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A new iterative scheme for numerical reckoning fixed points of total asymptotically nonexpansive mappings

Adoon Pansuwan and Wutiphol Sintunavarat*

*Correspondence:
wutiphol@mathstat.sci.tu.ac.th
Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani, 12121, Thailand

Abstract

In this paper, we propose a new iterative algorithm to approximate fixed points of total asymptotically nonexpansive mappings in CAT(0) spaces. We also provide two examples to illustrate the convergence behavior of the proposed algorithm and numerically compare the convergence of the proposed iteration scheme with the existing schemes.

MSC: 47H09; 47H10

Keywords: modified Picard-Ishikawa hybrid iteration; modified Picard-Mann hybrid iteration; total asymptotically nonexpansive mappings

1 Introduction and basic definitions

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping $c : [0,l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$ and $d(c(a), c(b)) = |a - b|$ for all $a, b \in [0,l]$. It is easy to see that c is an isometry and $d(x,y) = l$. The image $c([0,l])$ is called a geodesic (or metric) segment joining x and y and is denoted by $[x,y]$ if it is unique.

The metric space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset C of X is said to be convex if C includes every geodesic segment joining any two of its point.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points $x_1, x_2, x_3 \in X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Bridson and Haefliger [1] have shown that such a triangle always exists.

A geodesic space is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom:

- Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle} \subseteq \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{1.1}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [2]).

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which is a unique point with

$$d(y_1, y_0) = d(y_0, y_2) = \frac{1}{2} d(y_1, y_2), \tag{1.2}$$

then the CAT(0) inequality (1.1) implies

$$d^2(x, y_0) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \tag{1.3}$$

This inequality is called the (CN) inequality which due to Bruhat and Titz [3].

In fact (see [3], p.163), a geodesic metric space (X, d) is a CAT(0) space if and only if it satisfies the (CN) inequality. If (X, d) is a CAT(0) space and $x, y \in X$, then for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \tag{1.4}$$

For convenience, from now on we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (1.4).

Definition 1.1 Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d).

1. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) := \inf_{x \in X} \big\{ r\big(x, \{x_n\}\big) \big\},$$

where $r(x, \{x_n\}) := \limsup_{n \to \infty} d(x, x_n)$.

2. The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) := \big\{ x \in X : r\big(x, \{x_n\}\big) = r\big(\{x_n\}\big) \big\}.$$

In 2006, Dhompongsa *et al.* [4] showed that $A(\{x_n\})$ consists of exactly one point for each bounded sequence $\{x_n\}$ in a CAT(0) space (see Proposition 7 in [4])).

Next, we give the concept of Δ-convergent sequence in a CAT(0) spaces.

Definition 1.2 Let (X, d) be a CAT(0) space. A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if and only if x is the unique asymptotic center of all subsequences of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \to \infty} x_n = x$ and x is called the Δ-limit of $\{x_n\}$.

Let us recall some basics for nonlinear mappings on CAT(0) spaces.

Definition 1.3 Let C be a nonempty subset of a CAT(0) space (X, d). A mapping $T : C \to C$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \tag{1.5}$$

for all $x, y \in C$.

Definition 1.4 ([5]) Let C be a nonempty subset of a CAT(0) space (X,d). A mapping $T: C \to C$ is said to be asymptotically nonexpansive if there exists a sequences $\{k_n\} \subseteq [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(T^n x, T^n y) \leq k_n d(x,y) \tag{1.6}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Definition 1.5 Let C be a nonempty subset of a CAT(0) space (X,d). A mapping $T: C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L \geq 0$ such that

$$d(T^n x, T^n y) \leq L d(x,y) \tag{1.7}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Definition 1.6 ([6]) Let C be a nonempty subset of a CAT(0) space (X,d). A mapping $T: C \to C$ is said to be $(\{v_n\},\{u_n\},\zeta)$-total asymptotically nonexpansive (briefly, total asymptotically nonexpansive) if there exist nonnegative sequences $\{v_n\}$ and $\{u_n\}$ with $v_n \to 0$, $u_n \to 0$ and a strictly increasing continuous function $\zeta: [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + v_n \zeta\bigl(d(x,y)\bigr) + u_n \tag{1.8}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Remark 1.7 From Definitions 1.3, 1.4, 1.5, and 1.6, we note that each nonexpansive mapping is an asymptotically nonexpansive mapping with a sequence $\{k_n := 1\}$ for all $n \in \mathbb{N}$ and each asymptotically nonexpansive mapping is a $(\{v_n\},\{u_n\},\zeta)$-total asymptotically nonexpansive mapping with two sequences $\{v_n := k_n - 1\}$ and $\{u_n := 0\}$ for all $n \in \mathbb{N}$ and ζ is an identity mapping. Also, we see that each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L := \sup_{n\in\mathbb{N}}\{k_n\}$.

