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Common fixed point theorems under $(\mathcal{R}, \mathcal{S})$-contractivity conditions

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Abstract

Very recently, Roldán-López-de-Hierro and Shahzad introduced the notion of R-contractions as an extension of several notions given by different researchers (for instance, R-contractions generalize Meir-Keeler contractions, $\mathcal{Z}$-contractions - involving simulation functions - by Khojasteh *et al.*, manageable contractions by Du and Khojasteh, Geraghty's contractions, Banach contractions, *etc.*). In this manuscript, we use R-functions to present existence and uniqueness coincidence (and common fixed) point results under a contractivity condition that extend some celebrated contractive mappings. In our main theorems, we employ a binary relation on the metric space, which does not have to be a partial order. Finally, we illustrate our technique with an example in which other previous statements cannot be applied: in fact, we show how to apply our main results to a new kind of contractivity conditions which cannot be expressed in separate variables.

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Keywords: R-function; R-contraction; simulation function; manageable function; coincidence point theorem

1 Introduction

Taking into account its applications to several fields of study, fixed point theory has demonstrated to be a powerful branch of nonlinear analysis. All results in this area are inspired on the Banach contractive mapping principle, introduced in 1922. The way in which the most recent results generalize the initial theorem are diverse. Some manuscripts presented very general contractivity conditions (see [1–4]), especially using auxiliary functions (see [5–8]), other papers were developed in abstract metric spaces (see [9–11]), some contributions involved additional structures like partial orders (see [12, 13]) and even multidimensional fixed/coincidence points were introduced (see [14–17]).

Among other techniques, there are two basic ways in order to improve the original Banach theorem.

(1) On the one hand, most of authors have introduced contractivity conditions each time weaker. Thus, there are fewer requirements for checking that a mapping is contractive.

(2) On the other hand, several assumptions as regards the analytic and geometric elements that are considered in the statements have been appearing. For instance, there are many results in which the metric space is not necessarily complete (this

condition has been replaced by the completeness of an appropriate subset) and, even more, we do not need to consider a metric space (many results have been demonstrated by using quasi-metric spaces and pseudo-quasi-metric spaces).

Following the first line of research, in recent times, Khojasteh *et al.* [18] introduced the notion of $\mathcal{Z}$*-contraction* by using a new class of auxiliary functions called *simulation functions*. This kind of functions have attracted much attention because they are useful to express a great family of contractivity conditions that were well known in the field of fixed point theory. Immediately afterward, Roldán-López-de-Hierro *et al.* [19] extended the family of simulation functions by avoiding a symmetric condition that was implicitly considered in the original definition.

Very recently, inspired by $\mathcal{Z}$-contractions, Roldán-López-de-Hierro and Shahzad [20] introduced the notion of R-contractions as an extension of several notions given by different researchers. R-contractions do not only extend the class of $\mathcal{Z}$-contractions but they also generalize manageable contractions by Du and Khojasteh, Geraghty's contractions, Banach contractions, etc. Furthermore, these authors succeeded in proving that Meir-Keerler contractions are also R-contractions. Like $\mathcal{Z}$-contractions are based in manageable functions, the key piece of an R-contraction is its associated underlying R-function, which satisfies two independent conditions involving sequences of nonnegative real numbers. R-functions have only two arguments, in such way that they are appropriate in order to study contractivity conditions that only involve two elements: the distance between two points and the distance between their images by a self-mapping. Many contractivity conditions were introduced in the past by using these two terms but, in general, they always were conditions in separate variables (that is, these terms were the arguments of different auxiliary functions).

In 1986, Turinici [21] gave an initial result for guaranteeing existence of fixed points by involving a new algebraic structure: a partial order on the metric space. However, the most celebrated results in this line of research, with applications to matrix equations, were given by Ran and Reurings [12], and, later, by Nieto and Rodríguez-López [13]. Following these results, the contractivity condition does not need to hold for all pairs of points: it must only be satisfied by points that are related through the partial order. Thus, the continuity of the involved contractive mapping cannot be derived from the contractivity condition (as in the Banach theorem): in fact, Nieto and Rodríguez-López replaced such condition by the regularity of the partially ordered ambient metric space (which is a condition about the behavior of nondecreasing convergent sequences).

In this manuscript, we use R-functions to present existence and uniqueness coincidence (and common fixed) point results under a contractivity condition that extend some celebrated contractive mappings. In our main theorems, we employ a binary relation on the metric space that does not have to be either a partial order nor a transitive relation. Thus, we restrict very much the set of pairs of point for which the contractivity condition must hold. Furthermore, we replace the assumption as regards the completeness of the metric space by precompleteness of an appropriate subspace. Moreover, in our main results, we do not assume the existence of a point that serves as initial condition: we suppose a weaker condition about the existence of a Picard-Jungck sequence. Finally, we illustrate our technique with an example in which other previous statements (like the Dutta and Choudhury theorem, among others) cannot be applied: in fact, we show how to apply our main results to a new kind of contractivity conditions which cannot be expressed in separate variables.

2 Preliminaries

Let us introduce here basic notions and fundamental results. From now on, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ stands for the set of all nonnegative integers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{R}$ denotes the set of all real numbers. Henceforth, X stands for a nonempty set and $T, g : X \to X$ will denote two self-mappings. For simplicity, we write Tx instead of $T(x)$.

We will say that a point $x \in X$ is a:

- *fixed point of T* if $Tx = x$ ($\mathrm{Fix}(T)$ will denote the set of all fixed points of T);
- *coincidence point of T and g* if $Tx = gx$ ($\mathrm{Coin}(T, g)$ will denote the set of all coincidence points of T and g);
- *point of coincidence of T and g* if there exists $z \in \mathrm{Coin}(T, g)$ such that $x = Tz$;
- *common fixed point of T and g* if $Tx = gx = x$.

Inspired by [22], given a point $x_0 \in X$, a *Picard-Jungck sequence of the pair (T, g) based on x_0* is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.

