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Fixed point theorems for F-expanding mappings

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Abstract

Recently, Wardowski (Fixed Point Theory Appl. 2012:94, 2012) introduced a new concept of F-contraction and proved a fixed point theorem which generalizes the Banach contraction principle. Following this direction of research, in this paper, we present some new fixed point results for F-expanding mappings, especially on a complete G-metric space.

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1 Introduction

Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be expanding if

$$\forall_{x,y\in X} \quad d(Tx,Ty) \geq \lambda d(x,y), \quad \text{where} \quad \lambda > 1. \tag{1}$$

The condition $\lambda > 1$ is important, the function $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x + e^x$ satisfies the condition $|Tx - Ty| \geq |x - y|$ for all $x, y \in \mathbb{R}$, and T has no fixed point.

For an expanding map, the following result is well known.

Theorem 1.1 *Let (X,d) be a complete metric space, and let $T : X \to X$ be surjective and expanding. Then T is bijective and has a unique fixed point.*

It follows from the Banach contraction principle and the following very simple observation.

Lemma 1.2 *If $T : X \to X$ is surjective, then there exists a mapping $T^* : X \to X$ such that $T \circ T^*$ is the identity map on X.*

Proof For any point $x \in X$, let $y_x \in X$ be any point such that $Ty_x = x$. Let $T^*x = y_x$ for all $x \in X$. Then $(T \circ T^*)(x) = T(T^*x) = Ty_x = x$ for all $x \in X$. □

In the present paper, we introduce a new type of expanding mappings.

Definition 1.3 Let $\mathcal{F}$ be the family of all function $F : (0,+\infty) \to \mathbb{R}$ such that

$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0,+\infty)$, if $\alpha < \beta$, then $F(\alpha) < F(\beta)$;

$(F2)$ for each sequence $\{\alpha_n\} \subset (0, +\infty)$, the following holds:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3)$ there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0_+} \alpha^k F(\alpha) = 0$.

Definition 1.4 Let (X, d) be a metric space. A mapping $T : X \to X$ is called F-expanding if there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x, y \in X$,

$$d(x,y) > 0 \quad \Rightarrow \quad F\bigl(d(Tx,Ty)\bigr) \geq F\bigl(d(x,y)\bigr) + t. \tag{2}$$

When we consider in (2) the different types of the mapping $F \in \mathcal{F}$, then we obtain a variety of expanding mappings.

Example 1.5 Let $F_1(\alpha) = \ln\alpha$. It is clear that F_1 satisfies $(F1)$, $(F2)$, $(F3)$ for any $k \in (0,1)$. Each mapping $T : X \to X$ satisfying (2) is an F_1-expanding map such that

$$d(Tx,Ty) \geq e^t d(x,y) \quad \text{for all} \quad x, y \in X,\ d(x,y) > 0.$$

It is clear that for $x, y \in X$ such that $x = y$, the inequality $d(Tx,Ty) \geq e^t d(x,y)$ also holds.

Example 1.6 If $F_2(\alpha) = \ln\alpha + \alpha$, $\alpha > 0$, then F_1 satisfies $(F1)$, $(F2)$ and $(F3)$, and condition (2) is of the form

$$d(Tx,Ty)e^{d(Tx,Ty)-d(x,y)} \geq e^t d(x,y) \quad \text{for all} \quad x, y \in X.$$

Example 1.7 Consider $F_3(\alpha) = \ln(\alpha^2 + \alpha)$, $\alpha > 0$. F_3 satisfies $(F1)$, $(F2)$ and $(F3)$, and for F_3-expanding T, the following condition holds:

$$d(Tx,Ty) \cdot \frac{d(Tx,Ty) + 1}{d(x,y) + 1} \geq e^t d(x,y) \quad \text{for all} \quad x, y \in X.$$

Example 1.8 Consider $F_4(\alpha) = \arctan(-\frac{1}{\alpha})$, $\alpha > 0$. F_4 satisfies $(F1)$, $(F2)$ and $(F3)$, and for F_4-expanding T, the following condition holds:

$$d(Tx,Ty) \geq \left[\frac{1 + \frac{\tan t}{d(x,y)}}{1 - \tan t \cdot d(x,y)}\right] d(x,y) \quad \text{for some} \quad 0 < t < \frac{\pi}{2}.$$

Here, we have obtained a special type of nonlinear expanding map $d(Tx,Ty) \geq \varphi(d(x,y))d(x,y)$.