Lemma 1.8 ([7], Theorem 2.8) *Let C be a closed convex subset of a complete* CAT(0) *space (X,d) and $T: C \to C$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. If $\{x_n\}$ is a bounded sequence in C such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $\Delta\text{-}\lim_{n\to\infty} x_n = p$, *then $Tp = p$.*

In 2014, Panyanak [8] gave the following existence result of fixed points for total asymptotically nonexpansive mappings in CAT(0) spaces which is also need in our main results.

Theorem 1.9 ([8], Corollary 3.2) *Let C be a nonempty bounded closed convex subset of a complete* CAT(0) *space (X,d) and $T: C \to C$ be a continuous total asymptotically nonexpansive mapping. Then T has a fixed point.*

Recently, Thakur *et al.* [9] introduced the modified Picard-Mann hybrid iteration process $\{x_n\}$, which is given by

$$\left.\begin{aligned} &x_1 \in C, \\ &y_n = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \\ &x_{n+1} = T^n y_n \end{aligned}\right\} \tag{1.9}$$

for all $n \in \mathbb{N}$, where C is a nonempty bounded closed convex subset of a CAT(0) space (X,d), $\{\alpha_n\}$ is real sequence in the interval $[0,1]$ and $T: X \to X$ is a total asymptotically nonexpansive mapping. By using the iteration process (1.9) and Panyanak's fixed point result (Theorem 1.9), they proved Δ-convergence and strong convergence theorems for total asymptotically nonexpansive mappings on CAT(0) spaces. They also compare the convergence of the modified Picard-Mann hybrid iteration process (1.9) with the modified Mann iteration process $\{x_n\}$, which is given by

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n \end{aligned}\right\} \tag{1.10}$$

for all $n \in \mathbb{N}$, where C is a nonempty bounded closed convex subset of a CAT(0) space (X,d), $\{\alpha_n\}$ is real sequence in the interval $[0,1]$ and $T: X \to X$ is a total asymptotically nonexpansive mapping. The original idea of the modified Mann iteration process was introduced by Alber *et al.* [6].

Motivated by the above recorded studies, in this work, we introduce a new iterative algorithm called 'modified Picard-Ishikawa hybrid' to approximate fixed points of total asymptotically nonexpansive mappings on CAT(0) spaces. Our results are refinements and generalizations of many recent results from the current literature. We also provide two numerical examples to illustrate the convergence behavior of the proposed algorithm.

Before we show our main results in the next section, let us recall some useful lemmas.

Lemma 1.10 ([10], Lemma 2) *Let $\{a_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ be the sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1+\lambda_n)a_n + c_n.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. Moreover, if there exists a subsequence $\{a_{n_i}\} \subseteq \{a_n\}$ such that $a_{n_i} \to 0$ as $i \to \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.11 *Let (X,d) be a complete* CAT(0) *space. Then the following assertions hold*:

- (C_1) *every bounded sequence in X always has a Δ-convergent subsequence* [11], *p.*3690;
- (C_2) *if $\{x_n\}$ is a bounded sequence in a closed convex subset C of X, then the asymptotic center of $\{x_n\}$ is in C* [12], *Proposition* 2.1;
- (C_3) *if $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{p\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n,u)\}$ converges, then $p = u$* [13], *Lemma* 2.8.

