{lc}) be a *dCMS-BA*, $T\colon \chi \to \chi$ and $\alpha : X \times X \to [0, \infty)$ be mappings. Then T is a Perov-type α-quasi contraction mapping iff there exists $\mu \in P$ such that $0 \leq r(\mu) < 1$, and for all $u, v \in \chi$ with $\alpha(u, v) \geq 1$,

$$d_{\mathrm{lc}}(Tu, Tv) \preceq \mu.\varphi(u, v), \tag{3.1}$$

where $\varphi(u, v) \in \{d_{\mathrm{lc}}(u, v), d_{\mathrm{lc}}(u, Tu), d_{\mathrm{lc}}(v, Tv), d_{\mathrm{lc}}(u, Tv), d_{\mathrm{lc}}(v, Tu)\}$.

Lemma 3.15 *Let T be an α-admissible Perov-type α-quasi contraction mapping in a dCMS-BA (χ, d_{lc}), where $\alpha : X \times X \to [0, \infty)$, let x_p be the iterative sequence defined by $x_{p+1} = Tx_p$ for some arbitrary $x_0 \in \chi$ and all $p \in \mathbb{N}$ such that $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. If α is a triangular function or T is weak semi α-admissible, then for all $n \geq 1$, $p, q \in \mathbb{N}$, we have $\alpha(x_p, x_q) \geq 1$ for $0 \leq p \leq q \leq n$ and for all $1 \leq p \leq q \leq n$*

$$d_{\mathrm{lc}}(x_p, x_q) \preceq \mu(e - \mu)^{-1}\bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0)\bigr) \tag{3.2}$$

Proof Let $d_{\mathrm{lc}}(x_p, x_{p+1}) = d_p$ and $d_{\mathrm{lc}}(x_p, x_p) = d_{p,p}$. For the proof, we will make use of the principle of mathematical induction. Note that by Lemma 3.12 or Lemma 3.13 the case may be $\alpha(x_p, x_q) \geq 1$ for all $0 \leq p \leq q$. For $n = 1$, $p = q = 1$, since $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, x_1) \geq 1$ and $\alpha(x_1, x_1) \geq 1$, from (3.1) we have

$$\begin{aligned} d_{\mathrm{lc}}(x_1, x_1) = d_{\mathrm{lc}}(Tx_0, Tx_0) &\preceq \mu\bigl\{d_{\mathrm{lc}}(x_0, x_0), d_{\mathrm{lc}}(x_0, x_1), d_{\mathrm{lc}}(x_0, x_1), d_{\mathrm{lc}}(x_0, x_1), d_{\mathrm{lc}}(x_0, x_1)\bigr\} \\ &\preceq \mu(e - \mu)^{-1}\bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0)\bigr), \end{aligned}$$

and thus the result holds good. Now suppose (3.2) is true for $n = r$, i.e.

$$d_{\mathrm{lc}}(x_p, x_q) \preceq \mu(e-\mu)^{-1}\bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0)\bigr) \quad \text{for } 1 \le p, q \le r. \tag{3.3}$$

We will show that (3.2) is true for $n = r + 1$. It is enough to consider the case $1 \le p \le r + 1$ and $q = r + 1$. Note that

$$\mu(e-\mu)^{-1} = \mu \sum_{i=0}^{\infty} \mu^i \preceq \sum_{i=0}^{\infty} \mu^i = (e-\mu)^{-1} \tag{3.4}$$

and

$$e + \mu(e-\mu)^{-1} = e + \sum_{i=1}^{\infty} \mu^i = (e-\mu)^{-1}. \tag{3.5}$$

Since $\alpha(x_{p-1}, x_r) \ge 1$, from (3.1) we have

$$\begin{aligned} d_{\mathrm{lc}}(x_p, x_{r+1}) &= d_{\mathrm{lc}}(Tx_{p-1}, Tx_r) \\ &\preceq \mu\bigl\{d_{\mathrm{lc}}(x_{p-1}, x_r), d_{\mathrm{lc}}(x_{p-1}, x_p), d_{\mathrm{lc}}(x_r, x_{r+1}), d_{\mathrm{lc}}(x_{p-1}, x_{r+1}), d_{\mathrm{lc}}(x_r, x_p)\bigr\}. \end{aligned}$$

We will analyse each term on the right-hand side of the above inequality as follows.

(i) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_{p-1}, x_r)$.

Case i(a): $p = 1$.

$$\begin{aligned} d_{\mathrm{lc}}(x_1, x_{r+1}) &\preceq \mu d_{\mathrm{lc}}(x_0, x_r) \\ &\preceq \mu\bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_1, x_r)\bigr) \\ &\preceq \mu(e-\mu)^{-1}\bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0)\bigr) \quad \bigl(\text{by (3.3) and (3.5)}\bigr). \end{aligned}$$

Case i(b): $2 \le p \le r$. By (3.3) and (3.4) we get

$$d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu(e-\mu)^{-1}\bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0)\bigr).$$

Case i(c): $p = r + 1$. In this case $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_r, x_r)$, and the result follows from (3.3) and (3.4).

(ii) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_{p-1}, x_p)$.

Case ii(a): $p = 1$.

$$\begin{aligned} d_{\mathrm{lc}}(x_1, x_{r+1}) &\preceq \mu d_{\mathrm{lc}}(x_0, x_1) \\ &\preceq \mu(e-\mu)^{-1}\bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0)\bigr). \end{aligned}$$

Case ii(b): $2 \le p \le r$. The result follows from (3.3).

