Włodarczyk *Fixed Point Theory and Applications* (2016) 2016:85
DOI 10.1186/s13663-016-0575-7

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Periodic and fixed points of the Leader-type contractions in quasi-triangular spaces

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Abstract

Let $C = \{C_\alpha\}_{\alpha\in\mathcal{A}} \in [1;\infty)^{\mathcal{A}}$ with index set $\mathcal{A}$. A quasi-triangular space $(X,\mathcal{P}_{C;\mathcal{A}})$ is a set X with family $\mathcal{P}_{C;\mathcal{A}} = \{p_\alpha : X^2 \to [0,\infty), \alpha \in \mathcal{A}\}$ satisfying $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{p_\alpha(u,w) \le C_\alpha[p_\alpha(u,v) + p_\alpha(v,w)]\}$. In $(X,\mathcal{P}_{C;\mathcal{A}})$, using the left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ ($\mathcal{P}_{C;\mathcal{A}}$ is a particular case of $\mathcal{J}_{C;\mathcal{A}}$), we establish theorems concerning left (right) $\mathcal{P}_{C;\mathcal{A}}$-convergence, existence, periodic point, fixed point, and (when $(X,\mathcal{P}_{C;\mathcal{A}})$ is separable) uniqueness for $\mathcal{J}_{C;\mathcal{A}}$-contractions and weak $\mathcal{J}_{C;\mathcal{A}}$-contractions $T : X \to X$ satisfying $\forall_{x,y\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(T^{[s]}(x),T^{[l]}(y)) < \eta+\varepsilon \Rightarrow C_\alpha J_\alpha(T^{[s+r]}(x),T^{[l+r]}(y)) < \varepsilon\}$ and $\exists_{w^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(T^{[s]}(w^0), T^{[l]}(w^0)) < \eta+\varepsilon \Rightarrow C_\alpha J_\alpha(T^{[s+r]}(w^0), T^{[l+r]}(w^0)) < \varepsilon\}$, respectively. The spaces $(X,\mathcal{P}_{C;\mathcal{A}})$, in particular, generalize metric, ultrametric, quasi-metric, ultra-quasi-metric, *b*-metric, partial metric, partial *b*-metric, pseudometric, quasi-pseudometric, ultra-quasi-pseudometric, partial quasi-pseudometric, topological, uniform, quasi-uniform, gauge, ultra gauge, partial gauge, quasi-gauge, ultra-quasi-gauge, and partial quasi-gauge spaces. Results are new in all these spaces. Examples are provided.

MSC: 47H10; 47H09; 37C25; 54E99

Keywords: quasi-triangular space; Leader-type contraction; convergence; periodic point; fixed point; uniqueness

1 Introduction

Contractions are among the most important objects studied in fixed point theory. The study concerning convergence and fixed points of several types contractions in metric spaces and also in various more general distance spaces has undergone remarkable developments in the last years. The effect has been a still-ongoing series of results that are far stronger and more general and optimal than those known before.

The following theorem is one of the first, important, central, simpler, and very inspired result of this theory.

Theorem 1.1 (Banach [1], Caccioppoli [2]) *Let (X,d) be a complete metric space. If $T : X \to X$ satisfies*

$$\exists_{0\le\lambda<1}\forall_{x,y\in X}\{d(T(x),T(y)) \le \lambda d(x,y)\}, \tag{1.1}$$

then: (i) *T has a unique fixed point w in X*; *and* (ii) *for each $w^0 \in X$*, $\lim_{m\to\infty} d(T^{[m]}(w^0), w) = 0$.

The maps satisfying condition (1.1) are called in the literature *Banach contractions*.

In metric spaces, not necessarily complete, by changing condition (1.1), there are many different interesting generalizations of Theorem 1.1 in the literature. Significant papers here are Burton [3], Rakotch [4], Geraghty [5, 6], Matkowski [7–9], Walter [10], Dugundji [11], Tasković [12], Dugundji and Granas [13], Browder [14], Krasnosel'skiĭ et al. [15], Boyd and Wong [16], Mukherjea [17], Meir and Keeler [18], Leader [19], Jachymski [20, 21], Jachymski and Jóźwik [22], and many others.

Among the papers mentioned, the following is especially remarkable.

Theorem 1.2 (Leader [19], Theorem 3) *Let (X,d) be a metric space, and let $T: X \to X$ be a map with a complete graph* (*i.e., closed in Y^2, where Y is the completion of X*). *The following hold*:

(A) *T has a contractive fixed point in X if and only if*

$$\forall_{x,y\in X}\forall_{\varepsilon>0}\exists_{\eta\in(0,\infty]}\exists_{r\in\mathbb{N}}\forall_{i,j\in\mathbb{N}}\bigl\{d\bigl(T^{[i]}(x), T^{[j]}(y)\bigr) < \varepsilon + \eta \\ \Rightarrow d\bigl(T^{[i+r]}(x), T^{[j+r]}(y)\bigr) < \varepsilon\bigr\}. \tag{1.2}$$

(B) *T has a fixed point in X if and only if*

$$\exists_{x\in X}\forall_{\varepsilon>0}\exists_{\eta\in(0,\infty]}\exists_{r\in\mathbb{N}}\forall_{i,j\in\mathbb{N}}\bigl\{d\bigl(T^{[i]}(x), T^{[j]}(x)\bigr) < \varepsilon + \eta \\ \Rightarrow d\bigl(T^{[i+r]}(x), T^{[j+r]}(x)\bigr) < \varepsilon\bigr\}. \tag{1.3}$$

Moreover, if x, ε, η, and r are as in (1.3) *and if* $\exists_{w\in X}\{\lim_{m\to\infty} d(T^{[m]}(x), w) = 0\}$, *then* $\forall_{i\in\mathbb{N}}\{d(T^{[i]}(x), T^{[i+r]}(x)) \leq \eta \Rightarrow d(T^{[i+r]}(x), w) \leq \varepsilon\}$.

By a *contractive fixed point* of $T: X \to X$ we mean a fixed point w of T in X such that, for each $w^0 \in X$, $\lim_{m\to\infty} d(T^{[m]}(w^0), w) = 0$. The maps satisfying conditions (1.2) and (1.3) are called in the literature *Leader contractions* and *weak Leader contractions*, respectively.

Remark 1.1 Let (X,d) be a metric space.

(a) We refer to Jachymski [20, 21] and Jachymski and Jóźwik [22] for a few theorems in (X,d) showing how natural Leader contractions are and how results of Leader [19] generalize the papers of Banach [1], Caccioppoli [2], Burton [3], Rakotch [4], Geraghty [5, 6], Matkowski [7–9], Walter [10], Dugundji [11], Tasković [12], Dugundji and Granas [13], Browder [14], Krasnosel'skiĭ et al. [15], Boyd and Wong [16], Mukherjea [17], Meir and Keeler [18], and many others.

(b) It is worth noticing that Leader's method does not require the complete assumption of (X,d) and that the statements become more elegant and the most general.

Remark 1.2 Notice that:

(a) Leader's proof was based on the observation that (X,d) are topological and Hausdorff, d vanishes on the diagonal, is symmetric, and satisfies triangle inequality, and the map d is continuous.

(b) If we remove some of these conditions, then the situation is much more complicated.

Definition 1.1 Let X be a nonempty set.

(A) A *quasi-pseudometric* on X is a map $p: X^2 \to [0,\infty)$ such that: $\forall_{u\in X}\{p(u,u)=0\}$; and $\forall_{u,v,w\in X}\{p(u,w) \le p(u,v)+p(v,w)\}$. For given quasi-pseudometric p on X, a pair (X,p) is called *quasi-pseudometric space*, and (X,p) is called *Hausdorff* if $\forall_{u,v\in X}\{u \neq v \Rightarrow p(u,v)>0 \vee p(v,u)>0\}$.

(B) Each family $\mathcal{P}=\{p_\alpha : \alpha\in\mathcal{A}\}$ of quasi-pseudometrics $p_\alpha : X^2 \to [0,\infty)$, $\alpha\in\mathcal{A}$, is called a *quasi-gauge* on X.

(C) Let the family $\mathcal{P}=\{p_\alpha : \alpha\in\mathcal{A}\}$ be a quasi-gauge on X. The topology $\mathcal{T}(\mathcal{P})$ having as a subbase the family $\mathcal{B}(\mathcal{P})=\{B(u,\varepsilon_\alpha): u\in X, \varepsilon_\alpha>0, \alpha\in\mathcal{A}\}$ of all balls $B(u,\varepsilon_\alpha)=\{v\in X: p_\alpha(u,v)<\varepsilon_\alpha\}$, $u\in X$, $\varepsilon_\alpha>0$, $\alpha\in\mathcal{A}$, is called the topology *induced* by $\mathcal{P}$ on X.

(D) (Dugundji [23], Reilly [24]) A topological space $(X,\mathcal{T})$ such that there is a quasi-gauge $\mathcal{P}$ on X with $\mathcal{T}=\mathcal{T}(\mathcal{P})$ is called a *quasi-gauge space* and is denoted by $(X,\mathcal{P})$.

(E) A quasi-gauge space $(X,\mathcal{P})$ is called *Hausdorff* if the quasi-gauge $\mathcal{P}$ has the property $\forall_{u,v\in X}\{u\neq v \Rightarrow \exists_{\alpha\in\mathcal{A}}\{p_\alpha(u,v)>0 \vee p_\alpha(v,u)>0\}\}$.

Remark 1.3 Each quasi-uniform space and each topological space is a quasi-gauge space (Reilly [24], Theorems 4.2 and 2.6). The quasi-gauge spaces are spaces with asymmetric structures.

Let X be a (nonempty) set. A *distance* on X is a map $p: X^2 \to [0;\infty)$. A set X, together with a distance on X, is called a *distance space* (see [25, 26]).

Before proceeding further, let us recall the following:

Definition 1.2 ([27]) Let X be a (nonempty) set, let $\mathcal{A}$ be an index set, and let $C=\{C_\alpha\}_{\alpha\in\mathcal{A}}\in[1;\infty)^{\mathcal{A}}$.

(A) We say that a family $P_{C;\mathcal{A}}=\{p_\alpha,\alpha\in A\}$ of distances $p_\alpha: X^2\to[0,\infty)$, $\alpha\in A$, is a quasi-triangular family on X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\bigl\{p_\alpha(u,w)\le C_\alpha\bigl[p_\alpha(u,v)+p_\alpha(v,w)\bigr]\bigr\}.$$

A quasi-triangular space $(X,P_{C;\mathcal{A}})$ is a set X together with a quasi-triangular family $P_{C;\mathcal{A}}$ on X.

(B) Let $(X,P_{C;\mathcal{A}})$ be a quasi-triangular space. We say that $P_{C;\mathcal{A}}$ is separating if

$$\forall_{u,w\in X}\bigl\{u\neq w\Rightarrow\exists_{\alpha\in\mathcal{A}}\bigl\{p_\alpha(u,w)>0\vee p_\alpha(w,u)>0\bigr\}\bigr\}. \tag{1.4}$$

(C) We say that a family $\mathcal{L}_{C;\mathcal{A}}=\{l_\alpha,\alpha\in\mathcal{A}\}$ of distances $l_\alpha: X^2\to[0,\infty)$, $\alpha\in\mathcal{A}$, is an *ultra-quasi-triangular family on* X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\bigl\{l_\alpha(u,w)\le C_\alpha\max\bigl\{l_\alpha(u,v),l_\alpha(v,w)\bigr\}\bigr\}.$$

An *ultra-quasi-triangular space* $(X,\mathcal{L}_{C;\mathcal{A}})$ is a set X together with the ultra-quasi-triangular family $\mathcal{L}_{C;\mathcal{A}}$ on X.

(D) We say that a family $\mathcal{S}_{C;\mathcal{A}} = \{s_\alpha, \alpha \in \mathcal{A}\}$ of distances $s_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathcal{A}$, is a *partial quasi-triangular* family on X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\bigl\{s_\alpha(u,w) \leq C_\alpha\bigl[s_\alpha(u,v) + s_\alpha(v,w)\bigr] - s_\alpha(v,v)\bigr\}.$$

A *partial quasi-triangular space* $(X, \mathcal{S}_{C;\mathcal{A}})$ is a set X together with a partial quasi-triangular family $\mathcal{S}_{C;\mathcal{A}}$ on X.

(E) We say that a family $P_{\mathcal{A}} = \{p_\alpha, \alpha \in A\}$ of distances $p_\alpha : X^2 \to [0,\infty)$, $\alpha \in A$, is a triangular family on X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\bigl\{p_\alpha(u,w) \leq p_\alpha(u,v) + p_\alpha(v,w)\bigr\}.$$

A triangular space $(X, P_{\mathcal{A}})$ is a set X together with a triangular family $P_{\mathcal{A}}$ on X.

Remark 1.4 There are several reasons for studying the quasi-triangular spaces.

(a) First, in the spaces $(X, \mathcal{P}_{C;\mathcal{A}})$, in general, the distances $p_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathcal{A}$, do not vanish on the diagonal, are asymmetric, and do not satisfy the triangle inequality (i.e., the properties $\forall_{\alpha\in\mathcal{A}}\forall_{u\in X}\{p_\alpha(u,u) = 0\}$, or $\forall_{\alpha\in\mathcal{A}}\forall_{u,w\in X}\{p_\alpha(u,w) = p_\alpha(w,u)\}$, or $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{p_\alpha(u,w) \leq p_\alpha(u,v) + p_\alpha(v,w)\}$ do not necessarily hold).

(b) Second, these spaces are not necessarily topological, or Hausdorff, or sequentially complete.