The mappings T and g are *commuting* if $Tgx = gTx$ for all $x \in X$. A pair $\{T, g\}$ is *weakly compatible* if $Tgx = gTx$ for all $x \in X$ such that $Tx = gx$.

A *binary relation on X* is a nonempty subset $\mathcal{S}$ of $X \times X$. For simplicity, we denote $x \,\mathcal{S}\, y$ if $(x, y) \in \mathcal{S}$ (in some cases, we also use the symbol $\prec$ to denote a binary relation on X because the notation $x \prec y$ can be more usual for the reader). We say that x and y are *$\mathcal{S}$-comparable* if $x \,\mathcal{S}\, y$ or $y \,\mathcal{S}\, x$. A binary relation $\mathcal{S}$ on X is *reflexive* if $x \,\mathcal{S}\, x$ for all $x \in X$; it is *transitive* if $x \,\mathcal{S}\, z$ for all $x, y, z \in X$ such that $x \,\mathcal{S}\, y$ and $y \,\mathcal{S}\, z$; and it is *antisymmetric* if $x \,\mathcal{S}\, y$ and $y \,\mathcal{S}\, x$ imply $x = y$. A *preorder* (or a *quasiorder*) is a reflexive and transitive binary relation. And a *partial order* is an antisymmetric preorder.

The notion of a *metric space* and the concepts of a *convergent sequence* and a *Cauchy sequence* in a metric space can be found, for instance, in [23]. We will write $\{x_n\} \to x$ when a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X converges to $x \in X$ in the metric space (X, d). A metric space (X, d) is *complete* if every Cauchy sequence in X converges to some point of X. The limit of a convergent sequence in a metric space is unique. If (X, d) is a metric space, we denote the range of d by

$$\mathrm{ran}(d) = \big\{ d(x, y) : x, y \in X \big\} \subseteq [0, \infty).$$

We say that a sequence $\{x_n\} \subseteq X$ is *asymptotically regular on (X, d)* if $\{d(x_n, x_{n+1})\} \to 0$.

In a metric space (X, d), a mapping $T : X \to X$ is *continuous at a point $z \in X$* if $\{Tx_n\} \to Tz$ for all sequence $\{x_n\}$ in X such that $\{x_n\} \to z$. T is *continuous* if it is continuous at every point of X.

Definition 1 A metric space (X, d) endowed with a binary relation $\mathcal{S}$ is *$\mathcal{S}$-nondecreasing-regular* if for all $\mathcal{S}$-nondecreasing sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to u \in X$, it follows that $x_n \,\mathcal{S}\, u$ for all $n \in \mathbb{N}$.

Lemma 2 (Roldán *et al.* [24], Lemma 16, Berzig *et al.* [7], Lemma 13) *Let (X, d) be a metric space and let $\{x_n\} \subseteq X$ be a sequence. If $\{x_n\}$ is not Cauchy in (X, d), then there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that*

$$k \le n(k) < m(k) < n(k+1) \quad \textit{and} \quad d(x_{n(k)}, x_{m(k)-1}) \le \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) \quad \textit{for all } k \in \mathbb{N}.$$

Furthermore, if $\{d(x_n, x_{n+1})\} \to 0$, *then*

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon_0.$$

The following notion was introduced in [20].

Definition 3 (Roldán-López-de-Hierro and Shahzad [20]) Let $D \subseteq \mathbb{R}$ be a nonempty subset and let $\varrho : D \times D \to \mathbb{R}$ be a function. We say that ϱ is an *R-function* if it satisfies the following two conditions.

(ϱ_1) If $\{a_n\} \subset (0,\infty) \cap D$ is a sequence such that $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.
(ϱ_2) If $\{a_n\}, \{b_n\} \subset (0,\infty) \cap D$ are two sequences converging to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

We denote by R_D the family of all R-functions whose domain is $D \times D$.

In some cases, given a function $\varrho : D \times D \to \mathbb{R}$, we will also consider the following properties.

(ϱ_3) If $\{a_n\}, \{b_n\} \subset (0,\infty) \cap D$ are two sequences such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.
(ϱ_4) If $\{a_n\}, \{b_n\} \subset [0,\infty) \cap D$ are two sequences such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.

Notice that (ϱ_4) $\Rightarrow$ (ϱ_3).

Proposition 4
1. *If a function* $\varrho : D \times D \to \mathbb{R}$ *verifies* $\varrho(t,s) \leq s - t$ *for all* $t, s \in D \cap (0,\infty)$, *then* ($\varrho_3$) *holds.*
2. *If a function* $\varrho : D \times D \to \mathbb{R}$ *verifies* $\varrho(t,s) \leq s - t$ *for all* $t, s \in D \cap [0,\infty)$, *then* ($\varrho_3$) *and* ($\varrho_4$) *holds.*

Proposition 5 *If* $\varrho \in R_D$, *then* $\varrho(t,t) \leq 0$ *for all* $t \in (0,\infty) \cap D$.

Definition 6 (Roldán-López-de-Hierro and Shahzad [20]) Let (X,d) be a metric space and let $T : X \to X$ be a mapping. We will say that T is an *R-contraction* if there exists an R-function $\varrho : D \times D \to \mathbb{R}$ such that $\operatorname{ran}(d) \subseteq D$ and

$$\varrho\big(d(Tx, Ty), d(x,y)\big) > 0 \quad \text{for all } x, y \in X \text{ such that } x \neq y. \tag{1}$$

In such a case, we will say that T is an *R-contraction with respect to* ϱ.

In [20], the authors give a wide range of R-functions and R-contractions. In fact, Meir-Keeler contractions [25, 26], $\mathcal{Z}$-contractions [18, 19], manageable contractions [27], and Geraghty contractions [5] are particular cases of R-contractions. Another example is the following one.