Other functions belonging to $\mathcal{F}$ are, for example, $F(\alpha) = \ln(\alpha^n)$, $n \in \mathbb{N}$, $\alpha > 0$; $F(\alpha) = \ln(\arctan\alpha)$, $\alpha > 0$.

Now we recall the following.

Definition 1.9 Let (X, d) be a metric space. A mapping $T : X \to X$ is an F-contraction on X if there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x, y \in X$,

$$d(Tx,Ty) > 0 \quad \Rightarrow \quad t + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y)\bigr). \tag{3}$$

For such mappings, Wardowski [1] proved the following theorem.

Theorem 1.10 *Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $u \in X$ and for every $x \in X$, a sequence $\{x_n = T^n x\}$ is convergent to u.*

2 The result

In this section, we give some fixed point theorem for F-expanding maps.

Theorem 2.1 *Let (X,d) be a complete metric space and $T : X \to X$ be surjective and F-expanding. Then T has a unique fixed point.*

Proof From Lemma 1.2, there exists a mapping $T^* : X \to X$ such that $T \circ T^*$ is the identity mapping on X. Let $x, y \in X$ be arbitrary points such that $x \neq y$, and let $z = T^*x$ and $w = T^*y$ (obviously, $z \neq w$). By using (2) applied to z and w, we have

$$F\bigl(d(Tz, Tw)\bigr) \geq F\bigl(d(z,w)\bigr) + t.$$

Since $Tz = T(T^*x) = x$ and $Tw = T(T^*y) = y$, then

$$F\bigl(d(x,y)\bigr) \geq F\bigl(d\bigl(T^*x, T^*y\bigr)\bigr) + t,$$

so $T^* : X \to X$ is an F-contraction. By Theorem 1.10, T^* has a unique fixed point $u \in X$. In particular, u is also a fixed point of T because $T^*u = u$ implies that $Tu = T(T^*u) = u$.

Let us observe that T has at most one fixed point. If $u, v \in X$ and $Tu = u \neq v = Tv$, then we would get the contradiction

$$F\bigl(d(Tu, Tv)\bigr) \geq F\bigl(d(u,v)\bigr) + t,$$
$$0 = F\bigl(d(Tu, Tv)\bigr) - F\bigl(d(u,v)\bigr) \geq t > 0,$$

so the fixed point of T is unique. □

Remark 2.2 If T is not surjective, the previous result is false. For example, let $X = [0, \infty)$ endowed with the metric $d(x,y) = |x - y|$ for all $x, y \in X$, and let $T : X \to X$ be defined by $Tx = 2x + 1$ for all $x \in X$. Then T satisfies the condition $d(Tx, Ty) \geq 2d(x,y)$ for all $x, y \in X$ and T is fixed point free.

3 Applications to G-metric spaces

In 2006 Mustafa and Sims (see [2] and the references therein) introduced the notion of a G-metric space and investigated the topology of such spaces. The G-metric space is as follows.

Definition 3.1 Let X be a nonempty set. A function $G : X \times X \times X \to [0, \infty)$ satisfying the following axioms:

(G_1) $G(x,y,z) = 0$ if $x = y = z$,
(G_2) $G(x,x,y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G_3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,
(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),
(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$,

is called a G-metric on X, and the pair (X,G) is called a G-metric space.

Recently, Samet et al. [3] observed that some fixed point theorems in the context of G-metric spaces can be concluded from existence results in the setting of quasi-metric spaces. Especially, the following theorem is a simple consequence of Theorem 1.10.

Theorem 3.2 *Let (X,G) be a complete G-metric space, and let $T: X \to X$ satisfy one of the following conditions*:

(a) *T is an F-contraction of type I on a G-metric space X, i.e., there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x,y \in X$,*

$$G(Tx,Ty,Ty) > 0 \quad \Rightarrow \quad t + F\bigl(G(Tx,Ty,Ty)\bigr) \leq F\bigl(G(x,y,y)\bigr); \tag{4}$$

(b) *T is an F-contraction of type II on a G-metric space X, i.e., there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x,y,z \in X$,*

$$G(Tx,Ty,Tz) > 0 \quad \Rightarrow \quad t + F\bigl(G(Tx,Ty,Tz)\bigr) \leq F\bigl(G(x,y,z)\bigr). \tag{5}$$

Then T has a unique fixed point $u \in X$, and for any $x \in X$, a sequence $\{x_n = T^n x\}$ is G-convergent to u.