(c) Finally, these spaces generalize ultra-quasi-triangular and partial quasi-triangular spaces (in particular, generalize metric, ultrametric (Roovij [28]), quasi-metric (Wilson [29]), ultra-quasi-metric, b-metric (Bakhtin [30], Czerwik [31]), partial metric (Matthews [32]), partial b-metric (Shukla [33]), pseudometric, quasi-pseudometric, ultra-quasi-pseudometric (Künzi and Otafudu [34]), partial quasi-pseudometric, topological, uniform, quasi-uniform, gauge (Dugundji [11]), ultra-gauge, partial gauge, quasi-gauge (Reilly [24]), ultra-quasi-gauge, and partial quasi-gauge spaces).

Remarks 1.1-1.4 are a well-motivated argument to study several fundamental problems concerning the fixed point theory in these very general spaces $(X, \mathcal{P}_{C;\mathcal{A}})$. Some results in this direction are given in Włodarczyk [27, 35]. Based on some ideas presented in [27, 35–38], we can ask the following question.

Question 1.1 Is there a theorem of Leader type in $(X, \mathcal{P}_{C;\mathcal{A}})$?

The main results of this paper (see Theorems 3.1 and 3.2) provide conditions under which the answer to this question is 'Yes.'

More precisely, the subject of this paper is the constructions of contractions and weak contractions of Leader type and the study of convergence, existence, approximation, periodic point, fixed point, and uniqueness properties of these contractions and weak contractions in quasi-triangular spaces $(X, \mathcal{P}_{C;\mathcal{A}})$. In these very general spaces, by elementary tools and by a natural argument, using radically new and original technique, we derive unexpectedly richer conclusions from very weak hypotheses (see Theorems 3.1 and 3.2). This paper shows that the answer indeed is affirmative and inspires also new ways of looking at old problems in quasi-triangular spaces.

2 Left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$, left (right) $\mathcal{J}_{C;\mathcal{A}}$-convergences, and left (right) $\mathcal{J}_{C;\mathcal{A}}$-sequential completeness in quasi-triangular spaces $(X, \mathcal{P}_{C;\mathcal{A}})$

Let $\mathcal{P}_{C;\mathcal{A}}$ be a quasi-triangular family on X. It is natural to define the notions of *left* (*right*) *families* $\mathcal{J}_{C;\mathcal{A}}$ *generated by* $\mathcal{P}_{C;\mathcal{A}}$, which provide new structures on X.

Definition 2.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space.

(A) The family $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\}$ of distances $J_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathcal{A}$, is said to be a *left* (*right*) *family generated by* $\mathcal{P}_{C;\mathcal{A}}$ if:

($\mathcal{J}1$) $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{J_\alpha(u,w) \le C_\alpha[J_\alpha(u,v) + J_\alpha(v,w)]\}$; and furthermore

($\mathcal{J}2$) for any sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_m,u_n) = 0\Big\} \quad \Big(\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_n,u_m) = 0\Big\}\Big)$$

and

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha(v_m,u_m) = 0\Big\} \quad \Big(\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha(u_m,v_m) = 0\Big\}\Big),$$

we have

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} p_\alpha(v_m,u_m) = 0\Big\} \quad \Big(\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} p_\alpha(u_m,v_m) = 0\Big\}\Big).$$

(B) $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ ($\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) is the set of all left (right) families $\mathcal{J}_{C;\mathcal{A}}$ on X generated by $\mathcal{P}_{C;\mathcal{A}}$.

Remark 2.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Then we have:

(a) $\mathcal{P}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cap \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$.

(b) The structures on X determined by left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ are more general than the structure on X determined by $\mathcal{P}_{C;\mathcal{A}}$.

(c) If $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$, then $(X, \mathcal{J}_{C;\mathcal{A}})$ is a quasi-triangular space.

Definition 2.2 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space, and let $\mathcal{J}_{C;\mathcal{A}}$ be a left (right) family generated by $\mathcal{P}_{C;\mathcal{A}}$.

(A) We say that a sequence $(u_m : m \in \mathbb{N}) \subset X$ is *left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$-*Cauchy sequence* if

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_m,u_n) = 0\Big\} \quad \Big(\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_n,u_m) = 0\Big\}\Big).$$

(B) Let $u \in X$ and $(u_m : m \in \mathbb{N}) \subset X$. We say that a sequence $(u_m : m \in \mathbb{N})$ is *left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$-*convergent to* u if

$$u \in LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \emptyset \quad \big(u \in LIM^{R-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \emptyset\big),$$

where

$$LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} = \Big\{x \in X : \forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha(x,u_m) = 0\Big\}\Big\}$$
$$\Big(LIM^{R-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} = \Big\{x \in X : \forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha(u_m,x) = 0\Big\}\Big\}\Big).$$

(C) We say that a sequence $(u_m : m \in \mathbb{N}) \subset X$ is *left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$*-convergent in* X if

$$LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing \quad \bigl(LIM^{R-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing\bigr).$$

(D) If every left (right) $\mathcal{J}_{C;\mathcal{A}}$-Cauchy sequence $(u_m : m \in \mathbb{N}) \subset X$ is left (right) $\mathcal{J}_{C;\mathcal{A}}$-convergent in X (i.e., $LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing$)), then $(X, \mathcal{P}_{C;\mathcal{A}})$ is called *left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$*-sequential complete.*

(E) We say that $(X, \mathcal{P}_{C;\mathcal{A}})$ is *left* (*right*) *Hausdorff* if for each left (right) $\mathcal{P}_{C;\mathcal{A}}$-convergent in X sequence $(u_m : m \in \mathbb{N})$, the set

$$LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \quad \bigl(LIM^{R-\mathcal{P}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})}\bigr)$$

is a singleton.

Remark 2.2 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. It is clear that if $(u_m : m \in \mathbb{N})$ is left (right) $\mathcal{P}_{C;\mathcal{A}}$-convergent in X, then

$$LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \subset LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(v_m:m\in\mathbb{N})} \quad \bigl(LIM^{R-\mathcal{P}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \subset LIM^{R-\mathcal{P}_{C;\mathcal{A}}}_{(v_m:m\in\mathbb{N})}\bigr)$$

for each subsequence $(v_m : m \in \mathbb{N})$ of $(u_m : m \in \mathbb{N})$.

The following relations between $\mathcal{J}_{C;\mathcal{A}}$ and $\mathcal{P}_{C;\mathcal{A}}$ are interesting.

Theorem 2.1 ([27]) *Let* $(X, \mathcal{P}_{C;\mathcal{A}})$ *be a quasi-triangular space. Let* $E \subset X$ *be a set containing at least two different points, and let* $\{\mu_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$ *where* $\forall_{\alpha\in\mathcal{A}}\{\mu_\alpha \geq \delta_\alpha(E)/(2C_\alpha)\}$ *and* $\forall_{\alpha\in\mathcal{A}}\{\delta_\alpha(E) = \sup\{p_\alpha(u,w) : u, w \in E\}\}$. *If* $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\}$ *where, for each* $\alpha \in \mathcal{A}$, *the distance* $J_\alpha : X^2 \to [0,\infty)$ *is defined by*

$$J_\alpha(u,w) = \begin{cases} p_\alpha(u,w) & \textit{if } E \cap \{u,w\} = \{u,w\}, \\ \mu_\alpha & \textit{if } E \cap \{u,w\} \neq \{u,w\}, \end{cases}$$

then $\mathcal{J}_{C;\mathcal{A}}$ *is the left and right family generated by* $\mathcal{P}_{C;\mathcal{A}}$.

Remark 2.3 This result shows that Definition 2.1 is correct and that $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \setminus \{\mathcal{P}_{C;\mathcal{A}}\} \neq \varnothing$ and $\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})} \setminus \{\mathcal{P}_{C;\mathcal{A}}\} \neq \varnothing$.

Theorem 2.2 ([27]) *Let* $(X, \mathcal{P}_{C;\mathcal{A}})$ *be a quasi-triangular space, and let* $\mathcal{J}_{C;\mathcal{A}}$ *be the left* (*right*) *family generated by* $\mathcal{P}_{C;\mathcal{A}}$. *If* $\mathcal{P}_{C;\mathcal{A}}$ *is separating on* X (*i.e.*, (1.4) *holds*), *then* $\mathcal{J}_{C;\mathcal{A}}$ *is separating on* X, *that is,*

$$\forall_{u,w\in X}\bigl\{u \neq w \Rightarrow \exists_{\alpha\in\mathcal{A}}\bigl\{J_\alpha(u,w) > 0 \vee J_\alpha(w,u) > 0\bigr\}\bigr\}.$$

3 Statement of results

Recall that *a single-valued dynamic system* is defined as a pair (X, T), where X is a certain space, and T is a single-valued map $T : X \to X$, that is, $\forall_{x\in X}\{T(x) \in X\}$. By $Fix(T)$ and $Per(T)$ we denote the sets of all *fixed points* and *periodic points* of T, respectively, that is, $Fix(T) = \{w \in X : w = T(w)\}$ and $Per(T) = \{w \in X : w = T^{[q]}(w)$ for some $q \in \mathbb{N}\}$. For each

$w^0 \in X$, a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is called a *Picard iteration starting at* w^0 of the system (X, T).

In this section, in the quasi-triangular spaces $(X, \mathcal{P}_{C;\mathcal{A}})$, using left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$, we construct the $\mathcal{J}_{C;\mathcal{A}}$*-contractions* (X, T) and *weak* $\mathcal{J}_{C;\mathcal{A}}$*-contractions* (X, T) of Leader type, and we formulate the left (right) $\mathcal{P}_{C;\mathcal{A}}$-convergence, existence, approximation, uniqueness, periodic point, and fixed point theorems for such contractions.

The following terminology will be much used in the sequel.

Definition 3.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Suppose that $\mathcal{J}_{C;\mathcal{A}}$ is the left (right) family generated by $\mathcal{P}_{C;\mathcal{A}}$.

(A) A single-valued dynamic system (X, T) is said to be *left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$*-admissible on a set* $W^{L-\mathcal{J}_{C;\mathcal{A}}}$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}})$ if the following two conditions hold:

(A1) For each $w^0 \in X$ satisfying

$$\forall_{\alpha \in \mathcal{A}} \Big\{ \lim_{m \to \infty} \sup_{n > m} J_\alpha \big(w^m, w^n\big) = 0 \Big\} \tag{3.1}$$

$$\Big(\forall_{\alpha \in \mathcal{A}} \Big\{ \lim_{m \to \infty} \sup_{n > m} J_\alpha \big(w^n, w^m\big) = 0 \Big\} \Big), \tag{3.2}$$

we have

$$\exists_{w \in X} \forall_{\alpha \in \mathcal{A}} \Big\{ \lim_{m \to \infty} J_\alpha \big(w, w^m\big) = 0 \Big\} \quad \big(\text{i.e. } LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m : m \in \{0\} \cup \mathbb{N})} \neq \varnothing\big) \tag{3.3}$$

$$\Big(\exists_{w \in X} \forall_{\alpha \in \mathcal{A}} \Big\{ \lim_{m \to \infty} J_\alpha \big(w^m, w\big) = 0 \Big\} \quad \big(\text{i.e. } LIM^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m : m \in \{0\} \cup \mathbb{N})} \neq \varnothing\big) \Big), \tag{3.4}$$

where $w^m = T^{[m]}(w^0)$, $m \in \{0\} \cup \mathbb{N}$; and

(A2) There exists $W^{L-\mathcal{J}_{C;\mathcal{A}}} \in 2^X$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}} \in 2^X)$ satisfying

$$W^{L-\mathcal{J}_{C;\mathcal{A}}} = \big\{ w^0 \in X : w^0 \text{ satisfies (3.1) and (3.3)} \big\}$$

$$\big(W^{R-\mathcal{J}_{C;\mathcal{A}}} = \big\{ w^0 \in X : w^0 \text{ satisfies (3.2) and (3.4)} \big\} \big).$$

(B) If (X, T) is left (right) $\mathcal{J}_{C;\mathcal{A}}$-admissible on $W^{L-\mathcal{J}_{C;\mathcal{A}}} \in 2^X$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}} \in 2^X)$ and $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ $(w^0 \in W^{R-\mathcal{J}_{C;\mathcal{A}}})$, then we say that (X, T) is *left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$*-admissible in a point* w^0.

Here 2^X denotes the family of all nonempty subsets of a space X.

Remark 3.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space, and let $\mathcal{J}_{C;\mathcal{A}}$ be the left (right) family generated by $\mathcal{P}_{C;\mathcal{A}}$. Let (X, T) be a single-valued dynamic system. If $(X, \mathcal{P}_{C;\mathcal{A}})$ is left (right) $\mathcal{J}_{C;\mathcal{A}}$-sequentially complete, then (X, T) is left (right) $\mathcal{J}_{C;\mathcal{A}}$-admissible on some set $W^{L-\mathcal{J}_{C;\mathcal{A}}} \in 2^X$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}} \in 2^X)$.

Next, we introduce the following concept of *left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$*-closed* single-valued dynamic systems in $(X, \mathcal{P}_{C;\mathcal{A}})$.

Definition 3.2 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Let (X, T) be a single-valued dynamic system, $T : X \to X$, and let $q \in \mathbb{N}$. The single-valued dynamic system $(X, T^{[q]})$

is said to be *left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$*-closed on* $U \in 2^X$ ($V \in 2^X$) if for each $w^0 \in U$ ($w^0 \in V$) such that the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left (right) $\mathcal{P}_{C;\mathcal{A}}$-converging in X (thus, $LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$)) and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[q]}(u_m)\}$, the following property holds: there exists $x \in LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ ($y \in LIM^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$) such that $x = T^{[q]}(x)$ ($y = T^{[q]}(y)$).

For the definition and properties of *closed* maps in topological spaces, see [39, 40].

The main results of this paper are the following theorems.

Theorem 3.1 *Let* $(X, \mathcal{P}_{C;\mathcal{A}})$ *be a quasi-triangular space, and let* (X, T) *be a single-valued dynamic system with* $T : X \to X$. *Assume that the following three conditions hold*:

(H1) $\mathcal{J}_{C;\mathcal{A}}$ *is the left* (*right*) *family generated by* $\mathcal{P}_{C;\mathcal{A}}$.