Theorem 7 (Roldán-López-de-Hierro and Shahzad [20]) *Let* $\psi, \varphi : [0,\infty) \to [0,\infty)$ *be two functions such that* ψ *is nondecreasing and continuous from the right,* φ *is lower semi-*

continuous and $\varphi^{-1}(\{0\}) = \{0\}$. *Let* $\varrho_{\psi,\varphi} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *be the function defined by*

$$\varrho_{\psi,\varphi}(t,s) = \psi(s) - \varphi(s) - \psi(t) \quad \text{for all } t,s \in [0,\infty).$$

Then $\varrho_{\psi,\varphi}$ *is an R-function on* $[0,\infty)$. *Furthermore,* $\varrho_{\psi,\varphi}$ *satisfies condition* (ϱ_3).

As a consequence of the previous result and their main theorems, the authors obtained the following consequence.

Corollary 8 (Roldán-López-de-Hierro and Shahzad [20]) *Let* (X,d) *be a complete metric space and let* $T : X \to X$ *be a self-mapping. Assume that there exist two functions* $\psi,\phi : [0,\infty) \to [0,\infty)$ *such that*

$$\psi\bigl(d(Tx,Ty)\bigr) \le \psi\bigl(d(x,y)\bigr) - \phi\bigl(d(x,y)\bigr) \quad \text{for all } x,y \in X.$$

If ψ *is nondecreasing and continuous from the right,* φ *is lower semi-continuous, and* $\varphi^{-1}(\{0\}) = \{0\}$, *then* T *has a unique fixed point.*

The previous statement generalizes the well-known Dutta and Choudhury theorem.

Theorem 9 (Dutta and Choudhury [20], Theorem 2.1) *Let* (X,d) *be a complete metric space and let* $T : X \to X$ *be a self-mapping satisfying the inequality*

$$\psi\bigl(d(Tx,Ty)\bigr) \le \psi\bigl(d(x,y)\bigr) - \phi\bigl(d(x,y)\bigr) \quad \text{for all } x,y \in X,$$

where $\psi,\varphi : [0,\infty) \to [0,\infty)$ *are both continuous and monotone nondecreasing functions with* $\psi(t) = 0 = \phi(t)$ *if, and only if,* $t = 0$.

Then T *has a unique fixed point.*

3 Binary relations on a set

Throughout this section, T and g will always denote self-mappings on X and $\mathcal{S}$ will be a binary relation on X. Recall that we will write $x \mathcal{S}^* y$ when $x \mathcal{S} y$ and $x \neq y$. We introduce some properties that a binary relation can verify.

Definition 10 Given a nonempty subset $A \subseteq X$, we will say that the binary relation $\mathcal{S}$ is:

- *transitive on* A if $x \mathcal{S} z$ for all $x,y,z \in A$ such that $x \mathcal{S} y$ and $y \mathcal{S} z$;
- *transitive* if it is transitive on X;
- g*-transitive* if it is transitive on $g(X)$ (that is, $gx \mathcal{S} gy$ and $gy \mathcal{S} gz$ imply $gx \mathcal{S} gz$);
- (T,g)*-transitive* if $gx \mathcal{S} Ty$ for all $x,y \in X$ such that $gx \mathcal{S} gy$ and $gy \mathcal{S} Ty$;
- (T,g)*-compatible* if $Tx = Ty$ for all $x,y \in X$ such that $gx = gy$ and $gx \mathcal{S} gy$;
- g*-closed* if $gx \mathcal{S} gy$ for all $x,y \in X$ such that $x \mathcal{S} y$.

Proposition 11 *Every transitive binary relation is* g*-transitive and* (T,g)*-transitive, whatever* T *and* g.

The following examples show that the notions of g-transitivity and (T,g)-transitivity properly extend the notion of transitivity throughout independent notions.

Example 12 Let $X = \{0,1,2\}$ and let us define the self-mappings T and g and the binary relation $\mathcal{S}$ as follows:

$$Tx = 2; \qquad gx = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \in \{1,2\}; \end{cases} \qquad x\,\mathcal{S}\,y \quad \Leftrightarrow \quad \begin{cases} x = y \quad \text{or} \\ (x,y) \in \{(0,1),(1,2)\}. \end{cases}$$

We claim that $\mathcal{S}$ is g-transitive but it is neither (T,g)-transitive nor transitive. Let $x,y,z \in X$ be such that $gx\,\mathcal{S}\,gy$ and $gy\,\mathcal{S}\,gz$. Therefore, $gx \leq gy$ and $gy \leq gz$, so $gx \leq gz$. Moreover, $gx, gz \in \{0,1\}$. Therefore, $gx\,\mathcal{S}\,gz$, which implies that $\mathcal{S}$ is g-transitive. To prove that $\mathcal{S}$ is not (T,g)-transitive, let $x_0 = 0$ and $y_0 = 1$. Then $gx_0 = 0\,\mathcal{S}\,1 = gy_0$ and $gy_0 = 1\,\mathcal{S}\,2 = Ty_0$. However, $gx_0 \not\mathcal{S}\, Ty_0$ because $0 \not\mathcal{S}\, 2$. Hence, $\mathcal{S}$ is not (T,g)-compatible. By Proposition 11, $\mathcal{S}$ cannot be transitive.

Example 13 Let $X = \{0,1,2\}$ and let us define the self-mappings T and g and the binary relation $\mathcal{S}$ as follows:

$$Tx = 1; \qquad gx = x; \qquad x\,\mathcal{S}\,y \quad \Leftrightarrow \quad \begin{cases} x = y \quad \text{or} \\ (x,y) \in \{(0,1),(1,2)\}. \end{cases}$$

We claim that $\mathcal{S}$ is (T,g)-transitive but it is neither g-transitive nor transitive. Let $x,y \in X$ be such that $gx\,\mathcal{S}\,gy$ and $gy\,\mathcal{S}\,Ty = 1$. Then $y = gy \in \{0,1\}$. From $gx\,\mathcal{S}\,gy$ it follows that $x \leq y$, so $x \in \{0,1\}$. Anyway, $gx\,\mathcal{S}\,1 = Ty$, so $\mathcal{S}$ is (T,g)-transitive. However, if we take $x_0 = 0$, $y_0 = 1$, and $z_0 = 2$, we have $gx_0 = 0\,\mathcal{S}\,1 = gy_0$ and $gy_0 = 1\,\mathcal{S}\,2 = gz_0$, but it is false that $gx_0\,\mathcal{S}\,gz_0$ because $0 \not\mathcal{S}\, 2$. Hence, $\mathcal{S}$ is neither g-transitive nor transitive.