The previous ideas lead also to analogous fixed point theorems for F-expanding mappings on G-metric spaces.

Definition 3.3 A mapping $T: X \to X$ from a G-metric space (X,G) into itself is said to be

(a) F-expanding of type I on a G-metric space X if there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x,y \in X$,

$$G(x,y,y) > 0 \quad \Rightarrow \quad F\bigl(G(Tx,Ty,Ty)\bigr) \geq F\bigl(G(x,y,y)\bigr) + t; \tag{6}$$

(b) F-expanding of type II on a G-metric space X if there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x,y,z \in X$,

$$G(x,y,z) > 0 \quad \Rightarrow \quad F\bigl(G(Tx,Ty,Tz)\bigr) \geq F\bigl(G(x,y,z)\bigr) + t. \tag{7}$$

Theorem 3.4 *Let (X,G) be a complete G-metric space and $T: X \to X$ be a surjective and F-expanding mapping of type I (or type II). Then T has a unique fixed point.*

Proof Let T be an F-expanding mapping of type I. From Lemma 1.2, there exists a mapping $T^*: X \to X$ such that $T \circ T^*$ is the identity mapping on X. Let $x,y \in X$ be arbitrary points such that $x \neq y$, and let $\xi = T^*x$ and $\eta = T^*y$. Obviously, $\xi \neq \eta$ and $G(\xi,\eta,\eta) > 0$. By using (6) applied to ξ and η, we have

$$F\bigl(G(T\xi,T\eta,T\eta)\bigr) \geq F\bigl(G(\xi,\eta,\eta)\bigr) + t.$$

Since $T\xi = T(T^*x) = x$ and $T\eta = T(T^*y) = y$, then

$$F\big(G(x,y,y)\big) \geq F\big(G\big(T^*x, T^*y, T^*y\big)\big) + t,$$

so T^* is an F-contraction of type I on a G-metric space (X,G). Theorem 3.2 guarantees that T^* has a unique fixed point $u \in X$. The point u is also a fixed point of T because $Tu = T(T^*u) = u$.

Now, we prove the uniqueness of the fixed point. Assume that v is another fixed point of T different from u: $Tu = u \neq v = Tv$. This means $G(u,v,v) > 0$, so by (6)

$$0 < t \leq F\big(G(Tu,Tv,Tv)\big) - F\big(G(u,v,v)\big) = 0,$$

which is a contradiction, and hence $u = v$.

For F-expanding mappings of type II, it is necessary to take $z = y$ and apply the proof for F-expanding mappings of type I. □

As a corollary of Theorem 3.4, taking $F_1 \in \mathcal{F}$, see Examples 1.5, we obtain the following.

Corollary 3.5 ([2], Corollary 9.1.4) *Let (X,G) be a complete G-metric space and $T : X \to X$ be surjective, and let there exist $\lambda > 1$ such that*

$$G(Tx,Ty,Ty) \geq \lambda G(x,y,y) \quad \textit{for all} \quad x,y \in X,$$

or

$$G(Tx,Ty,Tz) \geq \lambda G(x,y,z) \quad \textit{for all} \quad x,y,z \in X.$$

Then T has a unique fixed point.

Remark 3.6 If T is not surjective, the previous results are false. Consider $X = (-\infty,-1] \cup [1,\infty)$ endowed with the G-metric $G(x,y,z) = |x-y| + |x-z| + |y-z|$ for all $x,y,z \in X$ and the mapping $T : X \to X$ defined by $Tx = -2x$. Then $G(Tx,Ty,Tz) \geq 2G(x,y,z)$ for all $x,y,z \in X$ and T has no fixed point.

Now, we will improve some results contained in the book [2]. We will use the following observation: if $T : X \to X$ is a surjective mapping, based on each $x_0 \in X$, there exists a sequence $\{x_n\}$ such that $Tx_{n+1} = x_n$ for all $n \geq 0$. Generally, a sequence $\{x_n\}$ verifying the above condition is not necessarily unique.