(H2) (X, T) *is a* $\mathcal{J}_{C;\mathcal{A}}$*-contraction on* X, *that is,*

$$\begin{aligned}&\forall_{x,y\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{J_\alpha\bigl(T^{[s]}(x), T^{[l]}(y)\bigr) < \varepsilon + \eta \\ &\quad\Rightarrow C_\alpha J_\alpha\bigl(T^{[s+r]}(x), T^{[l+r]}(y)\bigr) < \varepsilon\bigr\}.\end{aligned} \tag{3.5}$$

(H3) *There exists a set* $W^{L-\mathcal{J}_{C;\mathcal{A}}} \in 2^X$ ($W^{R-\mathcal{J}_{C;\mathcal{A}}} \in 2^X$) *such that* (X, T) *is left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$*-admissible on* $W^{L-\mathcal{J}_{C;\mathcal{A}}}$ ($W^{R-\mathcal{J}_{C;\mathcal{A}}}$).

Then the following statements hold:

(A) *For each* $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ ($w^0 \in W^{R-\mathcal{J}_{C;\mathcal{A}}}$), *there exists a point* $w \in X$ *such that the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$*-convergent to* w.

(B) *If the single-valued dynamic system* $(X, T^{[q]})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$*-closed on* $W^{L-\mathcal{J}_{C;\mathcal{A}}}$ ($W^{R-\mathcal{J}_{C;\mathcal{A}}}$) *for some* $q \in \mathbb{N}$, *then*:

  (B1) $Fix(T^{[q]}) \neq \varnothing$;

  (B2) *For each* $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ ($w^0 \in W^{R-\mathcal{J}_{C;\mathcal{A}}}$), *there exists a point* $w \in Fix(T^{[q]})$ *such that the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$*-convergent to* w; *and*

  (B3) $\forall_{\alpha\in\mathcal{A}}\forall_{v\in Fix(T^{[q]})}\{J_\alpha(v, T(v)) = J_\alpha(T(v), v) = 0\}$.

(C) *If the family* $\mathcal{P}_{C;\mathcal{A}} = \{p_\alpha, \alpha \in \mathcal{A}\}$ *is separating on* X *and if the single-valued dynamic system* $(X, T^{[q]})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$*-closed on* $W^{L-\mathcal{J}_{C;\mathcal{A}}}$ ($W^{R-\mathcal{J}_{C;\mathcal{A}}}$) *for some* $q \in \mathbb{N}$, *then*:

  (C1) *There exists a point* $w \in X$ *such that* $Fix(T^{[q]}) = Fix(T) = \{w\}$;

  (C2) *For each* $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ ($w^0 \in W^{R-\mathcal{J}_{C;\mathcal{A}}}$), *the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$*-convergent to* w; *and*

  (C3) $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$.

Theorem 3.2 *Let* $(X, \mathcal{P}_{C;\mathcal{A}})$ *be a quasi-triangular space, and let* (X, T) *be a single-valued dynamic system with* $T : X \to X$. *Assume that the following three conditions hold*:

(H1) $\mathcal{J}_{C;\mathcal{A}}$ *is the left* (*right*) *family generated by* $\mathcal{P}_{C;\mathcal{A}}$.

(H2) (X, T) *is a weak* $\mathcal{J}_{C;\mathcal{A}}$*-contraction on* X, *that is, there exists* $w^0 \in X$ *such that*

$$\begin{aligned}&\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{J_\alpha\bigl(T^{[s]}\bigl(w^0\bigr), T^{[l]}\bigl(w^0\bigr)\bigr) < \varepsilon + \eta \\ &\quad\Rightarrow C_\alpha J_\alpha\bigl(T^{[s+r]}\bigl(w^0\bigr), T^{[l+r]}\bigl(w^0\bigr)\bigr) < \varepsilon\bigr\}.\end{aligned} \tag{3.6}$$

(H3) *There exists a set* $W^{L-\mathcal{J}_{C;\mathcal{A}}} \in 2^X$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}} \in 2^X)$ *such that* (X,T) *is left* (*right*) $\mathcal{J}_{C;\mathcal{A}}$-*admissible on* $W^{L-\mathcal{J}_{C;\mathcal{A}}}$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}})$ *and* $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ $(w^0 \in W^{R-\mathcal{J}_{C;\mathcal{A}}})$.

Then the following statements hold:

(A) *There exists a point* $w \in X$ *such that the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$-*convergent to* w.

(B) *If the single-valued dynamic system* $(X, T^{[q]})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$-*closed on* $W^{L-\mathcal{J}_{C;\mathcal{A}}}$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}})$ *for some* $q \in \mathbb{N}$, *then*:

(B1) $Fix(T^{[q]}) \neq \varnothing$;

(B2) *There exists a point* $w \in Fix(T^{[q]})$ *such that the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$-*convergent to* w; *and*

(B3) *If* $w^0 \in Fix(T^{[q]})$, *then* $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w^0, T(w^0)) = J_\alpha(T(w^0), w^0) = 0\}$.

(C) *If the family* $\mathcal{P}_{C;\mathcal{A}} = \{p_\alpha, \alpha \in \mathcal{A}\}$ *is separating on* X *and if the single-valued dynamic system* $(X, T^{[q]})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$-*closed on* $W^{L-\mathcal{J}_{C;\mathcal{A}}}$ $(W^{R-\mathcal{J}_{C;\mathcal{A}}})$ *for some* $q \in \mathbb{N}$, *then*:

(C1) *There exists a point* $w \in X$ *such that* $Fix(T^{[q]}) = Fix(T) = \{w\}$;

(C2) *The sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*right*) $\mathcal{P}_{C;\mathcal{A}}$-*convergent to* w; *and*

(C3) *If* $w^0 = w$, *then* $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w^0, w^0) = 0\}$.

4 Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1 The proof is divided into eleven steps and is only in the case of 'left'; we omit the proof in the case of 'right' since it is based on an analogous technique.

For all $u^0, v^0 \in X$, $\alpha \in \mathcal{A}$, and $k \in \mathbb{N}$, we define

$$\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0, v^0) = \inf\{\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0, v^0, n) : n \in \mathbb{N}\}, \tag{4.1}$$

$$\gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0, v^0) = \inf\{\Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0, v^0, n) : n \in \mathbb{N}\}, \tag{4.2}$$

$$\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0, v^0, n) = \max\{J_\alpha(u^s, v^l) : n \le s, l \le n+k\}, \quad n \in \mathbb{N}, \tag{4.3}$$

$$\Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0, v^0, n) = \max\{J_\alpha(v^s, u^l) : n \le s, l \le n+k\}, \quad n \in \mathbb{N}, \tag{4.4}$$

where $u^m = T^{[m]}(u^0)$ and $v^m = T^{[m]}(v^0)$, $m \in \{0\} \cup \mathbb{N}$.

Step 1. *We have the following property:*

$$\begin{aligned}&\forall_{u^0,v^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\\ &\quad\{\exists_{r_1\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(u^s, v^l) < \varepsilon + \eta \Rightarrow C_\alpha J_\alpha(u^{s+r_1}, v^{l+r_1}) < \varepsilon\}\\ &\quad\wedge \exists_{r_2\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(v^s, u^l) < \varepsilon + \eta \Rightarrow C_\alpha J_\alpha(v^{s+r_2}, u^{l+r_2}) < \varepsilon\}\}.\end{aligned} \tag{4.5}$$

Indeed, let $u^0, v^0 \in X$ be arbitrary and fixed. If we assume that $\alpha \in \mathcal{A}$ and $\varepsilon > 0$ are arbitrary and fixed, then, using (3.5) for $x = u^0$ and $y = v^0$, we obtain $\exists_{\eta_1>0}\exists_{r_1\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(u^s, v^l) < \varepsilon + \eta_1 \Rightarrow C_\alpha J_\alpha(u^{s+r_1}, v^{l+r_1}) < \varepsilon\}$, and, using (3.5) for $x = v^0$ and $y = u^0$, we obtain $\exists_{\eta_2>0}\exists_{r_2\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(v^s, u^l) < \varepsilon + \eta_2 \Rightarrow C_\alpha J_\alpha(v^{s+r_2}, u^{l+r_2}) < \varepsilon\}$. Hence, putting $\eta = \min\{\eta_1, \eta_2\}$, we have

$$\exists_{r_1\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J_\alpha(u^s, v^l) < \varepsilon + \eta \Rightarrow C_\alpha J_\alpha(u^{s+r_1}, v^{l+r_1}) < \varepsilon\}$$

and

$$\exists_{r_2\in\mathbb{N}}\forall_{i,j\in\mathbb{N}}\{J_\alpha(v^s,u^l)<\varepsilon+\eta\Rightarrow C_\alpha J_\alpha(v^{s+r_2},u^{l+r_2})<\varepsilon\}.$$

This gives (4.5).

Step 2. *We show that*

$$\forall_{u^0,v^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\{\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0,v^0)=0\} \tag{4.6}$$

and

$$\forall_{u^0,v^0\in X}\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\{\gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(u^0,v^0)=0\}. \tag{4.7}$$

Indeed, if (4.6) is false, then

$$\exists_{u^0,v^0\in X}\exists_{\alpha_0\in\mathcal{A}}\exists_{k_0\in\mathbb{N}}\exists_{\varepsilon_0>0}\{\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0)=\varepsilon_0\}. \tag{4.8}$$

With this choice of u^0, v^0, α_0, and ε_0 we can use hypothesis (3.5); then there exist $\eta_0>0$ and $r_0\in\mathbb{N}$ such that

$$\forall_{s,l\in\mathbb{N}}\{J_{\alpha_0}(u^s,v^l)<\varepsilon_0+\eta_0\Rightarrow C_{\alpha_0}J_{\alpha_0}(u^{s+r_0},v^{l+r_0})<\varepsilon_0\}. \tag{4.9}$$

Further, by (4.1), $\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0)=\inf\{\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n):n\in\mathbb{N}\}$. This implies, using (4.8), that

$$\exists_{n_0\in\mathbb{N}}\{\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0)=\varepsilon_0\le\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n_0)<\varepsilon_0+\eta_0\}.$$

Next, in view of (4.3), we have that $\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n_0)=\max\{J_{\alpha_0}(u^s,v^l):n_0\le s,l\le n_0+k_0\}$. Thus,

$$\forall_{n_0\le s,l\le n_0+k_0}\{J_{\alpha_0}(u^s,v^l)<\varepsilon_0+\eta_0\},$$

and, using (4.9), we get $\forall_{n_0\le s,l\le n_0+k_0}\{C_{\alpha_0}J_{\alpha_0}(u^{s+r_0},v^{l+r_0})<\varepsilon_0\}$, which we can write as

$$\forall_{n_0+r_0\le s,l\le n_0+r_0+k_0}\{C_{\alpha_0}J_{\alpha_0}(u^s,v^l)<\varepsilon_0\}.$$

Now, note that

$$\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n_0+r_0)=\max\{J_{\alpha_0}(u^s,v^l):n_0+r_0\le s,l\le n_0+r_0+k_0\}$$

in view of (4.3). Consequently,

$$C_{\alpha_0}\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n_0+r_0)<\varepsilon_0. \tag{4.10}$$

Finally, from (4.8), (4.1), and (4.10) it follows that

$$\begin{aligned}\varepsilon_0&=\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0)=\inf\{\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n):n\in\mathbb{N}\}\\&\le\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n_0+r_0)<C_{\alpha_0}\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,k_0}(u^0,v^0,n_0+r_0)<\varepsilon_0,\end{aligned}$$

which is impossible. Therefore, (4.6) holds.

The proof of (4.7) is identical to the proof of (4.6) and is omitted.

Step 3. Let $u^0 \in X$, $\alpha \in \mathcal{A}$, and $\varepsilon > 0$ be arbitrary and fixed. Let $u^m = T^{[m]}(u^0)$ for $m \in \{0\} \cup \mathbb{N}$, and let

$$\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\left\{J_\alpha\left(u^s,u^l\right) < \varepsilon+\eta \Rightarrow C_\alpha J_\alpha\left(u^{s+r},u^{l+r}\right) < \varepsilon\right\}. \tag{4.11}$$

We show that

$$\exists_{n_0\in\mathbb{N}}\left\{C_\alpha \Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}\left(u^0,u^0,n_0\right) = C_\alpha \Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}\left(u^0,u^0,n_0\right) < \min\{\varepsilon,\eta\}\right\} \tag{4.12}$$

and

$$\forall_{s,l\geq n_0}\left\{J_\alpha\left(u^s,u^l\right) < 2\varepsilon\right\}. \tag{4.13}$$

By (4.1), (4.2), (4.6), and (4.7), $\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}(u^0,u^0) = \inf\{\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}(u^0,u^0,n) : n \in \mathbb{N}\} = 0$, $\gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}(u^0,u^0) = \inf\{\Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}(u^0,u^0,n) : n \in \mathbb{N}\} = 0$, and $\forall_{n\in\mathbb{N}}\{\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}(u^0,u^0,n) = \Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}(u^0,u^0,n)\}$. Then there exists $n_0 \in \mathbb{N}$ such that (4.12) holds.