Lemma 1.12 ([14], Lemma 4.5) *Let x be a given point in a* CAT(0) *space (X,d) and $\{t_n\}$ be a sequence in a closed interval $[a,b]$ with $0 < a \leq b < 1$ and $0 < a(1-b) \leq \frac{1}{2}$. Suppose that $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \leq r,$$

$$\limsup_{n\to\infty} d(y_n,x) \leq r,$$

$$\limsup_{n\to\infty} d\big((1-t_n)x_n \oplus t_n y_n, x\big) = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

2 Main results

In this section, we begin with the Δ-convergence theorem for a total asymptotically nonexpansive mapping T on a nonempty closed convex subset C of a CAT(0) space through the modified Picard-Ishikawa hybrid iteration process as follows:

$$\left.\begin{array}{l} x_1 \in C, \\ z_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ y_n = (1-\alpha_n)z_n \oplus \alpha_n T^n z_n, \\ x_{n+1} = T^n y_n \end{array}\right\} \tag{2.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real control sequences in the interval $[0,1]$.

Theorem 2.1 *Let C be a bounded closed convex subset of a complete* CAT(0) *space* (X,d) *and $T : C \to C$ be a uniformly L-Lipschitzian and $(\{v_n\},\{u_n\},\zeta)$-total asymptotically nonexpansive mapping. Suppose that the following conditions are satisfied*:

(S_1) $\sum_{n=1}^{\infty} v_n < \infty$ *and* $\sum_{n=1}^{\infty} u_n < \infty$;
(S_2) *there exist constants a, b with* $0 < a \leq \alpha_n \leq b < 1$ *for all* $n \in \mathbb{N}$ *and* $0 < a(1-b) \leq \frac{1}{2}$;
(S_3) *there exists a constant M^* such that $\zeta(r) \leq M^* r$ for all $r \geq 0$.*

Then the sequence $\{x_n\}$ which is defined by (2.1) *Δ-converges to some point $p \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Proof Since T is uniformly L-Lipschitzian, we have T is continuous. By using Theorem 1.9, we get $F(T) \neq \emptyset$. Next, we will divide the proof into three steps.

Step 1: First, we will prove that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$, where $\{x_n\}$ is defined by (2.1). Assume that $\{x_n\}$ is defined by (2.1) and let $p \in F(T)$. Then we obtain

$$\begin{aligned} d(z_n,p) &= d\big((1-\beta_n)x_n \oplus \beta_n T^n x_n, p\big) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d\big(T^n x_n, p\big) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n\big[d(x_n,p) + v_n\zeta\big(d(x_n,p)\big) + u_n\big] \\ &= d(x_n,p) + \beta_n\big[v_n\zeta\big(d(x_n,p)\big) + u_n\big] \\ &\leq d(x_n,p) + v_n\zeta\big(d(x_n,p)\big) + u_n \\ &\leq \big(1+v_nM^*\big)d(x_n,p) + u_n \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$. Also, we have

$$\begin{aligned} d(y_n,p) &= d\big((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p\big) \\ &\leq (1-\alpha_n)d(z_n,p) + \alpha_n d\big(T^n z_n, p\big) \\ &\leq (1-\alpha_n)d(z_n,p) + \alpha_n\big[d(z_n,p) + v_n\zeta\big(d(z_n,p)\big) + u_n\big] \\ &= d(z_n,p) + \alpha_n\big[v_n\zeta\big(d(z_n,p)\big) + u_n\big] \\ &\leq d(z_n,p) + v_n\zeta\big(d(z_n,p)\big) + u_n \\ &\leq \big(1+v_nM^*\big)d(z_n,p) + u_n \\ &\leq \big(1+v_nM^*\big)^2 d(x_n,p) + \big(1+v_nM^*\big)u_n + u_n \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. From (2.1), (2.2), and (2.3), for each $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_{n+1},p) &= d\bigl(T^n y_n,p\bigr) \\ &\leq d(y_n,p) + v_n \zeta\bigl(d(y_n,p)\bigr) + u_n \\ &\leq \bigl(1+v_nM^*\bigr)d(y_n,p) + u_n \\ &\leq \bigl(1+v_nM^*\bigr)^3 d(x_n,p) + \bigl(1+v_nM^*\bigr)^2 u_n + \bigl(1+v_nM^*\bigr)u_n + u_n \\ &= \bigl(1+v_nM^*\bigr)^3 d(x_n,p) + \bigl[\bigl(1+v_nM^*\bigr)^2 + \bigl(1+v_nM^*\bigr) + 1\bigr]u_n \\ &= (1+\lambda_n)d(x_n,p) + c_n, \end{aligned} \tag{2.4}$$

where $\lambda_n := 3v_nM^* + 3(v_nM^*)^2 + (v_nM^*)^3$ and $c_n := [3 + 3v_nM^* + (v_nM^*)^2]u_n$. The assumption (S_1) yields

$$\sum_{n=1}^{\infty} \lambda_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty. \tag{2.5}$$

By using Lemma 1.10 with assertions (2.4) and (2.5), we see that $\lim_{n\to\infty} d(x_n,p)$ exists.