Case ii(c): $p = r + 1$.

$$\begin{aligned} d_{\mathrm{lc}}(x_p, x_{r+1}) &\preceq \mu d_{\mathrm{lc}}(x_r, x_{r+1}) = \mu d_{\mathrm{lc}}(Tx_{r-1}, Tx_r) \\ &\preceq \mu^2\bigl\{d_{\mathrm{lc}}(x_{r-1}, x_r), d_{\mathrm{lc}}(x_{r-1}, x_r), d_{\mathrm{lc}}(x_r, x_{r+1}), d_{\mathrm{lc}}(x_{r-1}, x_{r+1}), d_{\mathrm{lc}}(x_r, x_r)\bigr\}. \end{aligned}$$

Case ii(c-1): $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_r, x_{r+1}) \preceq \mu^2 d_{\mathrm{lc}}(x_{r-1}, x_r)$.

By (3.3) and (3.4), we get $\mu^2 d_{\mathrm{lc}}(x_{r-1}, x_r) \preceq \mu(e-\mu)^{-1}(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0))$.

Case ii(c-2): $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_r, x_{r+1}) \preceq \mu^2 d_{\mathrm{lc}}(x_r, x_{r+1})$.

Now, $\mu d_{\mathrm{lc}}(x_r, x_{r+1}) \preceq \mu^2 d_{\mathrm{lc}}(x_r, x_{r+1})$ implies $\mu(e-\mu) d_{\mathrm{lc}}(x_r, x_{r+1}) \preceq \theta$. Note that $r(\mu) < 1$, and so $(e-\mu)$ is invertible and $(e-\mu)^{-1} > e$. Therefore we get $d_{\mathrm{lc}}(x_r, x_{r+1}) \preceq \theta$, i.e. $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \theta \preceq \mu(e-\mu)^{-1}(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0))$.

Case ii(c-3): $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_r, x_{r+1}) \preceq \mu^2 d_{\mathrm{lc}}(x_{r-1}, x_{r+1})$

Again by (3.1)

$$\begin{aligned} d_{\mathrm{lc}}(x_p, x_{r+1}) &\preceq \mu d_{\mathrm{lc}}(x_r, x_{r+1}) \\ &\preceq \mu^2 \bigl\{ d_{\mathrm{lc}}(x_{r-2}, x_r), d_{\mathrm{lc}}(x_{r-2}, x_{r-1}), d_{\mathrm{lc}}(x_r, x_{r+1}), d_{\mathrm{lc}}(x_{r-2}, x_{r+1}), d_{\mathrm{lc}}(x_r, x_{r-1}) \bigr\}. \end{aligned}$$

Continuing this process we will at most arrive at the following :

(i) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu^k d_{\mathrm{lc}}(x_p, x_q)$ for some $1 \leq p, q \leq r$, $2 \leq k \leq r$, and the result follows from this by (3.3) and (3.4).

(ii) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu^k d_{\mathrm{lc}}(x_r, x_{r+1})$, and the result follows by proceeding as in *Case ii(c-2).*

(iii) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu^r d_{\mathrm{lc}}(x_p, x_{r+1})$, which implies $(e-\mu) d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \theta$, and by the same argument as in *Case ii(c-2)* the result follows.

Case ii(c-4): $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_r, x_{r+1}) \preceq \mu^2 d_{\mathrm{lc}}(x_r, x_r)$. The result follows from (3.3) and (3.4).

(iii) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_r, x_{r+1}) = \mu d_{\mathrm{lc}}(Tx_{r-1}, Tx_r)$. The result follows proceeding as in Case ii(c).

(iv) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_r, x_p)$. The result follows from (3.3) and (3.4).

(v) $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_{p-1}, x_{r+1})$.

Using (3.1) and continuing in a similar manner as above, either we will get the desired result or we get $d_{\mathrm{lc}}(x_p, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_{p-1}, x_{r+1}) \preceq \mu^2 d_{\mathrm{lc}}(x_{p-2}, x_{r+1}) \cdots \preceq \mu^{p-1} d_{\mathrm{lc}}(x_1, x_{r+1})$.

Now

$$d_{\mathrm{lc}}(x_1, x_{r+1}) \preceq \mu \bigl\{ d_{\mathrm{lc}}(x_0, x_r), d_{\mathrm{lc}}(x_0, x_1), d_{\mathrm{lc}}(x_r, x_{r+1}), d_{\mathrm{lc}}(x_0, x_{r+1}), d_{\mathrm{lc}}(x_r, x_1) \bigr\}.$$

If $d_{\mathrm{lc}}(x_1, x_{r+1}) \preceq \mu\{d_{\mathrm{lc}}(x_0, x_r)$ or $d_{\mathrm{lc}}(x_0, x_1)$ or $d_{\mathrm{lc}}(x_r, x_{r+1})$ or $d_{\mathrm{lc}}(x_r, x_1)\}$, then the result follows by proceeding as in Case i(a) or ii(a) or ii(c) or by (3.3) and (3.4). If $d_{\mathrm{lc}}(x_1, x_{r+1}) \preceq \mu d_{\mathrm{lc}}(x_0, x_{r+1})$, then

$$\begin{aligned} d_{\mathrm{lc}}(x_1, x_{r+1}) &\preceq \mu d_{\mathrm{lc}}(x_0, x_{r+1}) \preceq \mu \bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_1, x_{r+1}) \bigr) \\ &\preceq \mu(e-\mu)^{-1} d_{\mathrm{lc}}(x_0, x_1) \preceq \mu(e-\mu)^{-1} \bigl(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0) \bigr). \end{aligned}$$