By (4.3) and (4.4) we see that (4.12) implies

$$\begin{aligned} C_\alpha \Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}\left(u^0,u^0,n_0\right) &= C_\alpha \Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}\left(u^0,u^0,n_0\right) \\ &= \max\left\{C_\alpha J_\alpha\left(u^s,u^l\right) : n_0 \leq s,l \leq n_0+r\right\} < \min\{\varepsilon,\eta\}. \end{aligned} \tag{4.14}$$

First, we establish that

$$\forall_{l\geq n_0}\left\{C_\alpha J_\alpha\left(u^{n_0+r},u^l\right) < \varepsilon\right\}. \tag{4.15}$$

If (4.15) is false, then $\exists_{l\geq n_0}\{C_\alpha J_\alpha(u^{n_0+r},u^l) \geq \varepsilon\}$, that is,

$$L = \left\{l\in\mathbb{N} : l \geq n_0 \wedge C_\alpha J_\alpha\left(u^{n_0+r},u^l\right) \geq \varepsilon\right\} \neq \varnothing. \tag{4.16}$$

Denote

$$l_0 = \min L. \tag{4.17}$$

Of course, in view of (4.14), this gives

$$l_0 > n_0. \tag{4.18}$$

Note that

$$\forall_{n_0\leq l<l_0}\left\{C_\alpha J_\alpha\left(u^{n_0+r},u^l\right) < \varepsilon\right\} \tag{4.19}$$

in view of (4.16)-(4.18).

Next, note that also

$$l_0 > n_0 + r. \tag{4.20}$$

Otherwise, $l_0 \leq n_0 + r$, and by (4.13) we get $C_\alpha J_\alpha(u^{n_0+r}, u^{l_0}) \leq \max\{C_\alpha J_\alpha(u^i, u^j) : n_0 \leq i,j \leq n_0 + r\} = C_\alpha \Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}(u^0, u^0, n_0) < \min\{\varepsilon, \eta\} \leq \varepsilon$, which, in view of (4.16), (4.17), and (4.19), is impossible. Thus, (4.20) holds.

In view of (4.18) and (4.20), we have that $n_0 < l_0 - r < l_0$, and, consequently, using (4.16) and (4.17), we conclude that

$$C_\alpha J_\alpha\big(u^{n_0+r}, u^{l_0-r}\big) < \varepsilon. \tag{4.21}$$

Next, using ($\mathcal{J}1$) of Definition 2.1, (4.13), (4.14), and (4.21), we obtain

$$\begin{aligned} &J_\alpha\big(u^{n_0}, u^{l_0-r}\big) \\ &\quad\leq C_\alpha\big[J_\alpha\big(u^{n_0}, u^{n_0+r}\big) + J_\alpha\big(u^{n_0+r}, u^{l_0-r}\big)\big] \\ &\quad< C_\alpha \Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,r}\big(u^0, u^0, n_0\big) + \varepsilon \\ &\quad< \eta + \varepsilon. \end{aligned}$$

Hence, since r satisfies (4.11), we get $C_\alpha J_\alpha(u^{n_0+r}, u^{l_0}) < \varepsilon$. In view of (4.16) and (4.17), this is impossible.

Consequently, (4.15) holds.

We can show in a similar way that

$$\forall_{s\geq n_0}\big\{C_\alpha J_\alpha\big(u^s, u^{n_0+r}\big) < \varepsilon\big\}; \tag{4.22}$$

the proof of (4.22) is identical to the proof of (4.15) and is omitted.

To establish (4.13), we see that by ($\mathcal{J}1$) of Definition 2.1, (4.22), and (4.15) we obtain

$$\forall_{s,l\geq n_0}\big\{J_\alpha\big(u^s, u^l\big) \leq C_\alpha J_\alpha\big(u^s, u^{n_0+r}\big) + C_\alpha J_\alpha\big(u^{n_0+r}, u^l\big) < \varepsilon + \varepsilon = 2\varepsilon\big\}.$$

Step 4. Let $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ be arbitrary and fixed. Define the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$. We show that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{s,l\geq n_0}\big\{J_\alpha\big(w^s, w^l\big) < \varepsilon/2\big\}. \tag{4.23}$$

Indeed, let α_0 and ε_0 be arbitrary and fixed. By (3.5) we get

$$\exists_{\eta_0>0}\exists_{r_0\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\big\{J_{\alpha_0}\big(w^s, w^l\big) < \varepsilon_0 + \eta_0 \Rightarrow C_{\alpha_0}J_{\alpha_0}\big(w^{s+r_0}, w^{l+r_0}\big) < \varepsilon_0\big\}.$$

Next, by (4.3) and (4.4) we have $\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\forall_{n\in\mathbb{N}}\{\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(w^0, w^0, n) = \Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(w^0, w^0, n)\}$. Moreover, by Step 2 we have $\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\{\delta_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(w^0, w^0) = \gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha,k}(w^0, w^0) = 0\}$. Hence, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &C_{\alpha_0}\Delta_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,r_0}\big(w^0, w^0, n_0\big) \\ &\quad= C_{\alpha_0}\Gamma_{\mathcal{J}_{C;\mathcal{A}};\alpha_0,r_0}\big(w^0, w^0, n_0\big) < \min\{\varepsilon_0/4, \eta_0\}. \end{aligned} \tag{4.24}$$

Now, from (4.24), using Step 3, we get $\forall_{s,l\geq n_0}\{J_{\alpha_0}(w^s, w^l) < 2(\varepsilon_0/4) = \varepsilon_0/2\}$. This proves that (4.23) holds.

Step 5. *Let* $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ *be arbitrary and fixed. Define the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$. *We show that*

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha\big(w^m, w^n\big) = 0\Big\} \tag{4.25}$$

and

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha\big(w^n, w^m\big) = 0\Big\}. \tag{4.26}$$

Indeed, in view of (4.23), we obtain, in particular, that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\big\{J_\alpha\big(w^m, w^n\big) < \varepsilon/2\big\}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\big\{J_\alpha\big(w^n, w^m\big) < \varepsilon/2\big\}.$$

From this it follows that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\Big\{\sup_{n>m} J_\alpha\big(w^m, w^n\big) \leq \varepsilon/2 < \varepsilon\Big\}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\Big\{\sup_{n>m} J_\alpha\big(w^n, w^m\big) \leq \varepsilon/2 < \varepsilon\Big\},$$

and hence (4.25) and (4.26) hold.

Step 6. *Statement* (A) *holds.*

Indeed, let $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ be arbitrary and fixed. Define the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$. By (4.25), Definition 2.2, and hypothesis (H3) we get that this sequence is left $\mathcal{J}_{C;\mathcal{A}}$-convergent in X, that is, there exists a nonempty set $LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset X$ such that

$$\forall_{w\in LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}}\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha\big(w, w^m\big) = 0\Big\}. \tag{4.27}$$

However, by hypothesis (H1), $\mathcal{J}_{C;\mathcal{A}}$ is left family generated by $\mathcal{P}_{C;\mathcal{A}}$. Therefore, fixing $w \in LIM^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$, defining $(u_m = w^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = w : m \in \{0\} \cup \mathbb{N})$, and using (4.25) and (4.27) we obtain

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_m, u_n) = 0\Big\} \quad \text{and} \quad \forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha(v_m, u_m) = 0\Big\}.$$

By Definition 2.1 this gives $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} p_\alpha(v_m, u_m) = 0\}$, which means that

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} p_\alpha(w, w^m) = 0\Big\}.$$

Therefore, $LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

Step 7. Conclusions (B1) *and* (B2) *hold.*

Let $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$ be arbitrary and fixed. Define the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$.

First, we prove that $Fix(T^{[q]}) \neq \varnothing$ and that there exists a point $w \in Fix(T^{[p]})$ such that this sequence is left $\mathcal{P}_{C;\mathcal{A}}$-convergent to w. Indeed, by statement (A), $LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$. Next, for this sequence, we have $w^{mq+k} = T^{[q]}(w^{(m-1)q+k})$ for $k = 1, 2, \ldots, q$ and $m \in \mathbb{N}$. Let, in the sequel, $k = 1, 2, \ldots, q$ be arbitrary and fixed. Defining $(z_m = w^{m-1+q} : m \in \mathbb{N})$, we see that $\varnothing \neq LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(z_m:m\in\{0\}\cup\mathbb{N})}$ and that the sequences $(v_m = w^{mq+k} : m \in \mathbb{N})$ and $(u_m = w^{(m-1)q+k} : m \in \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{v_m = T^{[q]}(u_m)\}$ and, as subsequences of $(w^m : m \in \{0\} \cup \mathbb{N})$, are left $\mathcal{P}_{C;\mathcal{A}}$-convergent to each point of $w \in LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Clearly, by Remark 2.2, $LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(z_m:m\in\mathbb{N})} \subset LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(v_m:m\in\mathbb{N})}$ and $LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(z_m:m\in\mathbb{N})} \subset LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})}$. Therefore, since $T^{[q]}$ is left $\mathcal{P}_{C;\mathcal{A}}$-closed on $W^{L-\mathcal{J}_{C;\mathcal{A}}}$, in virtue of Definition 3.2, we get $\exists_{w\in LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}=LIM^{L-\mathcal{P}_{C;\mathcal{A}}}_{(z_m:m\in\mathbb{N})}}\{w = T^{[q]}(w)\}$. Consequently, $Fix(T^{[q]}) \neq \varnothing$, and there exists a point $w \in Fix(T^{[q]})$ such that the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left $\mathcal{P}_{C;\mathcal{A}}$-convergent to w, so (B1) and (B2) hold.

Step 8. Conclusion (B3) *holds.*

Suppose that

$$\exists_{\alpha_0\in\mathcal{A}}\exists_{w\in Fix(T^{[q]})}\left\{J_{\alpha_0}\left(w, T(w)\right) > 0 \vee J_{\alpha_0}\left(T(w), w\right) > 0\right\}.$$

If $J_{\alpha_0}(w, T(w)) > 0$, then, putting

$$\varepsilon_0 = J_{\alpha_0}\left(w, T(w)\right), \tag{4.28}$$

by (3.5) we get

$$\begin{aligned}&\exists_{\eta_0>0}\exists_{r_0\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\left\{J_{\alpha_0}\left(T^{[s]}(w), T^{[l]}(w)\right) < \varepsilon_0 + \eta_0\right.\\ &\quad\left.\Rightarrow C_{\alpha_0}J_{\alpha_0}\left(T^{[s+r_0]}(w), T^{[l+r_0]}(w)\right) < \varepsilon_0\right\}.\end{aligned} \tag{4.29}$$

Since

$$\forall_{w\in Fix(T^{[q]})}\forall_{m\in\mathbb{N}}\left\{T^{[mq]}(w) = w\right\}, \tag{4.30}$$

we have

$$J_{\alpha_0}\left(T^{[q]}(w), T^{[q+1]}(w)\right) = J_{\alpha_0}\left(w, T(w)\right) = \varepsilon_0 < \varepsilon_0 + \eta_0.$$

Hence, using (4.29) for $s = q$ and $l = q + 1$, we get

$$C_{\alpha_0}J_{\alpha_0}\left(T^{[q+r_0]}(w), T^{[q+1+r_0]}(w)\right) < \varepsilon_0 < \varepsilon_0 + \eta_0.$$

We note that then, in particular, since $C_{\alpha_0} \geq 1$,

$$J_{\alpha_0}\left(T^{[q+r_0]}(w), T^{[q+1+r_0]}(w)\right) < \varepsilon_0 < \varepsilon_0 + \eta_0. \tag{4.31}$$

Next, by (4.31), using (4.29) for $s = q + r_0$ and $l = q + 1 + r_0$, we have

$$C_{\alpha_0}J_{\alpha_0}\left(T^{[q+2r_0]}(w), T^{[q+1+2r_0]}(w)\right) < \varepsilon_0 < \varepsilon_0 + \eta_0.$$

Hence, since $C_{\alpha_0} \geq 1$,

$$J_{\alpha_0}\bigl(T^{[q+2r_0]}(w), T^{[q+1+2r_0]}(w)\bigr) < \varepsilon_0 < \varepsilon_0 + \eta_0. \tag{4.32}$$

Using (4.31) and (4.32), by induction we have that (4.28) gives

$$\begin{aligned}&\forall_{m\in\mathbb{N}}\bigl\{J_{\alpha_0}\bigl(T^{[q+mr_0]}(w), T^{[q+1+mr_0]}(w)\bigr)\\ &\quad\leq C_{\alpha_0}J_{\alpha_0}\bigl(T^{[q+mr_0]}(w), T^{[q+1+mr_0]}(w)\bigr) < \varepsilon_0 < \varepsilon_0 + \eta_0\bigr\}.\end{aligned} \tag{4.33}$$

Putting $m = q$ in (4.33), we find

$$J_{\alpha_0}\bigl(T^{[q+qr_0]}(w), T^{[q+1+qr_0]}(w)\bigr) < \varepsilon_0 < \varepsilon_0 + \eta_0. \tag{4.34}$$

Finally, by (4.28), (4.31), and (4.34) we obtain that

$$\varepsilon_0 = J_{\alpha_0}\bigl(w, T(w)\bigr) = J_{\alpha_0}\bigl(T^{[q+qr_0]}(w), T^{[q+1+qr_0]}(w)\bigr) < \varepsilon_0,$$

a contradiction. Therefore, $J_{\alpha_0}(w, T(w)) = 0$.

Similarly, we prove that $J_{\alpha_0}(T(w), w) = 0$. We proved that (B3) holds.

Step 9. Conclusion (C1) *holds.*

First, we show that $Fix(T^{[q]}) = Fix(T) \neq \varnothing$. Indeed, let $w \in Fix(T^{[q]})$. Then, by (B3), $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, T(w)) = J_\alpha(T(w), w) = 0\}$. By Theorem 2.2 this gives $w = T(w)$, i.e. $w \in Fix(T)$. Consequently, $Fix(T^{[q]}) = Fix(T) \neq \varnothing$.