Despite the above examples, there are some relationships between g-transitivity and (T,g)-transitivity.

Proposition 14 *If $\mathcal{S}$ is g-transitive and $T(X) \subseteq g(X)$, then $\mathcal{S}$ is (T,g)-transitive.*

Proof Let $x,y \in X$ be such that $gx\,\mathcal{S}\,gy$ and $gy\,\mathcal{S}\,Ty$. Since $Ty \in T(X) \subseteq g(X)$, there is $z \in X$ such that $Ty = gz$. Hence $gx\,\mathcal{S}\,gy$ and $gy\,\mathcal{S}\,gz$. As $\mathcal{S}$ is g-transitive, then $gx\,\mathcal{S}\,gz$, so $gx\,\mathcal{S}\,Ty$. This proves that $\mathcal{S}$ is (T,g)-transitive. □

The following example shows that g-transitivity and (T,g)-transitivity do not imply any of the properties that a partial order satisfies.

Example 15 Let $X = [0,\infty)$ and let define $T,g : X \to X$ by $gx = x + 3$ and $Tx = x + 4$ for all $x \in X$. Let $\mathcal{S}$ be the binary relation on X given by $x\,\mathcal{S}\,y$ if

$$3 \leq x \leq y \quad \text{or} \quad (x,y) \in \bigl\{(0,1),(1,0),(1,2)\bigr\}.$$

Then $\mathcal{S}$ is neither reflexive ($0 \not\mathcal{S}\, 0$), nor transitive ($0\,\mathcal{S}\,1$ and $1\,\mathcal{S}\,2$, but $0 \not\mathcal{S}\, 2$) nor anti-symmetric ($0\,\mathcal{S}\,1$ and $0\,\mathcal{S}\,1$, but $0 \neq 1$). However, $\mathcal{S}$ is g-transitive, (T,g)-transitive, and (T,g)-compatible.

Definition 16 We say that T is $(g,\mathcal{S})$-*nondecreasing* if $Tx\,\mathcal{S}\,Ty$ for all $x,y \in X$ such that $gx\,\mathcal{S}\,gy$.

Proposition 17 *If $\mathcal{S}$ is reflexive and antisymmetric, and T is $(g,\mathcal{S})$-nondecreasing, then $\mathcal{S}$ is (T,g)-compatible.*

Proof Let $x,y\in X$ be such that $gx=gy$ and $gx\,\mathcal{S}\,gy$. As $\mathcal{S}$ is reflexive, $gx\,\mathcal{S}\,gy$ and $gy\,\mathcal{S}\,gx$. Since T is $(g,\mathcal{S})$-nondecreasing, $Tx\,\mathcal{S}\,Ty$ and $Ty\,\mathcal{S}\,Tx$. As $\mathcal{S}$ is antisymmetric, $Tx=Ty$. □

Proposition 18 *If g is injective, then any binary relation $\mathcal{S}$ is (T,g)-compatible, whatever T.*

Proof Let $x,y\in X$ be such that $gx=gy$ and $gx\,\mathcal{S}\,gy$. As g is injective, $x=y$, so $Tx=Ty$. □

Definition 19 We will write $T(X)\subseteq_{\mathcal{S}} g(X)$ if for all $x\in X$ there exists $y\in X$ such that $Tx=gy$ and $gx\,\mathcal{S}\,gy$.

Clearly, if $T(X)\subseteq_{\mathcal{S}} g(X)$, then $T(X)\subseteq g(X)$.

Definition 20 A subset A of a metric space (X,d) is *precomplete* if each Cauchy sequence $\{a_n\}\subseteq A$ is convergent to a point of X.

Remark 21
1. The empty subset is precomplete.
2. Every complete subset of X is precomplete.
3. Every subset of a complete metric space is also precomplete.

Example 22 Although $X=(0,3)$, endowed with the Euclidean metric, is not complete, and $A=(1,2)$ is not complete, the set A is precomplete.

Proposition 23 *If $A\subseteq B\subseteq X$ and B is precomplete, then A is also precomplete.*

Remark 24 If $T(X)\subseteq g(X)$ and one of X, or $g(X)$ or $T(X)$ is complete, then $T(X)$ is precomplete.

When a binary relation $\mathcal{S}$ is not symmetric, we can consider '*right-notions*' and '*left-notions*' depending on the character of the involved sequences. For instance, '*right-notions*' corresponds to definitions in which a sequence $\{x_n\}\subseteq X$ satisfies $x_n\,\mathcal{S}\,x_m$ for all $n,m\in\mathbb{N}$ with $n<m$, and '*left-notions*' are associated to condition $x_n\,\mathcal{S}\,x_m$ with $n>m$. In this paper, we consider the first ones, and we introduce *right-regularity*, $(O,\mathcal{S})$*-right-compatibility*, $(T,g,\mathcal{S})$*-right-Picard-Jungck sequences* and $\mathcal{S}$*-right-continuity*. However, we will omit the term '*right*'.

Definition 25 Let $T,g:X\to X$ be two mappings, let $\{x_n\}_{n\geq 0}\subseteq X$ be a sequence and let $\mathcal{S}$ be a binary relation on X. We say that $\{x_n\}$ is a:
- (T,g)*-Picard-Jungck sequence* if

$$gx_{n+1}=Tx_n \quad \text{for all } n\in\mathbb{N} \tag{2}$$

(in such a case, we say that $\{x_n\}$ is *based on the initial point* x_0);

- $(T,g,\mathcal{S})$-*Picard-Jungck sequence* if it is a (T,g)-Picard-Jungck sequence and

$$gx_n \,\mathcal{S}\, gx_m \quad \text{for all } n,m \in \mathbb{N} \text{ such that } n < m. \tag{3}$$

The following result shows some sufficient conditions in order to guarantee the existence of a $(T,g,\mathcal{S})$-Picard-Jungck sequence.