Step 2: In this step, we will prove that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Without loss of generality, we may assume that

$$r := \lim_{n\to\infty} d(x_n,p) \geq 0. \tag{2.6}$$

From (2.2), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq r. \tag{2.7}$$

It follows from T being a $(\{v_n\},\{u_n\},\zeta)$-total asymptotically nonexpansive mapping that

$$\begin{aligned} d\bigl(T^n z_n,p\bigr) &\leq d(z_n,p) + v_n\zeta\bigl(d(z_n,p)\bigr) + u_n \\ &\leq \bigl(1+v_nM^*\bigr)d(z_n,p) + u_n. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have

$$\limsup_{n\to\infty} d\bigl(T^n z_n,p\bigr) \leq r. \tag{2.9}$$

Similarly, we get

$$\limsup_{n\to\infty} d\bigl(T^n x_n,p\bigr) \leq r. \tag{2.10}$$

Since

$$d(x_{n+1},p) \leq \bigl(1+v_nM^*\bigr)d(y_n,p) + u_n,$$

we obtain

$$r \leq \limsup_{n\to\infty} d(y_n,p), \tag{2.11}$$

which implies that

$$r \leq \limsup_{n\to\infty} d(z_n, p). \tag{2.12}$$

From (2.7) and (2.12), we can conclude that

$$r = \limsup_{n\to\infty} d(z_n, p) = \limsup_{n\to\infty} d\big((1-\beta_n)x_n \oplus \beta_n T^n x_n, p\big). \tag{2.13}$$

By using Lemma 1.12 with (2.6), (2.10), and (2.13), we get

$$\lim_{n\to\infty} d\big(x_n, T^n x_n\big) = 0. \tag{2.14}$$

From (2.3) and (2.6), we have

$$\limsup_{n\to\infty} d(y_n, p) \leq r. \tag{2.15}$$

Combining (2.11) and (2.15), we get

$$r = \limsup_{n\to\infty} d(y_n, p) = \limsup_{n\to\infty} d\big((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p\big). \tag{2.16}$$

Again, by using Lemma 1.12 with (2.7), (2.9), and (2.16), we get

$$\lim_{n\to\infty} d\big(z_n, T^n z_n\big) = 0. \tag{2.17}$$

By using condition (1.8), we have

$$\begin{aligned} d\big(T^n z_n, T^n x_n\big) &\leq d(z_n, x_n) + \nu_n \zeta\big(d(z_n, x_n)\big) + u_n \\ &\leq \big(1 + \nu_n M^*\big) d(z_n, x_n) + u_n \\ &= \big(1 + \nu_n M^*\big) d\big((1-\alpha_n)x_n \oplus \alpha_n T^n x_n, x_n\big) + u_n \\ &\leq \big(1 + \nu_n M^*\big)\big[(1-\alpha_n)d(x_n, x_n) + \alpha_n d\big(T^n x_n, x_n\big)\big] + u_n \end{aligned} \tag{2.18}$$

for all $n \in \mathbb{N}$. From (2.14), we get

$$\lim_{n\to\infty} d\big(T^n z_n, T^n x_n\big) = 0. \tag{2.19}$$

Also, we have

$$\begin{aligned} d\big(T^n y_n, T^n z_n\big) &\leq d(y_n, z_n) + \nu_n \zeta\big(d(y_n, z_n)\big) + u_n \\ &\leq \big(1 + \nu_n M^*\big) d(y_n, z_n) + u_n \\ &= \big(1 + \nu_n M^*\big) d\big((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, z_n\big) + u_n \\ &\leq \big(1 + \nu_n M^*\big)\big[(1-\alpha_n)d(z_n, z_n) + \alpha_n d\big(T^n z_n, z_n\big)\big] + u_n \end{aligned} \tag{2.20}$$