Theorem 3.7 *Let (X,G) be a complete G-metric space, and let $T : X \to X$ be a surjective mapping. Suppose that there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x,y \in X$,*

$$G(x,Tx,y) > 0 \quad \Rightarrow \quad F\big(G\big(Tx,T^2x,Ty\big)\big) \geq F\big(G(x,Tx,y)\big) + t. \tag{8}$$

Then T has a unique fixed point.

Proof Let $x_0 \in X$ be arbitrary. Since T is surjective, there exists $x_1 \in X$ such that $Tx_1 = x_0$. By continuing this process, we can find a sequence $\{x_n = Tx_{n+1}\}$ for all $n = 0,1,2,\ldots$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0+1} is a fixed point of T.

Now assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Then $G(x_{n+1}, x_n, x_n) > 0$ for all $n \geq 0$, and from (8) with $x = x_{n+1}$ and $y = x_n$, we have, for all $n \geq 1$,

$$\begin{aligned} F\bigl(G(x_n, x_{n-1}, x_{n-1})\bigr) &= F\bigl(G\bigl(Tx_{n+1}, T^2x_{n+1}, Tx_n\bigr)\bigr) \\ &\geq F\bigl(G(x_{n+1}, Tx_{n+1}, x_n)\bigr) + t = F\bigl(G(x_{n+1}, x_n, x_n)\bigr) + t, \end{aligned}$$

and hence

$$t + F\bigl(G(x_{n+1}, x_n, x_n)\bigr) \leq F\bigl(G(x_n, x_{n-1}, x_{n-1})\bigr). \tag{9}$$

Using (9), the following holds for every $n \geq 1$:

$$\begin{aligned} F\bigl(G(x_{n+1}, x_n, x_n)\bigr) &\leq F\bigl(G(x_n, x_{n-1}, x_{n-1})\bigr) - t \\ &\leq F\bigl(G(x_{n-1}, x_{n-2}, x_{n-2})\bigr) - 2t \leq \cdots \leq F\bigl(G(x_1, x_0, x_0)\bigr) - nt. \end{aligned} \tag{10}$$

From (10) we obtain

$$\lim_{n\to\infty} F\bigl(G(x_{n+1}, x_n, x_n)\bigr) = -\infty,$$

which together with $(F2)$ gives

$$\lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = 0. \tag{11}$$

From $(F3)$ there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \bigl[G(x_{n+1}, x_n, x_n)\bigr]^k F\bigl(G(x_{n+1}, x_n, x_n)\bigr) = 0. \tag{12}$$

By (10), the following holds for all $n \geq 1$:

$$\begin{aligned} &\bigl[G(x_{n+1}, x_n, x_n)\bigr]^k F\bigl(G(x_{n+1}, x_n, x_n)\bigr) - \bigl[G(x_{n+1}, x_n, x_n)\bigr]^k F\bigl(G(x_1, x_0, x_0)\bigr) \\ &\quad\leq \bigl[G(x_{n+1}, x_n, x_n)\bigr]^k \bigl(F\bigl(G(x_1, x_0, x_0)\bigr) - nt\bigr) \\ &\qquad - \bigl[G(x_{n+1}, x_n, x_n)\bigr]^k F\bigl(G(x_1, x_0, x_0)\bigr) = -\bigl[G(x_{n+1}, x_n, x_n)\bigr]^k \cdot nt \leq 0. \end{aligned} \tag{13}$$

Letting $n \to \infty$ in (13) and using (11), (12), we obtain

$$\lim_{n\to\infty} \bigl[G(x_{n+1}, x_n, x_n)\bigr]^k \cdot n = 0. \tag{14}$$

Now, let us observe that from (14) there exists $n_1 \geq 1$ such that

$$\bigl[G(x_{n+1}, x_n, x_n)\bigr]^k \cdot n \leq 1 \quad \text{for all} \quad n \geq n_1.$$