Next, we show that $Fix(T) = \{w\}$ for some $w \in X$. Otherwise, $u, v \in Fix(T)$ and $u \neq v$ for some $u, v \in X$; recall that, by the preceding, $Fix(T) \neq \varnothing$. Then, by Theorem 2.2 there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(u, v) > 0$ or $J_{\alpha_0}(v, u) > 0$. Suppose $J_{\alpha_0}(u, v) > 0$. Then, for $\varepsilon_0 = J_{\alpha_0}(u, v) > 0$, by (3.5) there exist $\eta_0 > 0$ and $r_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\forall_{s,l\in\mathbb{N}}\bigl\{\bigl\{J_{\alpha_0}\bigl(T^{[s]}(u), T^{[l]}(v)\bigr) < \varepsilon_0 + \eta_0\bigr\}\\ &\quad\Rightarrow \bigl\{C_{\alpha_0}J_{\alpha_0}\bigl(T^{[s+r_0]}(u), T^{[l+r_0]}(v)\bigr) < \varepsilon_0\bigr\}\bigr\}.\end{aligned} \tag{4.35}$$

However, for all $s, l \in \mathbb{N}$, we have $J_{\alpha_0}(T^{[s]}(u), T^{[l]}(v)) = J_{\alpha_0}(u, v) = \varepsilon_0 < \varepsilon_0 + \eta_0$, and thus by (4.35) we get $0 < \varepsilon_0 = J_{\alpha_0}(u, v) = J_{\alpha_0}(T^{[s+r_0]}(u), T^{[l+r_0]}(v)) < C_{\alpha_0}J_{\alpha_0}(T^{[s+r_0]}(u), T^{[l+r_0]}(v)) < \varepsilon_0$, which is impossible. We obtain a similar implication in the case where $J_{\alpha_0}(v, u) > 0$. Therefore, $Fix(T) = \{w\}$ for some $w \in X$, so (C1) holds.

Step 10. Conclusion (C2) *holds.*

Step 9 with (B) means that, for each $w^0 \in W^{L-\mathcal{J}_{C;\mathcal{A}}}$, the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left $\mathcal{P}_{C;\mathcal{A}}$-convergent to w. Thus, (C2) holds.

Step 11. Conclusion (C3) *holds.*

Finally, we show that $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, w) = 0\}$, where $Fix(T) = \{w\}$. Indeed, if we assume that there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(w, w) > 0$, then, denoting $\varepsilon_0 = J_{\alpha_0}(w, w) > 0$, by (3.5) there exist $\eta_0 > 0$ and $r_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\forall_{s,l\in\mathbb{N}}\bigl\{\bigl\{J_{\alpha_0}\bigl(T^{[s]}(w), T^{[l]}(w)\bigr) < \varepsilon_0 + \eta_0\bigr\}\\ &\quad\Rightarrow \bigl\{C_{\alpha_0}J_{\alpha_0}\bigl(T^{[s+r_0]}(w), T^{[l+r_0]}(w)\bigr) < \varepsilon_0\bigr\}\bigr\}.\end{aligned} \tag{4.36}$$

However, for all $s, l \in \mathbb{N}$, we have $J_{\alpha_0}(T^{[s]}(w), T^{[l]}(w)) = J_{\alpha_0}(w, w) = \varepsilon_0 < \varepsilon_0 + \eta_0$. Thus, using (4.36), we obtain that

$$\begin{aligned} 0 < \varepsilon_0 = J_{\alpha_0}(w, w) &= J_{\alpha_0}\bigl(T^{[s+r_0]}(w), T^{[l+r_0]}(w)\bigr) \\ &< C_{\alpha_0} J_{\alpha_0}\bigl(T^{[s+r_0]}(w), T^{[l+r_0]}(w)\bigr) < \varepsilon_0, \end{aligned}$$

which is impossible. Therefore, (C3) holds.

The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.2 Assume that condition (3.6) holds. Then, defining $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$, where $w^0 \in X$ is as in (3.6), and next, using a similar argument as in the proof of Theorem 3.1 for this sequence, we have the assertions. □

5 Examples

Example 5.1 Let $X = (0; 3)$. For $A = (1/2; 2]$ and $\gamma > 0$, we let $p : X^2 \to [0; \infty)$ be a distance of the form

$$p(u, v) = \begin{cases} 0 & \text{if } A \cap \{u, v\} = \{u, v\}, \\ \gamma & \text{if } A \cap \{u, v\} \neq \{u, v\}. \end{cases} \tag{5.1}$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} 1 + x & \text{if } x \in (0; 1], \\ x/2 & \text{if } x \in (1; 2], \\ (1 + x)/2 & \text{if } x \in (2; 3). \end{cases} \tag{5.2}$$

(1) *Notice that* $(X, \mathcal{P}_{\{1\};\{1\}})$, $\mathcal{P}_{\{1\};\{1\}} = \{p\}$, *is a triangular space*. See Definition 1.2(E) and [27], Example 2, p.11; p does not vanish on the diagonal, is symmetric, and is triangular.

(2) *We show that* (X, T) *is* $\mathcal{P}_{\{1\};\{1\}}$*-contraction on* X, *that is*,

$$\forall_{x,y \in X} \forall_{\varepsilon > 0} \exists_{\eta > 0} \exists_{r \in \mathbb{N}} \forall_{s,l \in \mathbb{N}} \bigl\{ p\bigl(T^{[s]}(x), T^{[l]}(y)\bigr) < \varepsilon + \eta \Rightarrow p\bigl(T^{[s+r]}(x), T^{[l+r]}(y)\bigr) < \varepsilon \bigr\}. \tag{5.3}$$

First, we claim that

$$\forall_{m \in \mathbb{N}} \bigl\{ T^{[m]}(X) \subset A \bigr\}. \tag{5.4}$$

Indeed, observing that

$$T^{[2k+1]}(x) = \begin{cases} 2 - (1 - x)/2^k & \text{if } x \in (0; 1], \\ 1 - (2 - x)/2^{k+1} & \text{if } x \in (1; 2], \\ 2 - (3 - x)/2^{k+1} & \text{if } x \in (2; 3), \end{cases} \quad k \in \{0\} \cup \mathbb{N}, \tag{5.5}$$

and

$$T^{[2k]}(x) = \begin{cases} 1 - (1 - x)/2^k & \text{if } x \in (0; 1], \\ 2 - (2 - x)/2^k & \text{if } x \in (1; 2], \\ 1 - (3 - x)/2^{k+1} & \text{if } x \in (2; 3), \end{cases} \quad k \in \mathbb{N}, \tag{5.6}$$

we see that (5.5) and (5.6) imply (5.4). Now, using (5.4) and (5.1), we obtain

$$\forall_{x,y\in X}\forall_{m,n\in\mathbb{N}}\bigl\{p\bigl(T^{[m]}(x),T^{[n]}(y)\bigr)=0\bigr\}. \tag{5.7}$$

In view of (5.7), we conclude that (5.3) holds.

(3) *We show that* (X,T) *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-admissible in each point* $w^0\in X$; *thus, by Definition* 3.1, $W^{L-\mathcal{P}_{\{1\};\{1\}}}=W^{R-\mathcal{P}_{\{1\};\{1\}}}=X$. Indeed, it is clear that, for arbitrary and fixed $w^0\in X$, the sequence $(w^m:m\in\{0\}\cup\mathbb{N})$, where $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1}=T(w^m)\}$ satisfies

$$\forall_{m\in\mathbb{N}}\bigl\{w^m\in A\bigr\}. \tag{5.8}$$

Then, in view of (5.8) and (5.1),

$$\lim_{m\to\infty}\sup_{n>m}p\bigl(w^m,w^n\bigr)=\lim_{m\to\infty}\sup_{n>m}p\bigl(w^n,w^m\bigr)=0,$$

and also

$$\forall_{w\in A}\Bigl\{\lim_{m\to\infty}p\bigl(w,w^m\bigr)=\lim_{m\to\infty}p\bigl(w^m,w\bigr)=0\Bigr\}.$$

Thus, $LIM^{L-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}=LIM^{R-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}=A\neq\emptyset$.

(4) *We show that the single-valued dynamic system* $(X,T^{[2]})$ *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-closed on* $W^{L-\mathcal{P}_{\{1\};\{1\}}}=W^{R-\mathcal{P}_{\{1\};\{1\}}}=X$. Indeed, if $w^0\in X$ is arbitrary and fixed and if $(w^m=T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})$ is a left or right $\mathcal{P}_{\{1\};\{1\}}$-converging sequence in X having subsequences $(v_m:m\in\mathbb{N})$ and $(u_m:m\in\mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m=T^{[2]}(u_m)\}$, then by (5.4), (5.1), and (5.6) we have that $A=LIM^{L-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}=LIM^{R-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $\{1=T^{[2]}(1),2=T^{[2]}(2)\}\subset A$. Hence, in virtue of Definition 3.2, $U=V=W^{L-\mathcal{P}_{\{1\};\{1\}}}=W^{R-\mathcal{P}_{\{1\};\{1\}}}=X$.

(5) $\mathcal{P}_{\{1\};\{1\}}=\{p\}$ *is not separating on* X. This follows from Definition 1.2(B) since, for each $x,y\in X$ such that $A\cap\{x,y\}\neq\{x,y\}$, we have $p(x,y)=p(y,x)=\gamma>0$.

Claim *It follows from* (1)-(5) *that, for* $(X,\mathcal{P}_{\{1\};\{1\}})$, $\mathcal{P}_{\{1\};\{1\}}=\{p\}$, (X,T), *and* $\mathcal{J}_{\{1\};\{1\}}=\mathcal{P}_{\{1\};\{1\}}$ *defined by* (5.1) *and* (5.2) *and for* $q=2$, *statements* (A) *and* (B) *of Theorem* 3.1 *hold*: (a) *Statement* (A) *holds since, for each* $w^0\in X$, *the sequence* $(w^m=T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})$ *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-convergent to each point* $w\in A$; $W^{L-\mathcal{P}_{\{1\};\{1\}}}=W^{R-\mathcal{P}_{\{1\};\{1\}}}=X$ *by* (3). (b) *We have that* $Fix(T^{[2]})=\{1,2\}$; *thus, conclusion* (B1) *holds*. (c) *Conclusion* (B2) *follows from statement* (A) *and conclusion* (B1) *since* $Fix(T^{[2]})\subset A$. (d) *Conclusion* (B3) *holds*; *by* (5.1), *we have* $\forall_{v\in Fix(T^{[2]})=\{1,2\}}\{p(v,T(v))=p(T(v),v)=0\}$ *since* $T(1)=2$, $T(2)=1$, *and thus* $T(\{1,2\})=\{1,2\}=Fix(T^{[2]})\subset A$. (e) *By* (5), $(X,\mathcal{P}_{\{1\};\{1\}})$ *is not separable*. (f) *We see that* $Fix(T)=\emptyset$ *and that statement* (C) *does not hold*.

Example 5.2 Let $X=(0;1)\cup(1;2]$, and let $p:X^2\to[0;\infty)$ be of the form

$$p(u,v)=\begin{cases}0 & \text{if } u\geq v,\\ (v-u)^4 & \text{if } u<v,\end{cases}\qquad u,v\in X. \tag{5.9}$$

Define $T:X\to X$ by

$$T(x)=\begin{cases}1+x^2 & \text{if } x\in(0;1),\\ 1+x/2 & \text{if } x\in(1;2].\end{cases} \tag{5.10}$$

(1) *Notice that* $(X, \mathcal{P}_{\{8\};\{1\}})$, $\mathcal{P}_{\{8\};\{1\}} = \{p\}$, *is a quasi-triangular space*. See Definition 1.2(A) and [27], Example 1, p.10; p vanishes on the diagonal, is asymmetric, and is quasi-triangular since $\forall_{u,v,w\in X}\{p(u,w) \leq 8[p(u,v) + p(v,w)]\}$.

(2) *We show that* (X, T) *is* $\mathcal{P}_{\{8\};\{1\}}$*-contraction, that is,*

$$\begin{aligned}&\forall_{x,y\in X}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{p\bigl(T^{[s]}(x), T^{[l]}(y)\bigr) < \varepsilon + \eta \\ &\quad \Rightarrow 8 \cdot p\bigl(T^{[s+r]}(x), T^{[l+r]}(y)\bigr) < \varepsilon\bigr\}. \end{aligned} \tag{5.11}$$

Indeed, by (5.10),

$$T^{[m]}(x) = \begin{cases} 2 - (1 - x^2)/2^{m-1} & \text{if } x \in (0;1), \\ 2 - (2 - x)/2^m & \text{if } x \in (1;2], \end{cases} \qquad m \in \mathbb{N}. \tag{5.12}$$

Next, in the sequel, let $\varepsilon > 0$ be arbitrary and fixed, and let $\eta = \varepsilon$. Without loss of generality, to prove (5.11), it suffices to prove that if $x, y \in X$ and $s, l \in \mathbb{N}$ are such that $W_{x,y}^{s,l} < 2\varepsilon$, then there exists $r \in \mathbb{N}$ such that $8 \cdot W_{x,y}^{s+r,l+r} < \varepsilon$, where

$$\forall_{x,y\in}\forall_{s,l\in\mathbb{N}}\bigl\{W_{x,y}^{s,l} = p\bigl(T^{[s]}(x), T^{[l]}(y)\bigr)\bigr\}. \tag{5.13}$$

With this aim, we consider the following cases:

Case A. Let $x, y \in (0;1)$.

(A1) If $s, l, r \in \mathbb{N}$ and $T^{[s]}(x) = 2 - (1 - x^2)/2^{s-1} \geq T^{[l]}(y) = 2 - (1 - y^2)/2^{l-1}$, then $T^{[s+r]}(x) = 2 - (1 - x^2)/2^{s+r-1} \geq T^{[l+r]}(y) = 2 - (1 - y^2)/2^{l+r-1}$, and by (5.9) and (5.13) we get that $W_{x,y}^{s,l} = W_{x,y}^{s+r,l+r} = 0$, that is, (5.11) holds.