for all $n \in \mathbb{N}$. From (2.17) and (2.20), we obtain

$$\lim_{n\to\infty} d\bigl(T^n y_n, T^n z_n\bigr) = 0. \tag{2.21}$$

By using the triangle inequality, we have

$$d(x_n, x_{n+1}) = d\bigl(x_n, T^n y_n\bigr) \le d\bigl(x_n, T^n x_n\bigr) + d\bigl(T^n x_n, T^n z_n\bigr) + d\bigl(T^n z_n, T^n y_n\bigr)$$

for all $n \in \mathbb{N}$. Taking the limit $n \to \infty$ in the above inequality with (2.14), (2.19), and (2.21), we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.22}$$

By using the triangle inequality with (2.14) and (2.22), we have

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T^{n+1}x_{n+1}\bigr) + d\bigl(T^{n+1}x_{n+1}, T^{n+1}x_n\bigr) + d\bigl(T^{n+1}x_n, Tx_n\bigr) \\ &\le d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T^{n+1}x_{n+1}\bigr) + Ld(x_{n+1}, x_n) + Ld\bigl(T^n x_n, x_n\bigr) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

So Step 2 is proved.

Step 3: Now to claim that the sequence $\{x_n\}$ Δ-converges to a fixed point of T, we prove that

$$W_\Delta(x_n) := \bigcup_{\{w_n\}\subseteq\{x_n\}} A\bigl(\{u_n\}\bigr) \subseteq F(T)$$

and $W_\Delta(x_n)$ consists of exactly one point. Assume that $w \in W_\Delta(x_n)$. From the definition of $W_\Delta(x_n)$, there is a subsequence $\{w_n\}$ of $\{x_n\}$ such that $A(\{w_n\}) = \{w\}$. By Lemma 1.11(C_1), there exists a subsequence $\{z_n\}$ of $\{w_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} z_n = z \in C$. Using Lemma 1.8, we get $z \in F(T)$. Since $\{d(w_n, z)\}$ converges, by Lemma 1.11(C_2), we obtain $w = z$. It yields $W_\Delta(x_n) \subseteq F(T)$. Finally, we show that $W_\Delta(x_n)$ consists of exactly one point. Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $w = z \in F(T)$. Since $\{d(x_n, z)\}$ converges, by Lemma 1.11 (C_3), we have $x = z \in F(T)$, that is, $W_\Delta(x_n) = \{x\}$.

This completes the proof. □

By using the conclusion in Step 1 of Theorem 2.1 and the same technique as in the proof of Theorem 3.2 of Thakur *et al.* [9], we get the strong convergence result (Theorem 2.2). Then, in order to avoid repetition, the details are omitted.

Theorem 2.2 *Let C be a bounded closed convex subset of a complete* CAT(0) *space* (X, d) *and $T : C \to C$ be a uniformly L-Lipschitzian and $(\{v_n\}, \{u_n\}, \zeta)$-total asymptotically nonexpansive mapping. Suppose that the following conditions are satisfied*:

(S_1) $\sum_{n=1}^{\infty} v_n < \infty$ *and* $\sum_{n=1}^{\infty} u_n < \infty$;
(S_2) *there exist constants a, b with $0 < a \le \alpha_n \le b < 1$ for all $n \in \mathbb{N}$ and $0 < a(1-b) \le \frac{1}{2}$*;
(S_3) *there exists a constant M^* such that $\zeta(r) \le M^* r$ for all $r \ge 0$.*

Then the sequence $\{x_n\}$ *which is defined by* (2.1) *converges strongly to a fixed point of* T *if and only if*

$$\liminf_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0,$$

where $d(x, F(T)) := \inf\{d(x,p) : p \in F(T)\}$.

In [15], Senter and Dotson introduced the concept of special self mapping as follows.

Definition 2.3 ([15]) Let C be a nonempty subset of a CAT(0) space (X,d). A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy condition (I) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$ such that

$$d(x, Tx) \geq f\bigl(d\bigl(x, F(T)\bigr)\bigr)$$

for all $x \in C$.

Using the result in Step 2 of Theorem 2.1 with Condition (I) and the same technique as in the proof of Theorem 3.3 of Thakur *et al.* [9], we now state the following strong convergence result for total asymptotically nonexpansive mappings without the proof.