Thus (3.2) is true for $n = r+1$, and hence by the principle of mathematical induction it is true for all n. □

Theorem 3.16 *Let (χ, d_{lc}) be a complete dCMS-BA, $T\colon \chi \to \chi$ and $\alpha : X \times X \to [0, \infty)$ be mappings such that*

(i) *T is a Perov-type α-quasi contraction mapping.*

(ii) *α is a triangular function or T is weak semi α-admissible.*

(iii) *T is α-admissible.*

(iv) *There exists* $x_0 \in \chi$ *such that* $\alpha(x_0, x_0) \geq 1$ *and* $\alpha(x_0, Tx_0) \geq 1$.
(v) (χ, d_{lc}) *is* α*-regular.*
Then T *has a fixed point.*

Proof Consider the iterative sequence defined by $x_{p+1} = Tx_p$ for all $p \in \mathbb{N}$. Let $d_{\mathrm{lc}}(x_p, x_{p+1}) = d_p$ and $d_{\mathrm{lc}}(x_p, x_p) = d_{p,p}$. Note that $d_{p,p} \preceq 2d_{p-1}$ and $d_{p,p} \preceq 2d_{p+1}$. We will show that $\{x_p\}$ is a Cauchy sequence. For $1 < p < q$, let $\Gamma_{p,q} = \{d_{\mathrm{lc}}(x_i, x_j) : p \leq i \leq j \leq q\}$. Then, using (3.1) and by the same argument as that in the proof of Lemma 12 in [13], we can find $u_1 \in \Gamma_{p-1,q}, u_2 \in \Gamma_{p-2,q}, \ldots, u_{p-1} \in \Gamma_{1,q}$ satisfying

$$d_{\mathrm{lc}}(x_p, x_q) \preceq \mu u_1 \preceq \mu^2 u_2 \preceq \cdots \preceq \mu^{p-1} u_{p-1}.$$

By Lemma 3.15, $u_{p-1} \preceq \mu(e - \mu)^{-1}(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0))$. Therefore $d_{\mathrm{lc}}(x_p, x_q) \preceq \mu^p(e - \mu)^{-1}(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0))$. Since $r(\mu) < 1$, by Lemmas 2.8 and 2.7, we see that $\mu^p(e - \mu)^{-1}(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0))$ is a c-sequence. Now let $\theta \ll c$ be arbitrary in $\mathcal{A}$. By Definition 2.1, there exists a natural number p_0 such that $\mu^p(e - \mu)^{-1}(d_{\mathrm{lc}}(x_0, x_1) + d_{\mathrm{lc}}(x_0, x_0)) \ll c$ for all $p \geq p_0$. Thus we get $d_{\mathrm{lc}}(x_p, x_q) \ll c$ for all $p \geq p_0$. Therefore $\{x_p\}$ is a Cauchy sequence, and by the completeness of (χ, d_{lc}) there exists $x_* \in \chi$ such that $\lim_{n\to\infty} x_p = x_*$. By Proposition 3.5 and Lemma 2.6, $d_{\mathrm{lc}}(x_p, x_*) \to \theta$, $d_{\mathrm{lc}}(x_{p-1}, x_p) \to \theta$ and $d_{\mathrm{lc}}(x_{p-1}, x_*) \to \theta$. Since (χ, d_{lc}) is α-regular, $\alpha(x_{p-1}, x_*) \geq 1$, and so by (3.1)

$$\begin{aligned}
&d_{\mathrm{lc}}(x_*, Tx_*) \\
&\quad\preceq d_{\mathrm{lc}}(x_*, x_p) + d_{\mathrm{lc}}(x_p, Tx_*) = d_{\mathrm{lc}}(x_*, x_p) + d_{\mathrm{lc}}(Tx_{p-1}, Tx_*) \\
&\quad\preceq d_{\mathrm{lc}}(x_*, x_p) + \mu\bigl\{d_{\mathrm{lc}}(x_{p-1}, x_*), d_{\mathrm{lc}}(x_{p-1}, x_p), d_{\mathrm{lc}}(x_*, Tx_*), d_{\mathrm{lc}}(x_{p-1}, Tx_*), d_{\mathrm{lc}}(x_*, x_p)\bigr\} \\
&\quad\preceq d_{\mathrm{lc}}(x_*, x_p) + \mu\bigl\{d_{\mathrm{lc}}(x_{p-1}, x_*), d_{\mathrm{lc}}(x_{p-1}, x_p), d_{\mathrm{lc}}(x_*, Tx_*), d_{\mathrm{lc}}(x_{p-1}, x_p) + d_{\mathrm{lc}}(x_p, Tx_*), \\
&\qquad d_{\mathrm{lc}}(x_*, x_p)\bigr\} \\
&\quad\preceq d_{\mathrm{lc}}(x_*, x_p) + \mu\bigl\{d_{\mathrm{lc}}(x_{p-1}, x_*), d_{\mathrm{lc}}(x_{p-1}, x_p), \theta, (e - \mu)^{-1} d_{\mathrm{lc}}(x_{p-1}, x_p), d_{\mathrm{lc}}(x_*, x_p)\bigr\}.
\end{aligned}$$

By Proposition 3.5 and Lemma 2.6, $d_{\mathrm{lc}}(x_p, x_*) \to \theta$ and $d_{\mathrm{lc}}(x_{p-1}, x_p) \to \theta$. Thus $d_{\mathrm{lc}}(x_*, Tx_*) \preceq \theta$ and so $Tx_* = x_*$. □

In Theorem 3.16 condition (v) can be replaced with another condition as in Popescu [12]. We have the following.