Proposition 26 *Suppose that the binary relation $\mathcal{S}$ is g-transitive or (T,g)-transitive.*

1. *If $TX \subseteq g(X)$, T is $(g,\mathcal{S})$-nondecreasing and there exists a point $x_0 \in X$ such that $gx_0 \,\mathcal{S}\, Tx_0$, then there exists a $(T,g,\mathcal{S})$-Picard-Jungck sequence on X based on x_0.*
2. *If $T(X) \subseteq_{\mathcal{S}} g(X)$, then there exists a $(T,g,\mathcal{S})$-Picard-Jungck sequence on X based on each arbitrary point $x_0 \in X$.*

Proof Step 1. *We claim that there exists a sequence $\{x_n\} \subseteq X$ such that $gx_{n+1} = Tx_n$ and $gx_n \,\mathcal{S}\, gx_{n+1}$ for all $n \in \mathbb{N}$.*

(1) Since $Tx_0 \in T(X) \subseteq g(X)$, we can find $x_1 \in X$ such that $Tx_0 = gx_1$. As $Tx_1 \in T(X) \subseteq g(X)$, there is $x_2 \in X$ such that $Tx_1 = gx_2$. Moreover, as $gx_0 \,\mathcal{S}\, Tx_0 = gx_1$ and T is $(g,\mathcal{S})$-nondecreasing, $Tx_0 \,\mathcal{S}\, Tx_1$, which means that $gx_1 \,\mathcal{S}\, gx_2$. By induction, we can consider a sequence $\{x_n\} \subseteq X$ such that $gx_{n+1} = Tx_n$ and $gx_n \,\mathcal{S}\, gx_{n+1}$ for all $n \in \mathbb{N}$.

(2) Let $x_0 \in X$ be a point. Since $Tx_0 \in T(X) \subseteq_{\mathcal{S}} g(X)$, there is $x_1 \in X$ verifying $Tx_0 = gx_1$ and $gx_0 \,\mathcal{S}\, gx_1$. Repeating this argument, from $Tx_1 \in T(X) \subseteq_{\mathcal{S}} g(X)$ it follows that there is $x_2 \in X$ satisfying $Tx_1 = gx_2$ and $gx_1 \,\mathcal{S}\, gx_2$. By induction, Step 1 holds.

Step 2. *$\{x_n\}$ is a $(T,g,\mathcal{S})$-Picard-Jungck sequence.* On the one hand, assume that $\mathcal{S}$ is g-transitive. Then, for all $n,m \in \mathbb{N}$ such that $n < m$,

$$gx_n \,\mathcal{S}\, gx_{n+1}, \qquad gx_{n+1} \,\mathcal{S}\, gx_{n+2}, \qquad \ldots, \qquad gx_{m-1} \,\mathcal{S}\, gx_m \quad \Rightarrow \quad gx_n \,\mathcal{S}\, gx_m.$$

On the other hand, assume that $\mathcal{S}$ is (T,g)-transitive. Given $n \in \mathbb{N}$, we observe that

$$gx_n \,\mathcal{S}\, gx_{n+1}, \qquad gx_{n+1} \,\mathcal{S}\, gx_{n+2} = Tx_{n+1} \quad \Rightarrow \quad gx_n \,\mathcal{S}\, Tx_{n+1} = gx_{n+2}.$$

Repeating this argument,

$$gx_n \,\mathcal{S}\, gx_{n+2}, \qquad gx_{n+2} \,\mathcal{S}\, gx_{n+3} = Tx_{n+2} \quad \Rightarrow \quad gx_n \,\mathcal{S}\, Tx_{n+2} = gx_{n+3}.$$

By induction, $gx_n \,\mathcal{S}\, gx_{n+m}$ for all $n \in \mathbb{N}$ and all $m \in \mathbb{N}^*$. □

The same proof is valid when the binary relation is omitted.

Proposition 27 *If $TX \subseteq g(X)$, then there exists a (T,g)-Picard-Jungck sequence on X based on each $x_0 \in X$.*

From now on, let (X,d) be a metric space.

Definition 28 The map $T : X \to X$ is $\mathcal{S}$-*continuous* if $\{Tx_n\} \to Tu$ for all sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to u \in X$ and $x_n \,\mathcal{S}\, x_m$ for all $n,m \in \mathbb{N}$ with $n < m$.

Remark 29 If $T : X \to X$ is continuous and X is endowed with a binary relation $\mathcal{S}$, then T is $\mathcal{S}$-continuous.

Definition 30 If A is a subset of a metric space (X, d) endowed with a binary relation $\mathcal{S}$, we say that A is *$(d,\mathcal{S})$-regular* if for all sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to u \in X$ and $x_n \,\mathcal{S}\, x_m$ for all $n, m \in \mathbb{N}$ with $n < m$, we have $x_n \,\mathcal{S}\, u$ for all $n \in \mathbb{N}$.

The following definition extends some ideas that can be found on [28–30].

Definition 31 If (X, d) is a metric space and $\mathcal{S}$ is a binary relation on X, two mappings $T, g : X \to X$ are *$(O,\mathcal{S})$-compatible* if

$$\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$$

provided that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $gx_n \,\mathcal{S}\, gx_m$ for all $n < m$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

Clearly, commutativity implies $(O,\mathcal{S})$-compatibility.

4 Some coincidence point theorems under $(R,\mathcal{S})$-contractivity conditions

In this section, we employ a binary relation $\mathcal{S}$ to present some coincidence point results under R-contractivity conditions. If ϱ denotes an R-function, we will consider the following three classes of contractivity conditions.