Theorem 2.4 *Let* C *be a bounded closed convex subset of a complete* CAT(0) *space* (X,d) *and* $T : C \to C$ *be a uniformly* L*-Lipschitzian and* $(\{v_n\},\{u_n\},\zeta)$*-total asymptotically nonexpansive mapping. Suppose that the following conditions are satisfied*:

(S_1) $\sum_{n=1}^{\infty} v_n < \infty$ *and* $\sum_{n=1}^{\infty} u_n < \infty$;
(S_2) *there exist constants* a, b *with* $0 < a \leq \alpha_n \leq b < 1$ *for all* $n \in \mathbb{N}$ *and* $0 < a(1-b) \leq \frac{1}{2}$;
(S_3) *there exists a constant* M^* *such that* $\zeta(r) \leq M^* r$ *for all* $r \geq 0$;
(S_4) T *satisfies Condition* (I).

Then the sequence $\{x_n\}$ *which is defined by* (2.1) *converges strongly to a fixed point of* T.

3 Numerical example

In this section, using Example 3.1, we will compare the convergence of the modified Picard-Ishikawa hybrid iteration process (2.1) with the modified Mann iteration process (1.10) and the modified Picard-Mann hybrid iteration process (1.9).

Example 3.1 Let $X := \mathbb{R}$ be a usual metric space with the metric d, which is also a complete CAT(0) space, and $C := [0,2]$. We see that C is a bounded closed convex subset of X. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} 1 & \text{if } x \in [0,1]; \\ \frac{1}{\sqrt{3}}\sqrt{4-x^2} & \text{if } x \in [1,2]. \end{cases}$$

Recently, Kim [16] showed that T is a continuous uniformly L-Lipschitzian and total asymptotically nonexpansive mapping with $F(T) = \{1\}$. Also, he claimed that T is not Lipschitzian and then it is not an asymptotically nonexpansive mapping.

Table 1 Iterates of modified Mann, modified Picard-Mann hybrid, and modified Picard-Ishikawa hybrid iterations for $x_1 = 1.5$

Iterate	The modified Mann iteration process	The modified Picard-Mann hybrid iteration	The modified Picard-Ishikawa hybrid iteration
x_1	1.50000000000000	1.50000000000000	1.50000000000000
x_2	1.13188130791299	0.95198821855406	1.00000000000000
x_3	1.04396043597100	1.00000000000000	1.00000000000000
x_4	1.01099010899275	1.00000000000000	1.00000000000000
x_5	1.00219802179855	1.00000000000000	1.00000000000000
x_6	1.00036633696643	1.00000000000000	1.00000000000000
x_7	1.00005233385235	1.00000000000000	1.00000000000000
x_8	1.00000654173154	1.00000000000000	1.00000000000000
x_9	1.00000072685906	1.00000000000000	1.00000000000000
x_{10}	1.00000007268591	1.00000000000000	1.00000000000000
x_{11}	1.00000000660781	1.00000000000000	1.00000000000000
x_{12}	1.00000000055065	1.00000000000000	1.00000000000000
x_{13}	1.00000000004236	1.00000000000000	1.00000000000000
x_{14}	1.00000000000303	1.00000000000000	1.00000000000000
x_{15}	1.00000000000020	1.00000000000000	1.00000000000000

Table 2 Iterates of modified Mann, modified Picard-Mann hybrid, and modified Picard-Ishikawa hybrid iterations for $x_1 = 1.9$

Iterate	The modified Mann iteration process	The modified Picard-Mann hybrid iteration	The modified Picard-Ishikawa hybrid iteration
x_1	1.90000000000000	1.90000000000000	1.90000000000000
x_2	1.13027756377320	0.95262315543817	1.00000000000000
x_3	1.04342585459107	1.00000000000000	1.00000000000000
x_4	1.01085646364777	1.00000000000000	1.00000000000000
x_5	1.00217129272955	1.00000000000000	1.00000000000000
x_6	1.00036188212159	1.00000000000000	1.00000000000000
x_7	1.00005169744594	1.00000000000000	1.00000000000000
x_8	1.00000646218074	1.00000000000000	1.00000000000000
x_9	1.00000071802008	1.00000000000000	1.00000000000000
x_{10}	1.00000007180201	1.00000000000000	1.00000000000000
x_{11}	1.00000000652746	1.00000000000000	1.00000000000000
x_{12}	1.00000000054395	1.00000000000000	1.00000000000000
x_{13}	1.00000000004184	1.00000000000000	1.00000000000000
x_{14}	1.00000000000299	1.00000000000000	1.00000000000000
x_{15}	1.00000000000020	1.00000000000000	1.00000000000000