Consequently, we have

$$G(x_{n+1}, x_n, x_n) \leq \frac{1}{n^{1/k}} \quad \text{for all} \quad n \geq n_1.$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges, for any $\varepsilon > 0$, there exists $n_2 \geq 1$ such that $\sum_{i=n_2}^{\infty} \frac{1}{i^{1/k}} < \varepsilon$. In order to show that $\{x_n\}$ is a Cauchy sequence, we consider $m > n > \max\{n_1, n_2\}$. From [2], Lemma 3.1.2(4), we get

$$\begin{aligned} G(x_m, x_n, x_n) &\leq \sum_{j=n}^{m-1} G(x_{j+1}, x_j, x_j) \leq \sum_{j=n}^{\infty} G(x_{j+1}, x_j, x_j) \\ &\leq \sum_{j=n}^{\infty} \frac{1}{j^{1/k}} \leq \sum_{j=n_2}^{\infty} \frac{1}{j^{1/k}} < \varepsilon. \end{aligned}$$

Therefore by [2], Lemma 3.2.2 and axiom (G_4), $\{x_n\}$ is a Cauchy in a G-metric space (X, G). From the completeness of (X, G), there exists $u \in X$ such that $\{x_n\} \to u$. As T is surjective, there exists $w \in X$ such that $u = Tw$. From (8) with $x = x_{n+1}$ and $y = w$, we have, for all $n \geq 1$,

$$\begin{aligned} F\big(G(x_n, x_{n-1}, u)\big) &= F\big(G\big(Tx_{n+1}, T^2x_{n+1}, Tw\big)\big) \\ &\geq F\big(G(x_{n+1}, Tx_{n+1}, w)\big) + t = F\big(G(x_{n+1}, x_n, w)\big) + t, \end{aligned}$$

and hence

$$F\big(G(x_n, x_{n-1}, u)\big) > F\big(G(x_{n+1}, x_n, w)\big). \tag{15}$$

By $(F1)$ from (15), we have

$$G(x_n, x_{n-1}, u) > G(x_{n+1}, x_n, w) \quad \text{for all} \quad n \geq 1. \tag{16}$$

Using the fact that the function G is continuous on each variable ([2], Theorem 3.2.2), taking the limit as $n \to \infty$ in the above inequality, we get

$$G(u, u, w) = \lim_{n \to \infty} G(x_n, x_{n-1}, u) = 0,$$

that is, $u = w$. Then u is a fixed point of T because $u = Tw = Tu$.

To prove uniqueness, suppose that $u, v \in X$ are two fixed points. If $Tu = u \neq v = Tv$, then $G(u, u, v) > 0$. So, by (8),

$$\begin{aligned} F\big(G(u, u, v)\big) &= F\big(G\big(Tu, T^2u, Tv\big)\big) \\ &\geq F\big(G(u, Tu, v)\big) + t = F\big(G(u, u, v)\big) + t, \end{aligned}$$

which is a contradiction, because $t > 0$. Hence, $u = v$. □

Taking $F_1 \in \mathcal{F}$, see Example 1.5, we obtain the following.

Corollary 3.8 ([2], Theorem 9.1.2) *Let (X, G) be a complete G-metric space and $T : X \to X$ be a surjective mapping. Suppose that there exists $\lambda > 1$ such that*

$$G(Tx, T^2x, Ty) \geq \lambda G(x, Tx, y) \quad \text{for all} \quad x, y \in X.$$

Then T has a unique fixed point.

Next result does not guarantee the uniqueness of the fixed point.

Theorem 3.9 *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a surjective mapping. Suppose that there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x, y \in X$,*

$$G(x, Tx, T^2x) > 0 \quad \Rightarrow \quad F(G(Tx, Ty, T^2y)) \geq F(G(x, Tx, T^2x)) + t. \tag{17}$$

Then T has a fixed point.

Proof Let $x_0 \in X$ be arbitrary. Since T is surjective, there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we can find a sequence $\{x_n = Tx_{n+1}\}$ for all $n \geq 0$. If there exists $n_0 \geq 0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0+1} is a fixed point of T.