Theorem 3.17 *Let* (χ, d_{lc}) *be a complete dCMS-BA*, $T: \chi \to \chi$ *and* $\alpha : X \times X \to [0, \infty)$ *be mappings such that*
(i) T *is a Perov-type* α*-quasi contraction mapping.*
(ii) α *is a triangular function or* T *is weak semi* α*-admissible.*
(iii) T *is* α*-admissible.*
(iv) *There exists* $x_0 \in \chi$ *such that* $\alpha(x_0, x_0) \geq 1$ *and* $\alpha(x_0, Tx_0) \geq 1$.
(v) *If* $\{x_p\}$ *is a sequence in* χ *such that* $\alpha(x_p, x_{p+1}) \geq 1$ *for all* p *and* $x_p \to u \in \chi$ *as* $p \to \infty$, *then there exists a subsequence* $\{x_{p(k)}\}$ *of* $\{x_p\}$ *such that* $\alpha(x_{p(k)}, u) \geq 1$ *for all* k.
Then T *has a fixed point.*

Proof Proceeding as in the proof of Theorem 3.16, the Picard sequence $\{x_p\}$ starting with x_0 converges to $x_* \in \chi$. By (v) there exists a subsequence $\{x_{p(k)}\}$ of $\{x_p\}$ such that $\alpha(x_{p(k)}, u) \geq 1$ for all k. Thus we have

$$\begin{aligned} d_{\mathrm{lc}}(x_*, Tx_*) &\preceq d_{\mathrm{lc}}(x_*, x_{p(k)}) + d_{\mathrm{lc}}(x_{p(k)}, Tx_*) = d_{\mathrm{lc}}(x_*, x_{p(k)}) + d_{\mathrm{lc}}(Tx_{p(k)-1}, Tx_*) \\ &\preceq d_{\mathrm{lc}}(x_*, x_{p(k)}) + \mu\bigl\{d_{\mathrm{lc}}(x_{p(k)-1}, x_*), d_{\mathrm{lc}}(x_{p(k)-1}, x_{p(k)}), d_{\mathrm{lc}}(x_*, Tx_*), \\ &\quad d_{\mathrm{lc}}(x_{p(k)-1}, Tx_*), d_{\mathrm{lc}}\bigl(x_*, x_p(k)\bigr)\bigr\} \\ &\preceq d_{\mathrm{lc}}(x_*, x_{p(k)}) + \mu\bigl\{d_{\mathrm{lc}}(x_{p(k)-1}, x_*), d_{\mathrm{lc}}(x_{p(k)-1}, x_p), d_{\mathrm{lc}}(x_*, Tx_*), \\ &\quad d_{\mathrm{lc}}(x_{p(k)-1}, x_{p(k)}) + d_{\mathrm{lc}}(x_{p(k)}, Tx_*), d_{\mathrm{lc}}(x_*, x_{p(k)})\bigr\} \\ &\preceq d_{\mathrm{lc}}(x_*, x_{p(k)}) + \mu\bigl\{d_{\mathrm{lc}}(x_{p(k)-1}, x_*), d_{\mathrm{lc}}(x_{p(k)-1}, x_{p(k)}), \theta, \\ &\quad (e-\mu)^{-1} d_{\mathrm{lc}}(x_{p(k)-1}, x_{p(k)}), d_{\mathrm{lc}}(x_*, x_{p(k)})\bigr\}. \end{aligned}$$

By Proposition 3.5 and Lemma 2.6, $d_{\mathrm{lc}}(x_{p(k)}, x_*) \to \theta$ and $d_{\mathrm{lc}}(x_{p(k)-1}, x_{p(k)}) \to \theta$. Thus $d_{\mathrm{lc}}(x_*, Tx_*) \preceq \theta$, and so $Tx_* = x_*$. □

Theorem 3.18 *Let* (χ, d_{lc})*,* T *and* α *be as in Theorem* 3.16*. Suppose that all conditions of Theorem* 3.16 *or Theorem* 3.17 *are satisfied. If* T *is an* α*-identical function or if* T *is* α*-dominated, then* T *has a fixed point* $x_* \in \chi$ *and* $d_{\mathrm{lc}}(x_*, x_*) = \theta$*. Further, if* T *satisfies condition* (**G**)*, then the fixed point is unique.*

Proof As in the proof of Theorem 3.16 or Theorem 3.17, we see that T has a fixed point $x_* \in \chi$. If T is α-identical , then $\alpha(x_*, x_*) = \alpha(Tx_*, Tx_*) \geq 1$. If T is α-dominated, then $\alpha(x_*, x_*) = \alpha(x_*, Tx_*) \geq 1$. Then from (3.1) we have $d_{\mathrm{lc}}(x_*, x_*) = d_{\mathrm{lc}}(Tx_*, Tx_*) \preceq \mu\{d_{\mathrm{lc}}(x_*, x_*), d_{\mathrm{lc}}(x_*, x_*), d_{\mathrm{lc}}(x_*, x_*), d_{\mathrm{lc}}(x_*, x_*), d_{\mathrm{lc}}(x_*, x_*)\} = \mu d_{\mathrm{lc}}(x_*, x_*)$. Hence $d_{\mathrm{lc}}(x_*, x_*) = \theta$.