(C_1) $\varrho(d(Tx, Ty), d(gx, gy)) > 0$ for all $x, y \in X$ such that $Tx \,\mathcal{S}^*\, Ty$ and $gx \,\mathcal{S}^*\, gy$.
(C_2) $\varrho(d(Tx, Ty), d(gx, gy)) > 0$ for each $x, y \in X$ verifying $gx \,\mathcal{S}^*\, gy$.
(C_3) $\varrho(d(Tx, Ty), d(gx, gy)) > 0$ for each $x, y \in X$ satisfying $x \neq y$ and $gx \,\mathcal{S}\, gy$.

Obviously, (C_3) $\Rightarrow$ (C_2) $\Rightarrow$ (C_1) because the only difference between them is the kind of points for which the inequality holds. The contractivity condition can also be useful to prove some properties of the binary relation.

Proposition 32 *Suppose that $\varrho : D \times D \to \mathbb{R}$ is a function for which* (C_3) *holds, and assume that $\varrho(t, 0) \leq 0$ for all $t > 0$. Then $\mathcal{S}$ is (T, g)-compatible.*

Proof Let $x, y \in X$ be such that $gx = gy$ and $gx \,\mathcal{S}\, gy$. By contradiction, assume that $Tx \neq Ty$. Then, necessarily, $x \neq y$. Let $t_0 = d(Tx, Ty) \in (0,\infty) \cap \operatorname{ran}(d) \subseteq (0,\infty) \cap D$. By ($C_3$), $\varrho(d(Tx, Ty), d(gx, gy)) = \varrho(t_0, 0) > 0$, which contradicts $\varrho(t_0, 0) \leq 0$. Thus $Tx = Ty$. □

4.1 Coincidence point theorems under $\mathcal{S}$-continuity

The first main result of the present manuscript is the following one, in which we use the weaker contractivity condition.

Theorem 33 *Let $\mathcal{S}$ be a binary relation on a metric space (X, d) and let $T, g : X \to X$ be two $\mathcal{S}$-continuous mappings such that $T(X)$ is precomplete. Suppose that conditions* (A) *and* (B) *holds.*

(A) *There is on X a $(T, g,\mathcal{S})$-Picard-Jungck sequence.*

(B) *There is an R-function* $\varrho \in R_D$ *such that* $\operatorname{ran}(d) \subseteq D$ *and*

$$\varrho\big(d(Tx,Ty),d(gx,gy)\big) > 0 \quad \text{for all } x,y \in X \text{ such that } Tx\,\mathcal{S}^*\,Ty \text{ and } gx\,\mathcal{S}^*\,gy. \tag{4}$$

In addition to this, suppose, at least, one of the following assumptions holds.

(a) *The pair* (T,g) *is* $(O,\mathcal{S})$*-compatible*; *or*

(b) T *and* g *are commuting.*

Then T *and* g *have a coincidence point. In fact, if* $\{x_n\}$ *is any* $(T,g,\mathcal{S})$*-Picard-Jungck sequence, either* $\{gx_n\}$ *contains a coincidence point of* T *and* g, *or* $\{gx_n\}$ *converges to a coincidence point of* T *and* g.

We must point out the weakness of all hypotheses in the previous theorem because the following conditions, which we can find in many previous results in this field, are stronger.

- If T and g are continuous, then T and g are $\mathcal{S}$-continuous.
- If there is a subset A such that $T(X) \subseteq A \subseteq X$ and A is precomplete or complete, then $T(X)$ is precomplete. In particular, if one of $T(X)$, or $g(X)$, or X is complete, then $T(X)$ is precomplete.
- By Proposition 26, hypothesis (A) is guaranteed under the following conditions.

  (A′) $\mathcal{S}$ is (T,g)-transitive (or g-transitive), $TX \subseteq g(X)$, T is $(g,\mathcal{S})$-nondecreasing and there exists a point $x_0 \in X$ such that $gx_0\,\mathcal{S}\,Tx_0$.

  (A″) $\mathcal{S}$ is (T,g)-transitive (or g-transitive) and $T(X) \subseteq_{\mathcal{S}} g(X)$.

Proof By hypothesis (A), we can find a $(T,g,\mathcal{S})$-Picard-Jungck sequence $\{x_n\}_{n\geq 0}$ on X, that is, a sequence verifying $gx_{n+1} = Tx_n$ and $gx_n\,\mathcal{S}\,gx_m$ for all $n < m$, $n,m \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $gx_{n_0+1} = gx_{n_0}$, then x_{n_0} is a coincidence point of T and g. Assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Hence $gx_n\,\mathcal{S}^*\,gx_{n+1}$ and $Tx_n\,\mathcal{S}^*\,Tx_{n+1}$ for all $n \in \mathbb{N}$. Let $\{a_n\} \subset (0,\infty)$ be the sequence defined by $a_n = d(gx_n,gx_{n+1}) > 0$ for all $n \in \mathbb{N}$. Using the contractivity condition (4),

$$\begin{aligned}\varrho(a_{n+1},a_n) &= \varrho\big(d(gx_{n+1},gx_{n+2}),d(gx_n,gx_{n+1})\big)\\ &= \varrho\big(d(Tx_n,Tx_{n+1}),d(gx_n,gx_{n+1})\big) > 0\end{aligned}$$

for all $n \in \mathbb{N}$. Applying (ϱ_1) we deduce that $\{d(gx_n,gx_{n+1})\} = \{a_n\} \to 0$, that is, $\{gx_n\}$ is an asymptotically regular sequence.