Now, assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. From (17) with $x = x_{n+1}$ and $y = x_n$, we have $G(x_{n+1}, Tx_{n+1}, T^2x_{n+1}) = G(x_{n+1}, x_n, x_{n-1}) > 0$ and

$$\begin{aligned} F(G(x_n, x_{n-1}, x_{n-2})) &= F(G(Tx_{n+1}, Tx_n, T^2x_n)) \\ &\geq F(G(x_{n+1}, Tx_{n+1}, T^2x_{n+1})) + t = F(G(x_{n+1}, x_n, x_{n-1})) + t, \end{aligned}$$

and hence

$$\begin{aligned} F(G(x_{n+1}, x_n, x_{n-1})) &\leq F(G(x_n, x_{n-1}, x_{n-2})) - t \\ &\leq F(G(x_{n-1}, x_{n-2}, x_{n-3})) - 2t \\ &\leq \cdots \leq F(G(x_2, x_1, x_0)) - (n-1)t. \end{aligned} \tag{18}$$

From (18), we obtain

$$\lim_{n\to\infty} F(G(x_{n+1}, x_n, x_{n-1})) = -\infty,$$

which together with $(F2)$ gives

$$\lim_{n\to\infty} G(x_{n+1}, x_n, x_{n-1}) = 0.$$

Mimicking the proof of Theorem 3.7, we obtain

$$\lim_{n\to\infty} [G(x_{n+1}, x_n, x_{n-1})]^k \cdot (n-1) = 0;$$

and consequently, there exists $n_1 \geq 1$ such that

$$G(x_{n+1}, x_n, x_{n-1}) \leq \frac{1}{(n-1)^{1/k}} \quad \text{for all} \quad n > n_1.$$

Since the series $\sum_{i=1}^{\infty}\frac{1}{i^{1/k}}$ converges, for any $\varepsilon > 0$, there exists $n_2 \geq 1$ such that $\sum_{i=n_2}^{\infty}\frac{1}{i^{1/k}} < \varepsilon$. In order to show that $\{x_n\}$ is a Cauchy sequence, we consider $m > n > \max\{n_1, n_2\}$. From [2], Lemma 3.1.2(4) and axioms (G_3), (G_4), we get

$$\begin{aligned} G(x_m, x_n, x_n) &\leq \sum_{j=n}^{m-1} G(x_{j+1}, x_j, x_j) \leq \sum_{j=n}^{\infty} G(x_{j+1}, x_j, x_j) \\ &\leq \sum_{j=n}^{\infty} G(x_{j+1}, x_j, x_{j-1}) \leq \sum_{j=n}^{\infty}\frac{1}{j^{1/k}} \leq \sum_{j=n_2}^{\infty}\frac{1}{j^{1/k}} < \varepsilon. \end{aligned}$$

Therefore, by [2], Lemma 3.2.2, $\{x_n\}$ is a Cauchy in a G-metric space (X, G). From the completeness of (X, G), there exists $u \in X$ such that $\{x_n\} \to u$. As T is surjective, there exists $w \in X$ such that $u = Tw$. From (17) with $x = w$ and $y = x_{n+1}$, we have

$$F\big(G(u, x_n, x_{n-1})\big) = F\big(G\big(Tw, Tx_{n+1}, T^2x_{n+1}\big)\big) \geq F\big(G\big(w, Tw, T^2w\big)\big) + t,$$

so

$$F\big(G\big(w, Tw, T^2w\big)\big) \leq F\big(G(u, x_n, x_{n-1})\big) - t < F\big(G(u, x_n, x_{n-1})\big).$$

Using $(F1)$, we have

$$G\big(w, Tw, T^2w\big) < G(u, x_n, x_{n-1}) \quad \text{for all} \quad n \geq 1.$$

Using the fact that the function G is continuous on each variable ([2], Theorem 3.2.2), taking the limit as $n \to \infty$ in the above inequality, we get

$$G\big(w, Tw, T^2w\big) = \lim_{n\to\infty} G(u, x_n, x_{n-1}) = 0,$$

that is, $w = Tw = T^2w$. Hence, $u = Tu$. □

Taking $F_1 \in \mathcal{F}$, see Examples 1.5, we obtain the following.

Corollary 3.10 ([2], Theorem 9.1.3) *Let (X, G) be a complete G-metric space and $T : X \to X$ be a surjective mapping. Suppose that there exists $\lambda > 1$ such that*

$$G\big(Tx, Ty, T^2y\big) \geq \lambda G\big(x, Tx, T^2x\big) \quad \textit{for all} \quad x, y \in X.$$

Then T has, at least, a fixed point.

Competing interests
The author declares that they have no competing interests.

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