Now suppose y_* is another fixed point of T. Then as above $\alpha(y_*, y_*) \geq 1$ and $d_{\mathrm{lc}}(y_*, y_*) = \theta$. Since T satisfies condition (**G**), we have $\alpha(x_*, y_*) \geq 1$, and then by (3.1)

$$\begin{aligned} d_{\mathrm{lc}}(x_*, y_*) &= d_{\mathrm{lc}}(Tx_*, Ty_*) \\ &\preceq \mu\bigl\{d_{\mathrm{lc}}(x_*, y_*), d_{\mathrm{lc}}(x_*, x_*), d_{\mathrm{lc}}(y_*, y_*), d_{\mathrm{lc}}(x_*, y_*), d_{\mathrm{lc}}(y_*, x_*)\bigr\}. \end{aligned}$$

Thus $d_{\mathrm{lc}}(x_*, y_*) \preceq \theta$ and so $x_* = y_*$. □

Theorem 3.19 *Let* (χ, d_{lc})*,* T *and* α *be as in Theorem* 3.16*. Suppose that all conditions of Theorem* 3.16 *or Theorem* 3.17 *are satisfied. If* T *is an* α*-identical function or if* T *is* α*-dominated, then* T *has a fixed point* $x_* \in \chi$ *and* $d_{\mathrm{lc}}(x_*, x_*) = \theta$*. Further, if* T *satisfies condition* (**G**$'$)*, then the fixed point is unique.*

Proof As in the proof of Theorem 3.18, we see that T has a fixed point $x_* \in \chi$ and $d_{\mathrm{lc}}(x_*, x_*) = \theta$, and if y_* is another fixed point of T, then $\alpha(y_*, y_*) \geq 1$ and $d_{\mathrm{lc}}(y_*, y_*) = \theta$. Since T satisfies condition (**G**$'$), there exists $w \in \chi$ such that $\alpha(x_*, w) \geq 1$, $\alpha(y_*, w) \geq 1$, $\alpha(w, w) \geq 1$ and $\alpha(w, Tw) \geq 1$. By Theorem 3.16 the sequence $\{T^n w\}$ will converge to a fixed point say w_* of T. Since T is α-admissible, we get $\alpha(x_*, T^n w) \geq 1$ and $\alpha(y_*, T^n w) \geq 1$, and then by (3.1) we have

$$\begin{aligned}&d_{\mathrm{lc}}\bigl(x_*,T^{n+1}w\bigr)\\&\quad=d_{\mathrm{lc}}\bigl(Tx_*,TT^nw\bigr)\\&\quad\preceq\mu\bigl\{d_{\mathrm{lc}}\bigl(x_*,T^nw\bigr),d_{\mathrm{lc}}(x_*,x_*),d_{\mathrm{lc}}\bigl(T^nw,T^{n+1}w\bigr),d_{\mathrm{lc}}\bigl(x_*,T^nw\bigr),d_{\mathrm{lc}}\bigl(T^nw,x_*\bigr)\bigr\}.\end{aligned}$$

Then, as $n\to\infty$, using Proposition 3.5 and Lemma 2.6, we get $d_{\mathrm{lc}}(x_*,w_*)\preceq\theta$ and so $x_*=w_*$. Similarly, we can show that $y_*=w_*$. Therefore $x_*=y_*$. □

Remark 3.20 In Theorems 3.18 and 3.19 we can replace the requirement of condition (**G**) or condition (**G**′) with that of condition (**K**). But as in Examples 3.8 and 3.9, there exist functions α and T such that T is α-identical and T satisfies condition (**G**) and condition (**G**′) but does not satisfy condition (**K**). Hence our approach is new and justifiable.

Since every *CMS-BA* is a *dCMS-BA* and since in a cone metric space (χ,d_c), $d_c(x,y)=\theta$ for all $x,y\in\chi$, we give the following generalised results which are easily deduced from our main results.

Theorem 3.21 *Let (χ,d_c) be a complete CMS-BA, $T\colon\chi\to\chi$ and $\alpha:X\times X\to[0,\infty)$ be mappings such that*

(i) *T is a Perov-type α-quasi contraction mapping.*
(ii) *α is a triangular function.*
(iii) *T is α-admissible.*
(iv) *There exists $x_0\in\chi$ such that and $\alpha(x_0,Tx_0)\geq 1$.*
(v) *(χ,d_{lc}) is α-regular.*

Then T has a fixed point.

Theorem 3.22 *Let (χ,d_c) be a complete CMS-BA, $T\colon\chi\to\chi$ and $\alpha:X\times X\to[0,\infty)$ be mappings such that*

(i) *T is a Perov-type α-quasi contraction mapping.*
(ii) *α is a triangular function.*
(iii) *T is α-admissible.*
(iv) *There exists $x_0\in\chi$ such that and $\alpha(x_0,Tx_0)\geq 1$.*
(v) *If $\{x_p\}$ is a sequence in χ such that $\alpha(x_p,x_{p+1})\geq 1$ for all p and $x_p\to u\in\chi$ as $p\to\infty$, then there exists a subsequence $\{x_{p(k)}\}$ of $\{x_p\}$ such that $\alpha(x_{p(k)},u)\geq 1$ for all k.*

Then T has a fixed point.