Next we prove that $\{gx_n\}$ is Cauchy reasoning by contradiction. If it is not Cauchy, then there are $\varepsilon_0 > 0$ and two subsequences $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ of $\{gx_n\}$ such that

$$\begin{aligned}&k \leq n(k) < m(k), \quad d(gx_{n(k)},gx_{m(k)-1}) \leq \varepsilon_0 < d(gx_{n(k)},gx_{m(k)}) \quad \text{for all } k \in \mathbb{N},\\ &\lim_{k\to\infty} d(gx_{n(k)},gx_{m(k)}) = \lim_{k\to\infty} d(gx_{n(k)-1},gx_{m(k)-1}) = \varepsilon_0.\end{aligned}$$

As the sequences $\{d(gx_{n(k)},gx_{m(k)})\}$ and $\{d(gx_{n(k)-1},gx_{m(k)-1})\}$ converge to $\varepsilon_0 > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$d(gx_{n(k)},gx_{m(k)}) > 0 \quad \text{and} \quad d(gx_{n(k)-1},gx_{m(k)-1}) > 0$$

for all $k \geq k_0$. In particular, $d(Tx_{n(k)-1}, Tx_{m(k)-1}) = d(gx_{n(k)}, gx_{m(k)}) > 0$ for all $k \geq k_0$, which implies that

$$Tx_{n(k)-1}\, \mathcal{S}^*\, Tx_{m(k)-1} \quad \text{and} \quad gx_{n(k)-1}\, \mathcal{S}^*\, gx_{m(k)-1} \quad \text{for all } k \geq k_0.$$

Let $L = \varepsilon_0 > 0$, $\{a_k = d(gx_{n(k)}, gx_{m(k)})\}_{k\geq k_0} \to L$ and $\{b_k = d(gx_{n(k)-1}, gx_{m(k)-1})\}_{k\geq k_0} \to L$. Since $L = \varepsilon_0 < d(gx_{n(k)}, gx_{m(k)}) = a_k$ and

$$\begin{aligned}\varrho(a_k, b_k) &= \varrho\big(d(gx_{n(k)}, gx_{m(k)}), d(gx_{n(k)-1}, gx_{m(k)-1})\big)\\ &= \varrho\big(d(Tx_{n(k)-1}, Tx_{m(k)-1}), d(gx_{n(k)-1}, gx_{m(k)-1})\big) > 0\end{aligned}$$

for all $k \geq k_0$, condition (ϱ_2) guarantees that $\varepsilon_0 = L = 0$, which is a contradiction. As a result, the sequence $\{gx_n\}$ is Cauchy. Since $\{gx_{n+1} = Tx_n\} \subseteq T(X)$ and $T(X)$ is precomplete, there exists $z \in X$ such that $\{gx_n\} \to z$. Furthermore, as T and g are $\mathcal{S}$-continuous and $gx_n\, \mathcal{S}\, gx_m$ for all $n < m$, then $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Next, we consider two cases.

Case (a). *Assume the* $(O, \mathcal{S})$*-compatibility of the pair* (T, g). Taking into account that $gx_n\, \mathcal{S}\, gx_m$ for all $n < m$ and $\{Tx_n = gx_{n+1}\} \to z$, the $(O, \mathcal{S})$-compatibility of (T, g) yields

$$\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0.$$

Notice that

$$d(Tz, gz) = d\Big(\lim_{n\to\infty} Tgx_n, \lim_{n\to\infty} ggx_{n+1}\Big) = \lim_{n\to\infty} d(Tgx_n, gTx_n) = 0,$$

so $Tz = gz$ and z is a coincidence point of T and g.

Case (b). *Assume that T and g are commuting.* Then Case (a) is applicable because commutativity implies $(O, \mathcal{S})$-compatibility. □

The following consequence is obtained by using the binary relation $x\, \mathcal{S}\, y$ for all $x, y \in X$.

Corollary 34 *Let (X, d) be a metric space and let $T, g : X \to X$ be two continuous mappings such that $T(X)$ is precomplete. Suppose that conditions* (A) *and* (B) *holds.*

(A) *There exists on X a (T, g)-Picard-Jungck sequence.*
(B) *There exists an R-function $\varrho \in R_D$ such that* $\operatorname{ran}(d) \subseteq D$ *and*

$$\varrho\big(d(Tx, Ty), d(gx, gy)\big) > 0 \quad \textit{for all } x, y \in X \textit{ such that } Tx \neq Ty \textit{ and } gx \neq gy.$$

Additionally, assume that the pair (T, g) is O-compatible or T and g are commuting.

Then T and g have, at least, a coincidence point. In fact, if $\{x_n\}$ is any (T, g)-Picard-Jungck sequence, either $\{gx_n\}$ contains a coincidence point of T and g, or $\{gx_n\}$ converges to a coincidence point of T and g.

In the following result, we denote by $\prec$ a transitive binary relation (for instance, a preorder or a partial order), which is not necessarily reflexive. In this case, we replace hypothesis (A) by (A′) by virtue of Proposition 26 and the new condition (C).

Corollary 35 *Let (X,d) be a metric space endowed with a binary relation $\prec$ and let $T,g : X \to X$ be two $\prec$-continuous mappings such that $T(X)$ is precomplete. Suppose that conditions* (A) *and* (B) *holds.*

(A′) *$TX \subseteq g(X)$, T is $(g,\prec)$-nondecreasing and there exists a point $x_0 \in X$ such that $gx_0 \prec Tx_0$.*

(B) *There exists an R-function $\varrho \in R_D$ such that* $\operatorname{ran}(d) \subseteq D$ *and*

$$\varrho\big(d(Tx,Ty),d(gx,gy)\big) > 0 \quad \text{for all } x,y \in X \text{ such that } Tx \prec Ty \text{ and } gx \prec gy.$$

(C) *The binary relation $\prec$ is transitive (or g-transitive, or (T,g)-transitive).*

Additionally, assume that the pair (T,g) is $(O,\mathcal{S})$-compatible or T and g are commuting.

Then T and g have, at least, a coincidence point. In fact, if $\{x_n\}$ is any $(T,g,\prec)$-Picard-Jungck sequence, either $\{gx_n\}$ contains a coincidence point of T and g, or $\{gx_n\}$ converges to a coincidence point of T and g.

If $g = I_X$ is the identity mapping on X, then we derive the following result.