(A2) If $s, l, r \in \mathbb{N}$ and $T^{[s]}(x) = 2 - (1 - x^2)/2^{s-1} < T^{[l]}(y) = 2 - (1 - y^2)/2^{l-1}$, then $T^{[s+r]}(x) = 2 - (1 - x^2)/2^{s+r-1} < T^{[l+r]}(y) = 2 - (1 - y^2)/2^{l+r-1}$, and by (5.9) and (5.13) we get that

$$W_{x,y}^{s,l} = \bigl[\bigl(1 - y^2\bigr)/2^{l-1} - \bigl(1 - x^2\bigr)/2^{s-1}\bigr]^4 < 2\varepsilon$$

whenever s and l are sufficiently great, and, for such s and l,

$$8 \cdot W_{x,y}^{s+r,l+r} = \bigl(8/2^{4r}\bigr)\bigl[\bigl(1 - y^2\bigr)/2^{l-1} - \bigl(1 - x^2\bigr)/2^{s-1}\bigr]^4 < \varepsilon$$

whenever also r is sufficiently great. Thus, (5.11) holds.

Case B. Let $x, y \in (1;2]$.

(B1) If $s, l, r \in \mathbb{N}$ and $T^{[s]}(x) = 2 - (2 - x)/2^s \geq T^{[l]}(y) = 2 - (2 - y)/2^l$, then $T^{[s+r]}(x) = 2 - (2 - x)/2^{s+r} \geq T^{[l+r]}(y) = 2 - (2 - y)/2^{l+r}$, and by (5.9) and (5.13) we get that $W_{x,y}^{s,l} = W_{x,y}^{s+r,l+r} = 0$, that is, (5.11) holds.

(B2) If $s, l, r \in \mathbb{N}$ and $T^{[s]}(x) = 2 - (2 - x)/2^s < T^{[l]}(y) = 2 - (2 - y)/2^l$, then $T^{[s+r]}(x) = 2 - (2 - x)/2^{s+r} < T^{[l+r]}(y) = 2 - (2 - y)/2^{l+r}$, and by (5.9) and (5.13) we get that

$$W_{x,y}^{s,l} = \bigl[(2 - y)/2^l - (2 - x)/2^s\bigr]^4 < 2\varepsilon$$

whenever s and l are sufficiently great, and, for such s and l,

$$8 \cdot W_{x,y}^{s+r,l+r} = \bigl(8/2^{4r}\bigr)\bigl[(2 - y)/2^l - (2 - x)/2^s\bigr]^4 < \varepsilon$$

whenever also r is sufficiently great. Thus, (5.11) holds.

Case C. Let $x \in (0;1)$ and $y \in (1;2]$.

(C1) If $s, l, r \in \mathbb{N}$ and $T^{[s]}(x) = 2 - (1-x^2)/2^{s-1} \geq T^{[l]}(y) = 2 - (2-y)/2^l$, then $T^{[s+r]}(x) = 2 - (1-x^2)/2^{s+r-1} \geq T^{[l+r]}(y) = 2 - (2-y)/2^{l+r}$, and by (5.9) and (5.13) we get that $W_{x,y}^{s,l} = W_{x,y}^{s+r,l+r} = 0$, that is, (5.11) holds.

(C2) If $s, l, r \in \mathbb{N}$ and $T^{[s]}(x) = 2 - (1-x^2)/2^{s-1} < T^{[l]}(y) = 2 - (2-y)/2^l$, then $T^{[s+r]}(x) = 2 - (1-x^2)/2^{s+r-1} < T^{[l+r]}(y) = 2 - (2-y)/2^{l+r}$, and by (5.9) and (5.13) we get that

$$W_{x,y}^{s,l} = \left[(2-y)/2^l - \left(1-x^2\right)/2^{s-1}\right]^4 < 2\varepsilon$$

whenever s and l are sufficiently great, and, for such s and l,

$$8 \cdot W_{x,y}^{s+r,l+r} = \left(8/2^{4r}\right)\left[(2-y)/2^l - \left(1-x^2\right)/2^{s-1}\right]^4 < \varepsilon$$

whenever also r is sufficiently great. Thus, (5.11) holds.

Case D. Let $x \in (1;2]$ and $y \in (0;1)$. Then, with analogous consideration as in Case C, we obtain that (5.11) holds.

(3) *We show that* (X, T) *is left and right* $\mathcal{P}_{\{8\};\{1\}}$*-admissible in each point* $w^0 \in X$; *thus, by Definition* 3.1, $W^{L-\mathcal{P}_{\{8\};\{1\}}} = W^{R-\mathcal{P}_{\{8\};\{1\}}} = X$. Indeed, let $w^0 \in X$ be arbitrary and fixed, and let the sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ be defined by $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} = T(w^m)\}$. We consider the cases:

Case A. Let $w^0 \in (0;1)$. Then $w^m = 2 - [1-(w^0)^2]/2^{m-1} < w^n = 2 - [1-(w^0)^2]/2^{n-1}$ for $n > m$. Using (5.9), we then have $p(w^m, w^n) = [1-(w^0)^2]^4(1/2^{4(m-1)})(1-1/2^{n-m})^4$ and $p(w^n, w^m) = 0$. Hence, we conclude that

$$\begin{aligned}\lim_{m\to\infty}\sup_{n>m} p\left(w^m, w^n\right) &= \left[1-\left(w^0\right)^2\right]^4 \lim_{m\to\infty}\sup_{n>m}\left(1/2^{4(m-1)}\right)\left(1-1/2^{n-m}\right)^4 \\ &= \left[1-\left(w^0\right)^2\right]^4 \lim_{m\to\infty}\left(1/2^{4(m-1)}\right) = 0\end{aligned}$$

and $\lim_{m\to\infty}\sup_{n>m} p(w^n, w^m) = 0$, respectively.

Next, observe that

$$\{2\} \subset LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{L-\mathcal{P}_{\{8\};\{1\}}} \quad \text{and} \quad (0;1)\cup(1;2) \subset LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{R-\mathcal{P}_{\{8\};\{1\}}};$$

thus, $LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{L-\mathcal{P}_{\{8\};\{1\}}} \neq \varnothing$ and $LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{R-\mathcal{P}_{\{8\};\{1\}}} \neq \varnothing$. In fact, by (5.9) we have

$$\lim_{m\to\infty} p(2, w^m) = \lim_{m\to\infty}\left[1-(w^0)^2\right]^4/2^{4(m-1)} = 0.$$

Moreover, by (5.9), $\forall_{w\in(0;1)\cup(1;2)}\{\lim_{m\to\infty} p(w^m, w) = 0\}$.

Case B. Let $w^0 \in (1;2]$. Then $w^m = 2 - (2-w^0)/2^m < w^n = 2 - [2-w^0]/2^n$ for $n > m$, and thus, by (5.9), $p(w^m, w^n) = [2-w^0]^4(1/2^{4m})(1-1/2^{n-m})^4$, $p(w^n, w^m) = 0$. Consequently, we get

$$\begin{aligned}&\lim_{m\to\infty}\sup_{n>m} p\left(w^m, w^n\right) = \left[2-w^0\right]^4 \lim_{m\to\infty}\sup_{n>m}\left(1/2^{4m)}\right)\left(1-1/2^{n-m}\right)^4 = 0, \\ &\lim_{m\to\infty}\sup_{n>m} p\left(w^n, w^m\right) = 0.\end{aligned}$$

In addition, we see that $\lim_{m\to\infty} p(2, w^m) = \lim_{m\to\infty}[2 - w^0]^4/2^{4m} = 0$ and

$$\forall_{w\in(0;1)\cup(1;2)}\left\{\lim_{m\to\infty} p(w^m, w) = 0\right\}.$$

Thus, $\{2\} \subset LIM^{L-\mathcal{P}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and $(0;1) \cup (1;2) \subset LIM^{R-\mathcal{P}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(4) *We show that* (X, T) *is left* $\mathcal{P}_{\{8\};\{1\}}$*-closed on* $X = W^{L-\mathcal{P}_{\{8\};\{1\}}}$. Indeed, by Definition 3.2, if $w^0 \in X$ is arbitrary and fixed and if $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is a left $\mathcal{P}_{\{8\};\{1\}}$-converging sequence in X having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T(u_m)\}$, then by (5.9) (5.10), and (5.12) we have that $2 \in LIM^{L-\mathcal{P}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $2 = T(2)$. Therefore, $U = W^{L-\mathcal{P}_{\{8\};\{1\}}} = X$. Clearly, $V = \emptyset$.

(5) $\mathcal{P}_{\{8\};\{1\}} = \{p\}$ *is separating on* X. Indeed, for each $x, y \in X$ such that $x \neq y$, we have $p(x, y) > 0$ or $p(y, x) > 0$. Thus, (1.4) holds.

Claim *It follows from* (1)-(5) *that, for* $(X, \mathcal{P}_{\{8\};\{1\}})$, $\mathcal{P}_{\{8\};\{1\}} = \{p\}$, (X, T), *and* $\mathcal{J}_{\{8\};\{1\}} = \mathcal{P}_{\{8\};\{1\}}$ *defined by* (5.9) *and* (5.10) *and for* $q = 1$, *statements* (A), (B), *and* (C) *of Theorem* 3.1 *in the left case hold*: (a) *We have* $Fix(T) = \{2\}$. (b) *Conclusions* (B3) *and* (C3) *hold since by* (5.9) *we have* $p(2, T(2)) = p(T(2), 2) = p(2, 2) = 0$. (c) *For each* $w^0 \in X = W^{L-\mathcal{P}_{\{8\};\{1\}}}$, *the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* $\mathcal{P}_{\{8\};\{1\}}$*-convergent to* 2.

Example 5.3 Let $X = (0; 3]$, and let $p : X^2 \to [0; \infty)$ be of the form

$$p(u, v) = \begin{cases} 0 & \text{if } u \geq v, \\ (v - u)^4 & \text{if } u < v, \end{cases} \quad u, v \in X. \tag{5.14}$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} (1 + x)/2 & \text{if } x \in (0; 1], \\ (2 + x)/2 & \text{if } x \in (1; 2], \\ (3 + x)/2 & \text{if } x \in (2; 3]. \end{cases} \tag{5.15}$$

(1) *Observe that* $(X, \mathcal{P}_{\{8\};\{1\}})$, $\mathcal{P}_{\{8\};\{1\}} = \{p\}$, *is a separable quasi-triangular space.*

(2) *When* $\mathcal{J}_{\{8\};\{1\}} = \mathcal{P}_{\{8\};\{1\}}$, *we will show that:* (a) $Fix(T) = \{1, 2, 3\}$, (b) *for each* $w^0 \in X$, *there exists* $w \in Fix(T)$ *such that the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* $\mathcal{P}_{\{8\};\{1\}}$*-convergent to* w, *and* (c) $\forall_{w\in Fix(T)}\{p(w, w) = 0\}$.

To prove this, we see that (5.15) implies

$$T^{[m]}(x) = \begin{cases} T_1^{[m]}(x) = 1 - (1 - x)/2^m & \text{if } x \in X_1 = (0; 1], \\ T_2^{[m]}(x) = 2 - (2 - x)/2^m & \text{if } x \in X_2 = (1; 2], \\ T_3^{[m]}(x) = 3 - (3 - x)/2^m & \text{if } x \in X_3 = (2; 3], \end{cases} \quad m \in \mathbb{N}. \tag{5.16}$$

This means that, for each $k = \{1, 2, 3\}$, $T_k : X_k \to X_k$ where $X_k = (k - 1; k]$, and $(X_k, \mathcal{P}_{\{8\};\{1\}})$ is a separable quasi-triangular space.

It is not hard to show that, for each $k = \{1, 2, 3\}$, we have: (a) (X_k, T_k) is a $\mathcal{P}_{\{8\};\{1\}}$-contraction on X^k. (b) (X_k, T_k) is left and right $\mathcal{P}_{\{8\};\{1\}}$-admissible on $X_k = W^{L-\mathcal{P}_{\{8\};\{1\}}} = W^{R-\mathcal{P}_{\{8\};\{1\}}}$; if $w^0 \in X_k$ is arbitrary and fixed, then the sequence $(w^m = T_k^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfies $k \in LIM^{L-\mathcal{P}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $(k - 1; k) \subset LIM^{R-\mathcal{P}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. (c) (X_k, T_k) is left $\mathcal{P}_{\{8\};\{1\}}$-closed on $X_k = W^{L-\mathcal{P}_{\{8\};\{1\}}}$; indeed, if $w^0 \in X_k$ is arbitrary and fixed and if $(w^m =$

$T_k^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is a left $\mathcal{P}_{\{8\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T_k(u_m)\}$, then by (5.14)-(5.16) we have that $k \in LIM^{L-\mathcal{P}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $k = T_k(k)$.

Claim *For each* $k = \{1,2,3\}$, *all assumptions of Theorem* 3.1 *in the left case hold, and we see that*: (a) $Fix(T_k) = \{k\}$. (b) *For each* $w^0 \in X_k$, *the sequence* $(w^m = T_k^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* $\mathcal{P}_{\{8\};\{1\}}$*-convergent to* k. (c) $p(k,k) = 0$.