Theorem 3.23 *Let (χ,d_c), T and α be as in Theorem* 3.21. *Suppose that all conditions of Theorem* 3.21 *or Theorem* 3.22 *are satisfied. If T is an α-identical or α-dominated function, then T has a fixed point $x_*\in\chi$. Further, if T satisfies condition* (**G**), *then the fixed point is unique.*

Theorem 3.24 *Let (χ,d_c), T and α be as in Theorem* 3.21. *Suppose that all conditions of Theorem* 3.21 *or Theorem* 3.22 *are satisfied. If T is an α-identical or α-dominated function, then T has a fixed point $x_*\in\chi$. Further, if T satisfies condition* (**G**′), *then the fixed point is unique.*

Theorem 3.25 *Let (χ, d_c) be a complete CMS-BA, $T\colon \chi \to \chi$, A and B be nonempty subsets of χ such that $\chi = A \cup B$ and $T(A) \subseteq T(B)$ and $T(B) \subseteq T(A)$. If there exists $\mu \in P$ such that $0 \leq r(\mu) < 1$, and*

$$d_{\mathrm{lc}}(Tu, Tv) \preceq \mu.\varphi(u,v) \tag{3.6}$$

for all $u \in A$, $v \in B$ and $\varphi(u,v) \in \{d_{\mathrm{lc}}(u,v), d_{\mathrm{lc}}(u,Tu), d_{\mathrm{lc}}(v,Tv), d_{\mathrm{lc}}(u,Tv), d_{\mathrm{lc}}(v,Tu)\}$, then T has a unique fixed point in $A \cap B$.

Proof Let

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \{A \times B, B \times A\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then T is an α-admissible, weak semi α-admissible and α-dominated function and satisfies condition ($\mathbf{G}'$). Hence, by Theorem 3.24, T has a unique fixed point x_* in χ. Since T is α-dominated, $\alpha(x_*, x_*) = \alpha(x_*, Tx_*) \geq 1$. This is possible iff $x_* \in A \cap B$. □

Corollary 3.26 *Let (χ, d_{lc}) be a complete dCMS-BA and $T\colon \chi \to \chi$ be a mapping. If there exists $\mu \in P$ such that $0 \leq r(\mu) < 1$, and*

$$d_{\mathrm{lc}}(Tu, Tv) \preceq \mu.\varphi(u,v) \tag{3.7}$$

for all $u, v \in \chi$ and $\varphi(u,v) \in \{d_{\mathrm{lc}}(u,v), d_{\mathrm{lc}}(u,Tu), d_{\mathrm{lc}}(v,Tv), d_{\mathrm{lc}}(u,Tv), d_{\mathrm{lc}}(v,Tu)\}$, then T has a unique fixed point.

Proof The proof easily follows from Theorems 3.21 and 3.23 or Theorems 3.22 and 3.24 by taking $\alpha(x,y) = 1$ for all $x, y \in \chi$. □

Corollary 3.27 (Theorem 9, [13]) *Let (χ, d_c) be a complete CMS-BA and $T\colon \chi \to \chi$ be a mapping. If there exists $\mu \in P$ such that $0 \leq r(\mu) < 1$, and*

$$d_c(Tu, Tv) \preceq \mu.\varphi(u,v) \tag{3.8}$$

for all $u, v \in \chi$ and $\varphi(u,v) \in \{d_c(u,v), d_c(u,Tu), d_c(v,Tv), d_c(u,Tv), d_c(v,Tu)\}$, then T has a unique fixed point.

Proof Since every *CMS-BA* is a *dCMS-BA*, the proof follows from Corollary 3.26. □

Example 3.28 Let $\chi = [0,\infty)$ and $\mathcal{A}$ be as in Example 3.3 and $d_{\mathrm{lc}}(x,y)(t) = (|x-y|(1+t^2), |x-y|(1+t^2))$. Let $T : \chi \to \chi$ be given by

$$Tx = \begin{cases} \log(1 + \frac{x}{2}) & \text{if } x \in [0,1], \\ \log(1 + 2x) & \text{otherwise,} \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1 & (x,y) \in [0,1] \text{ or } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then α is a triangular function, T is α-admissible, α-identical and satisfies condition (**G**). Also, for all $\alpha(x,y) \geq 1$, we see that $d_{\mathrm{lc}}(Tx,Ty) \preceq q.d_{\mathrm{lc}}(x,y)$, where $q = \frac{1}{2}$. Thus T satisfies all conditions of Theorems 3.18 and 3.19 but does not satisfy conditions of Corollary 3.27. Further 0 is a unique common fixed point of T.

Theorem 3.29 *Let* (χ, d_{lc}) *be a complete dCMS-BA and* $T\colon \chi \to \chi$ *be a mapping. Let* $\alpha : X \times X \to [0,\infty)$ *be a mapping satisfying conditions* (iii), (iv) *and* (v) *of Theorem* 3.16 *or* 3.17. *If there exist* $\lambda, \mu, \nu \in P$ *such that* λ *commutes with* $\mu + 3\nu$, $\mu + \nu$ *commutes with* $\mu + 3\nu$, $r(\lambda + \mu + \nu) + r(\mu + 3\nu) < 1$ *and for all* $x, y \in \chi$ *with* $\alpha(x,y) \geq 1$

$$d_{\mathrm{lc}}(Tx,Ty) \preceq \lambda d_{\mathrm{lc}}(x,y) + \mu\big(d_{\mathrm{lc}}(x,Tx) + d(y,Ty)\big) + \nu\big(d_{\mathrm{lc}}(x,Ty) + d(y,Tx)\big), \tag{3.9}$$

then T *has a fixed point. Further, if* T *is an* α*-identical function or if* T *is* α*-dominated, then* T *has a fixed point* $x_* \in \chi$ *and* $d_{\mathrm{lc}}(x_*,x_*) = \theta$. *Moreover, if* T *satisfies condition* (**G**) *or* (**G**′), *then the fixed point is unique.*