Let $\alpha_n := \frac{n}{n+1}$ and $\beta_n := 1$ for all $n \in \mathbb{N}$. By using MATLAB, we computed the iterates of (1.10), (1.9), and (2.1) for two different initial points $x_1 = 1.5$ and $x_1 = 1.9$. The numerical experiments of all iterations for approximating the fixed point 1 are given in Tables 1 and 2. Moreover, the convergence behavior of all iteration is shown in Figure 1.

In Figures 2 and 3, we give the convergence behavior of the iterates of (1.10), (1.9), and (2.1) for some initial point under the different control conditions.

Next, we will give an example to show the nontrivial difference between the rate of convergence of the modified Picard-Ishikawa hybrid iteration process (2.1) with the modified Picard-Mann hybrid iteration process (1.9).

Example 3.2 Let $X := \mathbb{R}$ be a usual metric space with the metric d, which is also a complete CAT(0) space, and $C := [1, 999]$. We see that C is a bounded closed convex subset of X.

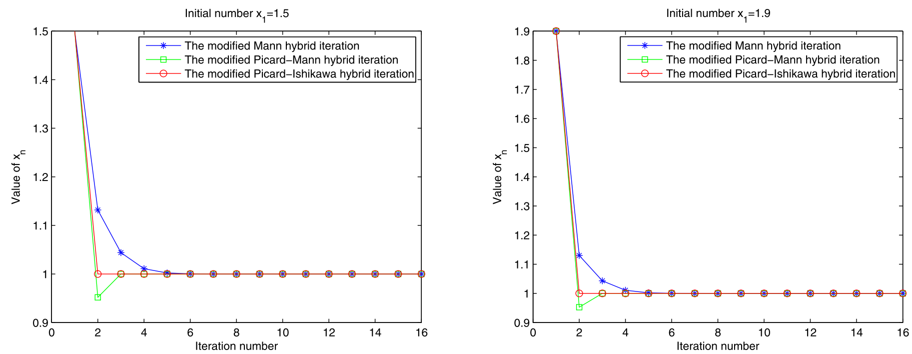


Figure 1 Convergence behavior for Example 3.1 in the cases $\alpha_n := \frac{n}{n+1}$ and $\beta_n := 1$ for all $n \in \mathbb{N}$.

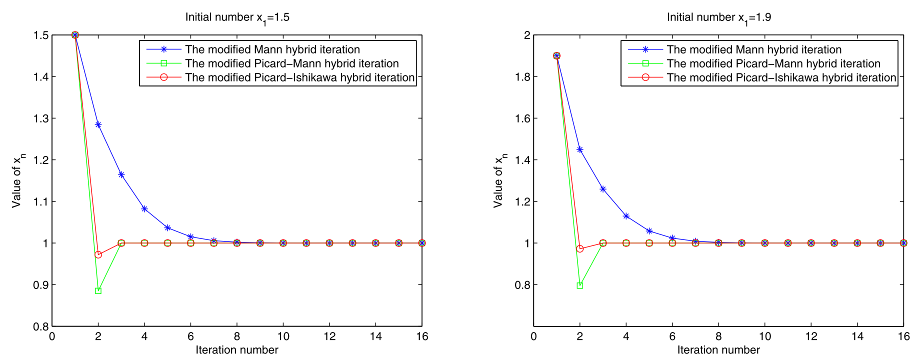


Figure 2 Convergence behavior for Example 3.1 in the cases $\alpha_n := 1 - \frac{1}{\sqrt{n+1}}$ and $\beta_n := \frac{1}{n+1}$ for all $n \in \mathbb{N}$.