Example 5.4 Let $X = (0;1) \cup (1;2)$. For $\gamma > 0$ and $A = \{1/2, 3/2\}$, set

$$p(u,v) = \begin{cases} 0 & \text{if } A \cap \{u,v\} = \{u,v\}, \\ \gamma & \text{if } A \cap \{u,v\} \neq \{u,v\}, \end{cases} \quad u,v \in X. \tag{5.17}$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} 1+x & \text{if } x \in (0;1), \\ 2-x & \text{if } x \in (1;2). \end{cases} \tag{5.18}$$

(1) *Observe that* $(X, \mathcal{P}_{\{1\};\{1\}})$, $\mathcal{P}_{\{1\};\{1\}} = \{p\}$, *is a triangular space.*
(2) (X,T) *is a weak* $\mathcal{P}_{\{1\};\{1\}}$*-contraction on* X. *More precisely, for each* $w^0 \in A$,

$$\begin{aligned} &\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{p\bigl(T^{[s]}\bigl(w^0\bigr), T^{[l]}\bigl(w^0\bigr)\bigr) < \varepsilon + \eta \\ &\quad \Rightarrow p\bigl(T^{[s+r]}\bigl(w^0\bigr), T^{[l+r]}\bigl(w^0\bigr)\bigr) < \varepsilon\bigr\}. \end{aligned} \tag{5.19}$$

In fact, by (5.18) we calculate that, for each $m \in \{0\} \cup \mathbb{N}$,

$$T^{[4m+1]}(x) = \begin{cases} 1+x & \text{if } x \in (0;1), \\ 2-x & \text{if } x \in (1;2), \end{cases} \tag{5.20}$$

$$T^{[4m+2]}(x) = \begin{cases} 1-x & \text{if } x \in (0;1), \\ 3-x & \text{if } x \in (1;2), \end{cases} \tag{5.21}$$

$$T^{[4m+3]}(x) = \begin{cases} 2-x & \text{if } x \in (0;1), \\ x-1 & \text{if } x \in (1;2), \end{cases} \tag{5.22}$$

$$T^{[4m+4]}(x) = \begin{cases} x & \text{if } x \in (0;1), \\ x & \text{if } x \in (1;2). \end{cases} \tag{5.23}$$

Putting $m \in \{0\} \cup \mathbb{N}$, this becomes

$$\begin{aligned} &Fix\bigl(T^{[4m+1]}\bigr) = Fix\bigl(T^{[4m+3]}\bigr) = \varnothing, \\ &Fix\bigl(T^{[4m+2]}\bigr) = \{1/2, 3/2\}, \qquad Fix\bigl(T^{[4m+4]}\bigr) = X, \qquad T^{[4m+4]} = I_X. \end{aligned} \tag{5.24}$$

Also, from (5.17) and (5.20)-(5.23) we conclude that

$$\forall_{w^0\in A=\{1/2,3/2\}}\forall_{m\in\mathbb{N}}\bigl\{T^{[m]}\bigl(w^0\bigr) \in A\bigr\}. \tag{5.25}$$

Hence, $\forall_{w^0\in A}\forall_{s,l\in\mathbb{N}}\{p(T^{[s]}(w^0), T^{[l]}(w^0)) = 0\}$. Consequently, (5.19) holds.

(3) (X,T) *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-admissible in each point* $w^0 \in A$; *thus,* $W^{L-\mathcal{P}_{\{1\};\{1\}}} = W^{R-\mathcal{P}_{\{1\};\{1\}}} = A$. To verify this, we take any $w^0 \in A$ and define $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$. Then, by (5.17) and (5.25),

$$\lim_{m\to\infty} \sup_{n>m} p\big(w^m, w^n\big) = 0\}, \qquad \lim_{m\to\infty} \sup_{n>m} p\big(w^n, w^m\big) = 0,$$
$$A = LIM^{L-\mathcal{P}_{\{1\};\{1\}}}_{(w^m : m\in\{0\}\cup\mathbb{N})} = \Big\{x \in X : \forall_{\alpha\in\mathcal{A}} \Big\{\lim_{m\to\infty} p\big(x, w^m\big) = 0\Big\}\Big\},$$
$$A = LIM^{R-\mathcal{P}_{\{1\};\{1\}}}_{(w^m : m\in\{0\}\cup\mathbb{N})} = \Big\{x \in X : \forall_{\alpha\in\mathcal{A}} \Big\{\lim_{m\to\infty} p\big(w^m, x\big) = 0\Big\}\Big\}.$$

(4) *We show that* $(X, T^{[2]})$ *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-closed on* $A = W^{L-\mathcal{P}_{\{1\};\{1\}}} = W^{R-\mathcal{P}_{\{1\};\{1\}}}$. Indeed, if $w^0 \in A$, then by (5.25) and (5.17) the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is contained in A, is left and right $\mathcal{P}_{\{1\};\{1\}}$-convergent in X to each point of A, has subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[2]}(u_m)\}$, and $LIM^{L-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = LIM^{R-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = A = Fix(T^{[2]})$ holds.

(5) *We show that the single-valued dynamic system* $(X, T^{[4]})$ *is also left and right* $\mathcal{P}_{\{1\};\{1\}}$*-closed on* $W^{L-\mathcal{P}_{\{1\};\{1\}}} = W^{R-\mathcal{P}_{\{1\};\{1\}}} = A$. Indeed, if $w^0 \in A$ is arbitrary and fixed and if $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is a left or right $\mathcal{P}_{\{1\};\{1\}}$-converging sequence in X having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[4]}(u_m)\}$, then by (5.4), (5.1), (5.6), and (5.24) we have that $A = LIM^{L-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = LIM^{R-\mathcal{P}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $A \subset X = Fix(T^{[4]})$. Hence, by virtue of Definition 3.2, $U = V = W^{L-\mathcal{P}_{\{1\};\{1\}}} = W^{R-\mathcal{P}_{\{1\};\{1\}}} = A$.

Claim I *By* (1)-(4) *it follows that we may use Theorem* 3.2 *in the left and right cases* (*when* $\mathcal{J}_{\{1\};\{1\}} = \mathcal{P}_{\{1\};\{1\}}$), *and we see that statements* (A) *and* (B) *of this theorem hold. We have*: (a) *For each* $w^0 \in A$ *and for each* $w \in A$, *the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-convergent to* w, *and thus statement* (A) *holds*. (b) $Fix(T^{[2]}) = \{1/2, 3/2\} \neq \emptyset$; *thus, conclusion* (B1) *holds*. (c) *For each* $w^0 \in A$, *the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-convergent to each point* $w \in Fix(T^{[2]})$, *and thus conclusion* (B2) *holds*. (d) $\forall_{w^0\in Fix(T^{[2]})}\{p(w^0, T(w^0)) = p(T(w^0), w^0) = 0\}$, *and thus conclusion* (B3) *holds*. (e) $(X, \mathcal{P}_{\{1\};\{1\}})$ *is not separable*, $Fix(T) = \emptyset$, *and statement* (C) *does not hold*.

Claim II *It follows from* (1)-(3) *and* (5) *that, for* $(X, \mathcal{P}_{\{1\};\{1\}})$, $\mathcal{P}_{\{1\};\{1\}} = \{p\}$, (X,T) *and* $\mathcal{J}_{\{1\};\{1\}} = \mathcal{P}_{\{1\};\{1\}}$ *defined by* (5.1) *and* (5.2) *and for* $q = 4$, *statements* (A) *and* (B) *of Theorem* 3.1 *hold*: (a) *Statement* (A) *holds since, for each* $w^0 \in X$, *the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-convergent to each point* $w \in A$; $W^{L-\mathcal{P}_{\{1\};\{1\}}} = W^{R-\mathcal{P}_{\{1\};\{1\}}} = A$ *by Theorem* 3.1. (b) *We have that* $Fix(T^{[4]}) = X$; *thus, conclusion* (B1) *holds*. (c) *Conclusion* (B2) *follows from statement* (A) *and conclusion* (B1) *since* $A \subset Fix(T^{[4]})$. (d) *Conclusion* (B3) *holds*; *by* (5.1) *we have* $\forall_{w^0\in A\subset Fix(T^{[4]})}\{p(w^0, T(w^0)) = p(T(w^0), w^0) = 0\}$. (e) $(X, \mathcal{P}_{\{1\};\{1\}})$ *is not separable*, $Fix(T) = \emptyset$, *and statement* (C) *does not hold*.

Example 5.5 Let $X = (0;3)$, $p : X^2 \to [0;\infty)$ be of the form

$$p(u,v) = \begin{cases} 0 & \text{if } u \geq v, \\ (v-u)^2 & \text{if } u < v, \end{cases} \qquad u, v \in X, \tag{5.26}$$

and $\mathcal{J}_{\{2\};\{1\}} = \{J\}$, $J : X^2 \to [0;\infty)$, be given by the formula

$$J(u,v) = \begin{cases} p(u,v) & \text{if } \{u,v\} \cap E = \{u,v\}, \\ \mu & \text{if } \{u,v\} \cap E \neq \{u,v\}, \end{cases} \quad u,v \in X, \tag{5.27}$$

where $\mu = 9/64$ and

$$E = \{e_m = 2 - (2/3)^{m-1} : m \in \mathbb{N}\} \cup \{e_\infty = 2\}. \tag{5.28}$$

Let $T : X \to X$ be given by

$$T(x) = \begin{cases} f_1(x) & \text{if } x \in (0;1/2), \\ f_2(x) & \text{if } x = 1/2, \\ f_3(x) & \text{if } x \in (1/2;1], \\ f_4(x) & \text{if } x \in (1;2], \\ f_5(x) & \text{if } x \in (2;3), \end{cases} \tag{5.29}$$

where

$$f_1(x) = 1/2 - [1/4 - x^2]^{1/2}, \qquad f_1 : (0;1/2) \to (0;1/2), \tag{5.30}$$

$$f_2(x) = 2, \qquad f_2 : 1/2 \to 2, \tag{5.31}$$

$$f_3(x) = 4 - 2x, \qquad f_3 : (1/2;1] \to [2;3), \tag{5.32}$$

$$f_4(x) = (3/2)x - 1 = (3/2)(x-2) + 2, \qquad f_4 : (1;2] \to (1/2;2], \tag{5.33}$$

$$f_5(x) = [1 - (3-x)^2]^{1/2} + 2, \qquad f_5 : (2;3) \to (2;3). \tag{5.34}$$

(1) $(X, \mathcal{P}_{\{2\};\{1\}})$, $\mathcal{P}_{\{2\};\{1\}} = \{p\}$, *is a separable quasi-triangular space*. See [27], Example 4, p.11; p vanishes on the diagonal, is asymmetric, and is quasi-triangular since $\forall_{u,v,w\in X}\{p(u,w) \leq 2[p(u,v) + p(v,w)]\}$.

(2) $\mathcal{J}_{\{2\};\{1\}}$ *is the left and right family generated by* $\mathcal{P}_{\{2\};\{1\}} = \{p\}$, *and* $\mathcal{J}_{\{2\};\{1\}}$ *is separating on* X (see Theorems 2.1 and 2.2).

(3) (X,T) *is a weak* $\mathcal{J}_{\{2\};\{1\}}$*-contraction on* X, *that is, there exists* $w^0 \in X$ *such that*

$$\begin{aligned} &\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{J(T^{[s]}(w^0), T^{[l]}(w^0)) < \varepsilon + \eta \\ &\quad \Rightarrow 2 \cdot J(T^{[s+r]}(w^0), T^{[l+r]}(w^0)) < \varepsilon\}. \end{aligned} \tag{5.35}$$

More precisely, we show that (5.35) *holds for each* $w^0 \in E$.

The proof proceeds in four steps.

Step 1. We construct $T^{[m]}$ *in* $(0;1] \cup (2;3)$, $m \in \mathbb{N}$. Using (5.28)-(5.34), for $m \in \mathbb{N}$, we have:

Case A. If $x \in (0;1/2)$, *then* $T^{[m]} = f_1^{[m]} : (0;1/2) \to (0;1/2)$.

Case B. If $x = 1/2$, *then* $T^{[m]}(1/2) = (f_4^{[m-1]} \circ f_2)(1/2) = f_4^{[m-1]}(2) = 2$.

Case C. If $x \in (1/2;1)$, *then* $T^{[m]} = f_5^{[m-1]} \circ f_3 : (1/2;1) \to (2;3)$.

Case D. If $x = e_1 = 1$, *then* $T^{[m]}(1) = (f_4^{[m-1]} \circ f_3)(1) = f_4^{[m-1]}(2) = 2$.

Case E. If $x \in (2;3)$, *then* $T^{[m]} = f_5^{[m]} : (2;3) \to (2;3)$.

Step 2. We construct $T^{[m]}$ in $(1;2]$, $m \in \mathbb{N}$. Using (5.33), we compute that if $m \in \mathbb{N}$, then $f_4^{[m]}(x) = (3/2)^m(x-2)+2, f_4^{[m]} : \mathbb{R} \to \mathbb{R}, f_4^{[m]}(e_m) = 1/2, f_4^{[m-1]}(e_m) = 1, e_m = 2-(2/3)^{m-1}$, and $f_4^{[m]}(e_\infty = 2) = 2$. Using (5.28)-(5.34), we therefore have:

Case A. Let $m = 1$. Since $f_4(x) = (3/2)x - 1 = (3/2)(x-2) + 2$, we see that $f_4 : \mathbb{R} \to \mathbb{R}$, $f_4 : (1;2] \to (1/2;2], f_4(e_1) = 1/2, e_1 = 1$, and $f_4(e_\infty = 2) = 2$. Consequently,

$$T|_{(1;2]} = f_4|_{(1;2]} : (e_1;2] \to (1/2;2], \qquad f_4(e_1 = 1) = 1/2, \qquad f_4(e_\infty = 2) = 2.$$

Case B. Let $m = 2$. Then $f_4^{[2]}(x) = (3/2)^2(x-2) + 2, f_4^{[2]} : \mathbb{R} \to \mathbb{R}, f_4^{[2]}(e_2) = 1/2$, $e_2 = 2 - (2/3) = 4/3, f_4(e_2) = 1, f_4(e_1 = 1) = 1/2, f_4^{[2]}(e_\infty = 2) = 2$, and in view of (5.29), this implies

$$\begin{aligned} &T^{[2]}|_{(e_1;e_2]} = f_3 \circ f_4 : (e_1;e_2] \to f_3\big((1/2;1]\big) \subset [2;3), \qquad f_3(1) = 2, \\ &T^{[2]}|_{(e_2;2]} = f_4^{[2]} : (e_2;2] \to (1/2;2]. \end{aligned}$$