Proof Consider the iterative sequence defined by $x_{p+1} = Tx_p$ for all $p \in \mathbb{N}$. Let $d_{\mathrm{lc}}(x_p,x_{p+1}) = d_p$ and $d_{\mathrm{lc}}(x_p,x_p) = d_{p,p}$. Note that $d_{p,p} \preceq 2d_{p-1}$ and $d_{p,p} \preceq 2d_{p+1}$. By Lemma 3.12 or 3.13, $\alpha(x_p,x_{p+1}) \geq 1$, and therefore using (3.9) we have

$$\begin{aligned} d_{p+1} &= d_{lcb}(x_{p+1},x_{p+2}) = d_{\mathrm{lc}}(Tx_p,Tx_{p+1}) \\ &\preceq \lambda d_{\mathrm{lc}}(x_p,x_{p+1}) + \mu\big(d_{\mathrm{lc}}(x_p,Tx_p) + d_{\mathrm{lc}}(x_{p+1},Tx_{p+1})\big) \\ &\quad + \nu\big(d_{\mathrm{lc}}(x_p,Tx_{p+1}) + d_{\mathrm{lc}}(x_{p+1},Tx_p)\big) \\ &= \lambda d_p + \mu(d_p + d_{p+1}) + \nu(d_p + d_{p+1} + 2d_p), \end{aligned}$$

i.e.

$$d_{p+1} \preceq \beta.d_p \quad \text{where } \beta = (\lambda + \mu + \nu)(e - \mu - 3\nu)^{-1}$$

or

$$d_{p+1} \preceq \beta^{p+1} d_p.$$

Since λ commutes with $\mu + 3\nu$, $\mu + \nu$ commutes with $\mu + 3\nu$, simple calculations show that $(\lambda + \mu + \nu)(e - \mu - 3\nu)^{-1} = (e - \mu - 3\nu)^{-1}(\lambda + \mu + \nu)$, and using Lemma 2.4 again by simple calculations we have $r(\beta) = r(\lambda + \mu + \nu)(e - \mu - 3\nu)^{-1} < 1$. Therefore $e - \beta$ is invertible and $(e - \beta)^{-1} = \sum_{i=0}^{\infty}(\beta)^i$. Hence

$$\begin{aligned} d_{\mathrm{lc}}(x_p,x_{p+n}) &\preceq d_p + d_{p+1} + \cdots d_{p+n-1} \\ &\preceq \beta^p\big\{e + \beta + \beta^2 + \cdots\big\} d_0 \\ &= \beta^p(e - \beta)^{-1} d_0. \end{aligned}$$

Also using Lemma 2.8 and Lemma 2.7, $\beta^p(e-\beta)^{-1}d_0$ is a c-sequence. Thus by Definition 2.1 for any $c \in A$ with $\theta \ll c$, there is $N_1 \in \mathbb{N}$ satisfying $n > N_1$ implies

$$d_{lcb}(x_p, x_{p+n}) \preceq \beta^p(e-\beta)^{-1}d_0 \ll c. \tag{3.10}$$

Thus $\{x_p\}$ is a Cauchy sequence, and since (χ, d_{lc}) is complete, we have $u \in \chi$ such that

$$\lim_{n\to\infty} x_p = u. \tag{3.11}$$

By Proposition 3.5 and Lemma 2.6, $d_{\mathrm{lc}}(x_p, u) \to \theta$, $d_{\mathrm{lc}}(x_{p-1}, x_p) \to \theta$ and $d_{\mathrm{lc}}(x_{p-1}, u) \to \theta$. Since (χ, d_{lc}) is α-regular, $\alpha(x_{p-1}, u) \geq 1$ and so by (3.1). Since $d_p \neq d_q$ whenever $p \neq q$, there exists $k \in \mathbb{N}$ such that $d_{\mathrm{lc}}(u, Tu) \neq \{d_k, d_{k+1}, \ldots\}$. Then, for any $p > k$,

$$\begin{aligned} d_{\mathrm{lc}}(u, Tu) &\preceq \bigl(d_{\mathrm{lc}}(u, x_p) + d_{\mathrm{lc}}(x_p, Tu)\bigr) \\ &= \bigl(d_{\mathrm{lc}}(u, x_p) + d_{\mathrm{lc}}(Tx_{p-1}, Tu)\bigr) \\ &\preceq d_{\mathrm{lc}}(u, x_p) + \lambda d_{\mathrm{lc}}(x_{p-1}, u) + \mu\bigl(d_{\mathrm{lc}}(x_{p-1}, x_p) + d_{\mathrm{lc}}(u, Tu)\bigr) \\ &\quad + \nu\bigl(d_{\mathrm{lc}}(x_{p-1}, Tu) + d_{\mathrm{lc}}(u, x_p)\bigr), \end{aligned}$$

i.e.

$$\begin{aligned} d_{\mathrm{lc}}(u, Tu) \preceq{}& \nu(e-\mu-\nu)^{-1}d_{\mathrm{lc}}(u, x_p) \\ &+ (e-\mu-\nu)^{-1}(\lambda+\nu)d_{\mathrm{lc}}(x_{p-1}, u) \\ &+ (e-\mu-\nu)^{-1}\mu\bigl(d_{\mathrm{lc}}(x_{p-1}, x_p) + d_{lcb}(u, Tu)\bigr). \end{aligned}$$

By Lemma 2.6, $\nu(e-\mu-\nu)^{-1}d_{\mathrm{lc}}(u, x_p) \to \theta$, $(e-\mu-\nu)^{-1}(\lambda+\nu)d_{\mathrm{lc}}(u, x_{p-1}) \to \theta$ and $(e-\mu-\nu)^{-1}\mu(d_{\mathrm{lc}}(x_{p-1}, x_p) + d_{\mathrm{lc}}(u, Tu)) \to \theta$. Hence $d_{\mathrm{lc}}(u, Tu) \to \theta$. Thus $Tu = u$.