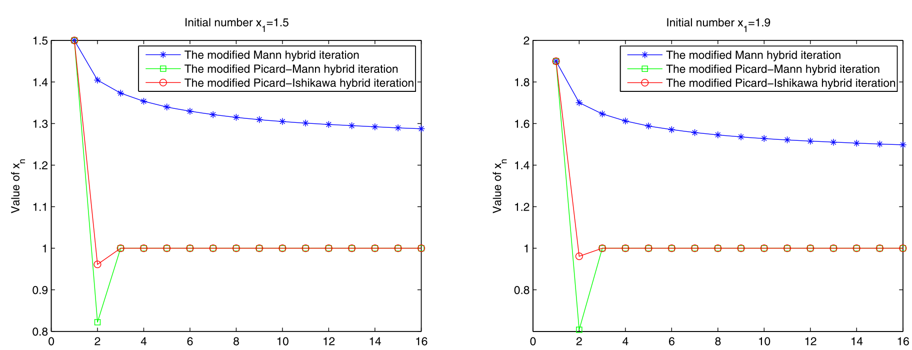


Figure 3 Convergence behavior for Example 3.1 in the cases $\alpha_n := \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$ and $\beta_n := \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Define a mapping $T : C \to C$ by

$$Tx = \sqrt{x^2 - 8x + 40}.$$

It is easy to see that T is a continuous uniformly L-Lipschitzian and a total asymptotically nonexpansive mapping with $F(T) = \{5\}$.

Figure 4 Convergence behavior for Example 3.2 in the cases $\alpha_n := \frac{n}{n+1}$ and $\beta_n := 1$ for all $n \in \mathbb{N}$.

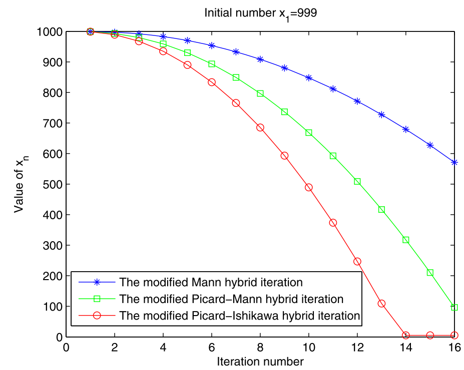


Figure 5 Convergence behavior for Example 3.2 in the cases $\alpha_n := 1 - \frac{1}{\sqrt{n+1}}$ and $\beta_n := \frac{1}{n+1}$ for all $n \in \mathbb{N}$.

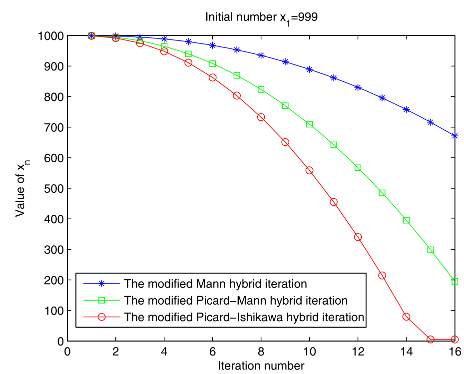
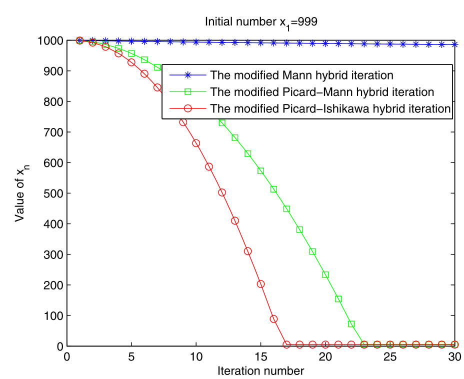


Figure 6 Convergence behavior for Example 3.2 in the cases $\alpha_n := \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$ and $\beta_n := \frac{n}{n+1}$ for all $n \in \mathbb{N}$.



Let $\alpha_n := \frac{n}{n+1}$ and $\beta_n := 1$ for all $n \in \mathbb{N}$. By using MATLAB, we computed the iterates of (1.10), (1.9), and (2.1) for an initial point $x_1 = 999$. The convergence behavior of all iterations for approximating the fixed point 5 are given in Figure 4.

In Figures 5 and 6, we give the convergence behavior of the iterates of (1.10), (1.9), and (2.1) for some initial point under the different control conditions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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