Case C. Let $m = 3$. We see that $f_4^{[3]}(x) = (3/2)^3(x-2) + 2$, $f_4^{[3]} : \mathbb{R} \to \mathbb{R}$, $f_4^{[3]}(e_3) = 1/2$, $e_3 = 2 - (2/3)^2 = 14/9, f_4^{[2]}(e_3) = 1, f_4^{[2]}(e_2) = 1/2$, and $f_4^{[3]}(e_\infty = 2) = 2$. Now (5.31) implies that

$$\begin{aligned} &T^{[3]}|_{(e_1;e_2]} = f_5 \circ f_3 \circ f_4 : (e_1;e_2] \to f_5\big(f_3\big((1/2;1]\big)\big) \subset f_5\big([2;3)\big) \subset [2;3), \\ &T^{[3]}|_{(e_2;e_3]} = f_3 \circ f_4^{[2]} : (e_2;e_3] \to f_3\big((1/2;1]\big) \subset [2;3), \\ &T^{[3]}|_{(e_3;2]} = f_4^{[3]} : (e_3;2] \to (1/2;2]. \end{aligned}$$

Case D. Let $m > 3$. Since $f_4^{[m]}(x) = (3/2)^m(x-2) + 2$, $f_4^{[m]} : \mathbb{R} \to \mathbb{R}$, $f_4^{[m]}(e_m) = 1/2$, $e_m = 2 - (2/3)^{m-1}, f_4^{[m-1]}(e_m) = 1, f_4^{[m-1]}(e_{m-1}) = 1/2$, and $f_4^{[m]}(e_\infty = 2) = 2$. Hence,

$$\begin{aligned} T^{[m]}|_{(e_1;e_2]} &= f_5^{[m-2]} \circ f_3 \circ f_4 : (e_1;e_2] \to f_5^{[m-2]}\big(f_3\big((1/2;1]\big)\big) \\ &\subset f_5^{[m-2]}\big([2;3)\big) \subset [2;3), \\ T^{[m]}|_{(e_2;e_3]} &= f_5^{[m-3]} \circ f_3 \circ f_4^{[2]} : (e_2;e_3] \to f_5^{[m-3]}\big(f_3\big((1/2;1]\big)\big) \\ &\subset f_5^{[m-3]}\big([2;3)\big) \subset [2;3), \\ T^{[m]}|_{(e_{k-1};e_k]} &= f_5^{[m-k]} \circ f_3 \circ f_4^{[k-1]} : (e_{k-1};e_k] \to f_5^{[m-k]}\big(f_3\big((1/2;1]\big)\big) \\ &\subset f_5^{[m-k]}\big([2;3)\big) \subset [2;3), \quad k = 4,5,\ldots,m, \\ T^{[m]}|_{(e_m;2]} &= f_4^{[m]} : (e_m;2] \to (1/2;2]. \end{aligned}$$

Step 3. We describe the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$, $w^0 \in E$. Let $w^0 \in E$ be arbitrary and fixed. We consider the following cases:

Case A. If $k = 0$ *and* $w^0 = e_1 = 1$, *then*

$$\forall_{m\in\mathbb{N}}\big\{T^{[m]}\big(w^0\big) = 2\big\}. \tag{5.36}$$

This is a consequence of Case B of Step 1.

Case B. If $k \in \mathbb{N}\backslash\{1\}$ *and* $w^0 = e_k$, *then*

$$\exists_{m_0=k+1}\forall_{m>m_0}\big\{w^m = 2\big\}. \tag{5.37}$$

In fact, by Case D of Step 2 we have

$$\begin{aligned}\forall_{m\in\mathbb{N}}\big\{w^{m+k+1} &= T^{[m+k+1]}(e_k) = T^{[m]}\big(T\big(T^{[k]}(e_k)\big)\big)\\ &= T^{[m]}\big(T\big(f_4^{[k]}(e_k)\big)\big) = T^{[m]}\big(T(1)\big) = T^{[m]}(2) = 2\big\}.\end{aligned}$$

Case C. If $w^0 = e_\infty = 2$ *and* $m \in \mathbb{N}$, *then*

$$w^m = 2. \tag{5.38}$$

Indeed, by Case D of Step 2 we obtain $\forall_{m\in\mathbb{N}}\{w^m = T^{[m]}(2) = f_4^{[m]}(2) = 2\}$.

*Step 4. We will show that (*5.35*) holds on E.* Let $w^0 \in E$ be arbitrary and fixed. Using (5.36)-(5.38) in Cases A-C of Step 3 and (5.26)-(5.28), we observe that

$$\begin{aligned}\forall_{w^0\in E}\exists_{m_0\in\mathbb{N}}\forall_{s,l\geq m_0}\big\{&J\big(T^{[s]}\big(w^0\big), T^{[l]}\big(w^0\big)\big) = p\big(T^{[s]}\big(w^0\big), T^{[l]}\big(w^0\big)\big)\\ &= p(2,2) = 0\big\}.\end{aligned}$$

In view of this, we conclude that (5.35) holds.

(4) (X, T) *is left and right* $\mathcal{J}_{\{2\};\{1\}}$*-admissible on*

$$W^{L-\mathcal{J}_{\{2\};\{1\}}} = W^{R-\mathcal{J}_{\{2\};\{1\}}} = E.$$

Indeed, by Step 3 it is clear that, for arbitrary and fixed $w^0 = e_k \in E$, $k \in \mathbb{N}\cup\{\infty\}$, the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\}\cup\mathbb{N})$ satisfies

$$\exists_{m_0=k+1}\forall_{m>m_0}\big\{w^m = 2 \in E\big\}. \tag{5.39}$$

Then, in view of (5.39), (5.26), and (5.27), we have

$$\lim_{m\to\infty}\sup_{n>m} J\big(w^m, w^n\big) = \lim_{m\to\infty}\sup_{n>m} p\big(w^m, w^n\big) = 0$$

and

$$\lim_{m\to\infty}\sup_{n>m} J\big(w^n, w^m\big) = \lim_{m\to\infty}\sup_{n>m} p\big(w^n, w^m\big) = 0.$$

Also, we have

$$\lim_{m\to\infty} J\big(2, w^m\big) = \lim_{m\to\infty} p\big(2, w^m\big) = 0$$

and

$$\forall_{w\in E}\Big\{\lim_{m\to\infty} J\big(w^m, w\big) = \lim_{m\to\infty} p\big(w^m, w\big) = 0\Big\},$$

that is,

$$LIM^{L-\mathcal{J}_{\{2\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = \{2\} \quad \text{and} \quad LIM^{R-\mathcal{J}_{\{2\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = E.$$

Therefore, (3.1)-(3.4) of Definition 3.1 hold.

(5) *We show that the single-valued dynamic system* (X,T) *is left and right* $\mathcal{P}_{\{2\};\{1\}}$*-closed on* $U = V = W^{L-\mathcal{J}_{\{2\};\{1\}}} = W^{R-\mathcal{J}_{\{2\};\{1\}}} = E$. Indeed, if $w^0 \in E$ then, by Step 3 and (5.26) we conclude that the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfies

$$[2;3) = LIM^{L-\mathcal{P}_{\{2\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \quad \text{and} \quad (0;2] = LIM^{R-\mathcal{P}_{\{2\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}.$$

Clearly, the subsequences $(v_m = w^{m+1} : m \in \mathbb{N})$ and $(u_m = w^m : m \in \mathbb{N})$ of $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{v_m = T(u_m)\}$. We have $2 \in Fix(T)$.

Claim *By* (1)-(5) *it follows that, for each* $w^0 \in E$, *we may use Theorem* 3.2 *in the cases of left and right* (*when* $\mathcal{J}_{\{2\};\{1\}} \neq \mathcal{P}_{\{2\};\{1\}}$), *and we see that statements* (A)-(C) *of this theorem hold. We have*: (a) $(X,\mathcal{P}_{\{2\};\{1\}})$ *is separable*. (b) *For each* $w^0 \in E$, *the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left and right* $\mathcal{P}_{\{2\};\{1\}}$*-convergent to* $w = 2 \in Fix(T) = \{2\}$. (c) *For* $w^0 = w = 2$, *we have* $J(w^0,w^0) = 0$.

Example 5.6 Let X, p, E, T, and $\mathcal{P}_{\{2\};\{1\}}$ be such as in Example 5.5.

(1) *We show that, for each* $w^0 \in X$,

$$\begin{aligned}&\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{p\bigl(T^{[s]}\bigl(w^0\bigr),T^{[l]}\bigl(w^0\bigr)\bigr) < \varepsilon + \eta \\ &\quad\Rightarrow 2\cdot p\bigl(T^{[s+r]}\bigl(w^0\bigr),T^{[l+r]}\bigl(w^0\bigr)\bigr) < \varepsilon\bigr\},\end{aligned} \tag{5.40}$$

that is, (X,T) *is a weak* $\mathcal{P}_{\{2\};\{1\}}$*-contraction on* X.

Indeed, let $w^0 \in (0;1/2) \cup [(1/2;3)\backslash E]$ be arbitrary and fixed. Then, for all $s,l \in \mathbb{N}$,

$$p\bigl(T^{[s]}\bigl(w^0\bigr),T^{[l]}\bigl(w^0\bigr)\bigr) = \begin{cases}0 & \text{if } T^{[s]}(w^0) \geq T^{[l]}(w^0),\\ [T^{[l]}(w^0) - T^{[s]}(w^0)]^2 & \text{if } T^{[s]}(w^0) < T^{[l]}(w^0).\end{cases}$$

Using Steps 1 and 2 of Example 5.5, we therefore have

$$\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{s,l\geq m_0}\bigl\{p\bigl(T^{[s]}\bigl(w^0\bigr),T^{[l]}\bigl(w^0\bigr)\bigr) < \varepsilon\bigr\},$$

which means that (5.40) holds.

If $w^0 \in \{1/2\} \cup E$, then by Steps 1 and 3 of Example 5.5 we see that

$$\forall_{w^0\in\{1/2\}\cup E}\exists_{m_0\in\mathbb{N}}\forall_{s,l\geq m_0}\bigl\{p\bigl(T^{[s]}\bigl(w^0\bigr),T^{[l]}\bigl(w^0\bigr)\bigr) = p(2,2) = 0\bigr\},$$

and therefore (5.40) also holds.

(2) *We may not use Theorems* 3.1 *and* 3.2 *for* $\mathcal{J}_{\{2\};\{1\}} = \mathcal{P}_{\{2\};\{1\}}$ *in the left and right cases since condition* (A1) *of Definition* 3.1 *in these cases does not hold.* Indeed, we consider two cases:

Case A. Let $w^0 \in (1/2;3)\backslash E$ be arbitrary and fixed. It is clear that

$$\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\bigl\{T^{[m]}\bigl(w^0\bigr) \in (2;3)\bigr\}$$

and that the sequence $(w^m = T^{[m]}(w^0) : m \geq m_0)$ is increasing. Hence,

$$\lim_{m\to\infty}\sup_{n>m} p\bigl(w^m,w^n\bigr) = \lim_{m\to\infty}\sup_{n>m}\bigl[T^{[n]}\bigl(w^0\bigr) - T^{[m]}\bigl(w^0\bigr)\bigr]^2 = \lim_{m\to\infty}\bigl[3 - T^{[m]}\bigl(w^0\bigr)\bigr]^2 = 0,$$

that is, property (3.1) holds, whereas $LIM^{L-\mathcal{P}_{\{2\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = \emptyset$. In fact, since

$$\forall_{w\in(0;3)}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\left\{p(w,w^m)=(w^m-w)^2\right\},$$

we have

$$\forall_{w\in(0;3)}\left\{\lim_{m\to\infty}p\left(w,w^m\right)=\lim_{m\to\infty}\left(w^m-w\right)^2=(3-w)^2\neq 0\right\},$$

that is, property (3.3) does not hold.

Case B. Let $w^0 \in (0;1/2)$ be arbitrary and fixed. It is clear that

$$\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\left\{T^{[m]}\left(w^0\right)\in(0;1/2)\right\}$$

and that the sequence $(w^m = T^{[m]}(w^0) : m \geq m_0)$ is decreasing. Hence,

$$\begin{aligned}\lim_{m\to\infty}\sup_{n>m}p\left(w^n,w^m\right)&=\lim_{m\to\infty}\sup_{n>m}\left[T^{[m]}\left(w^0\right)-T^{[n]}\left(w^0\right)\right]^2\\&=\lim_{m\to\infty}\left[T^{[m]}\left(w^0\right)-0\right]^2=0,\end{aligned}$$

that is, property (3.2) holds. Note, however, that $LIM^{R-\mathcal{P}_{\{2\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = \emptyset$. In fact, since $\forall_{w\in(0;3)}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\{p(w^m,w)=(w-w^m)^2\}$, we have

$$\forall_{w\in(0;3)}\left\{\lim_{m\to\infty}p\left(w^m,w\right)=\lim_{m\to\infty}\left(w-w^m\right)^2=(w-0)^2\neq 0\right\},$$

that is, property (3.4) does not hold.

Remark 5.1 We make the following remarks about Examples 5.5 and 5.6: (a) By Example 5.5 we observe that, for (X,T), we may apply Theorem 3.2 in $(X,\mathcal{P}_{C;\mathcal{A}})$ with the left and right family $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ where $\mathcal{J}_{C;\mathcal{A}}\neq\mathcal{P}_{C;\mathcal{A}}$; (b) By Example 5.6 we note, however, that, for (X,T) in the left and right cases, we do not apply Theorems 3.1 and 3.2 in $(X,\mathcal{P}_{C;\mathcal{A}})$ when $\mathcal{J}_{C;\mathcal{A}}=\mathcal{P}_{C;\mathcal{A}}$; (c) From (a) and (b) it follows that, in Theorems 3.1 and 3.2, the existence of the family $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ and such that $\mathcal{J}_{C;\mathcal{A}}\neq\mathcal{P}_{C;\mathcal{A}}$ is essential.

Competing interests
The author declares that he has no conflict of interests regarding the publication of this paper.

Received: 15 April 2016 Accepted: 4 August 2016 Published online: 18 August 2016

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