If T is an α-identical function or if T is α-dominated, then proceeding as in the proof of Theorem 3.18 and using (3.9), we get $d_{\mathrm{lc}}(u, u) = \theta$. Now suppose that there exists u^* such that $Tu^* = u^*$. Then, as above, $d_{\mathrm{lc}}(u^*, u^*) = \theta$. If T satisfies condition (**G**), we have $\alpha(u, u_*) \geq 1$, and then by (3.9)

$$\begin{aligned} d_{\mathrm{lc}}\bigl(u, u^*\bigr) &= d_{\mathrm{lc}}\bigl(Tu, Tu^*\bigr) \preceq (\lambda+2\nu)d_{\mathrm{lc}}\bigl(u, u^*\bigr) \\ &\preceq (\lambda+2\nu)^2 d_{\mathrm{lc}}\bigl(u, u^*\bigr) \cdots (\lambda+2\nu)^n d_{\mathrm{lc}}\bigl(u, u^*\bigr). \end{aligned}$$

As above, by Lemma 2.8 and Lemma 2.7, $(\lambda+2\nu)^n d_{\mathrm{lc}}(u, u^*)$ is a c-sequence, and so by Lemma 2.6 $(\lambda+2\nu)^n d_{\mathrm{lc}}(u, u^*) \to \theta$ as $n \to \infty$. Thus $u = u^*$.

If T satisfies condition (**G**$'$), then there exists $w \in \chi$ such that $\alpha(u, w) \geq 1$, $\alpha(u_*, w) \geq 1$, $\alpha(w, w) \geq 1$ and $\alpha(w, Tw) \geq 1$. Then, by replacing x_0 with w in condition (iv) and proceeding as above, the sequence $\{T^n w\}$ will converge to a fixed point say w_* of T and $d_{\mathrm{lc}}(w^*, w^*) = \theta$. Since T is α-admissible, we get $\alpha(u, T^n w) \geq 1$ and $\alpha(u_*, T^n w) \geq 1$, and then by (3.9) we have

$$\begin{aligned} d_{\mathrm{lc}}\bigl(u, T^{n+1}w\bigr) &= d_{\mathrm{lc}}\bigl(Tu, TT^n w\bigr) \\ &\preceq \lambda d_{\mathrm{lc}}\bigl(u, T^n w\bigr) + \mu\bigl\{d_{\mathrm{lc}}(u, u) + d_{\mathrm{lc}}\bigl(T^n w, T^{n+1}w\bigr)\bigr\} \\ &\quad + \nu\bigl\{d_{\mathrm{lc}}\bigl(u, T^n w\bigr) + d_{\mathrm{lc}}\bigl(T^n w, u\bigr)\bigr\}. \end{aligned}$$

Then, as $n \to \infty$, using Proposition 3.5 and Lemma 2.6, we get $d_{lc}(u, w_*) \preceq \theta$ and so $u = w_*$. Similarly, we can show that $u_* = w_*$. Therefore $u = u_*$. □

Theorem 3.30 *Let (χ, d_{lc}) be a complete dCMS-BA and $T: \chi \to \chi$ be a mapping. If there exists $\lambda \in P$ such that $0 \leq r(2\lambda) < 1$, and*

$$d_{lc}(Tx, Ty) \preceq \lambda\big[d_{lc}(x, Tx) + d_{lc}(y, Ty)\big] \tag{3.12}$$

for all $x, y \in \chi$ with $\alpha(x, y) \geq 1$, then T has a unique fixed point.

Proof Note that (3.12) implies (3.9). Hence the result follows from Theorem 3.29. □

Theorem 3.31 *Let (χ, d_{lc}) be a complete dCMS-BA and $T: \chi \to \chi$ be a mapping. If there exists $\lambda \in P$ such that $0 \leq r(4\lambda) < 1$, and*

$$d_{lc}(Tx, Ty) \preceq \lambda\big[d_{lc}(x, Ty) + d_{lc}(y, Ts)\big] \tag{3.13}$$

for all $x, y \in \chi$ with $\alpha(x, y) \geq 1$, then T has a unique fixed point.

Proof Note that (3.13) implies (3.9). Hence the result follows from Theorem 3.29. □

4 Conclusion

In this paper we have introduced the concept of dislocated cone metric space over Banach algebra and proved some generalised fixed point theorems in such a space. Some new properties of mappings such as α-identical mappings, semi α-admissible mappings, mappings satisfying condition (**G**) and condition (**G**′) are also introduced. Our work is a generalisation of some work already done on metric spaces and cone metric spaces over Banach algebra. There is further scope for extending and generalising various fixed point theorems in the setting of a dislocated cone metric space over Banach algebra.

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