Corollary 36 *Let $\mathcal{S}$ be a binary relation on a metric space (X,d) and let $T : X \to X$ be an $\mathcal{S}$-continuous mapping such that $T(X)$ is precomplete. Suppose that conditions* (A) *and* (B) *holds.*

(A) *There exists on X a Picard sequence $\{x_{n+1} = Tx_n\}$ such that $x_n \,\mathcal{S}\, x_m$ for all $n,m \in \mathbb{N}$ such that $n < m$.*

(B) *There exists an R-function $\varrho \in R_D$ such that* $\operatorname{ran}(d) \subseteq D$ *and*

$$\varrho\big(d(Tx,Ty),d(x,y)\big) > 0 \quad \text{for all } x,y \in X \text{ such that } x\,\mathcal{S}^*\,y \text{ and } Tx\,\mathcal{S}^*\,Ty.$$

Then T has, at least, a fixed point.

4.2 Coincidence point theorems under $\mathcal{S}$-regularity and condition (C_2)

In the following result, the contractivity condition is stronger because we do not assume that $Tx\,\mathcal{S}^*\,Ty$ and $gx\,\mathcal{S}^*\,gy$. However, the following result is applicable even if T and g are not $\mathcal{S}$-continuous.

Theorem 37 *Let $\mathcal{S}$ be a binary relation on a metric space (X,d) such that X (or $g(X)$) is $(d,\mathcal{S})$-regular, and let $T,g : X \to X$ be two mappings. Suppose that conditions* (A) *and* (B) *hold.*

(A) *There exists on X a $(T,g,\mathcal{S})$-Picard-Jungck sequence.*

(B) *There exists an R-function $\varrho \in R_D$ such that (ϱ_3) holds,* $\operatorname{ran}(d) \subseteq D$ *and*

$$\varrho\big(d(Tx,Ty),d(gx,gy)\big) > 0 \quad \text{for all } x,y \in X \text{ such that } gx\,\mathcal{S}^*\,gy. \tag{5}$$

In addition to this, suppose, at least, one of the following assumptions holds.

(c) *$(g(X),d)$ is complete;*

(d) *(X,d) is complete and $g(X)$ is closed;*

(e) (X,d) *is complete, the pair* (T,g) *is* $(O,\mathcal{S})$*-compatible,* $\mathcal{S}$ *is* g*-closed and* g *is injective on* $g(X)$ *and* $\mathcal{S}$*-continuous.*

Then T *and* g *have, at least, a coincidence point.*

Proof Since (5) $\Rightarrow$ (4), the proof of Theorem 33 can be followed, point by point, to prove that the sequence $\{gx_n\}$, which satisfies $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$, is a Cauchy sequence. Next, we distinguish some cases.

Case (c). *Assume that* $g(X)$ *is complete.* Since $g(X)$ is complete, there exists $u \in g(X)$ such that $\{gx_n\} \to u$. As $u \in g(X)$, then $g^{-1}(u)$ is not empty. We are going to show that any point $z \in g^{-1}(u)$ is a coincidence point of T and g. Indeed, let $z \in X$ be an arbitrary point such that $gz = u$. Since $gx_n \,\mathcal{S}\, gx_{n+1}$ for all $n \in \mathbb{N}$ and $\{gx_n\} \to gz$, the $(d,\mathcal{S})$-regularity of X (or $g(X)$) yields $gx_n \,\mathcal{S}\, gz$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $gx_{n_0} = gz$, then $gx_{n_0+1} \neq gz$. Therefore, the set $\{n \in \mathbb{N} : gx_n \neq gz\}$ is not finite. As a consequence, there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that

$$d(gx_{n(k)}, gz) > 0 \quad \text{for all } k \in \mathbb{N}.$$

In particular, $gx_{n(k)} \,\mathcal{S}^*\, gz$ for all $k \in \mathbb{N}$. Applying the contractivity condition (5),

$$\varrho\big(d(Tx_{n(k)}, Tz), d(gx_{n(k)}, gz)\big) > 0 \quad \text{for all } k \in \mathbb{N}. \tag{6}$$

Let $M = \{k \in \mathbb{N} : Tx_{n(k)} = Tz\}$.

Subcase (c.1). *Assume that* M *is finite.* In this case, there exists $k_0 \in \mathbb{N}$ such that $Tx_{n(k)} \neq Tz$ for all $k \geq k_0$. Let $\{a_k = d(Tx_{n(k)}, Tz)\}_{k \geq k_0}$ and $\{b_k = d(gx_{n(k)}, gz)\}_{k \geq k_0}$. Then $a_k > 0$ and $b_k > 0$ for all $k \geq k_0$. Moreover, $\{b_k\} \to 0$. As (6) means that $\varrho(a_k, b_k) > 0$ for all $k \geq k_0$, condition (ϱ_3) implies that $\{a_k\} \to 0$. It follows that $\{gx_{n(k)+1} = Tx_{n(k)}\} \to Tz$. As this is a subsequence of $\{gx_n\}$ and $\{gx_n\} \to gz$, we conclude that $Tz = gz$, that is, z is a coincidence point of T and g.

Subcase (c.2). *Assume that* M *is not finite.* In this case, there exists a subsequence $\{Tx_{n'(k)}\}$ of $\{Tx_{n(k)}\}$ such that $Tx_{n'(k)} = Tz$ for all $k \in \mathbb{N}$. Since $gx_{n'(k)+1} = Tx_{n'(k)} = Tz$ for all $k \in \mathbb{N}$, and $\{gx_n\} \to gz$, we also conclude that $Tz = gz$, that is, z is a coincidence point of T and g.

Case (d). *Assume that* $g(X)$ *is closed and* (X,d) *is complete.* In this case, we can apply item (c) because any closed subsets of complete spaces are also complete.

Case (e). *Assume that* (X,d) *is complete, the pair* (T,g) *is* $(O,\mathcal{S})$*-compatible,* $\mathcal{S}$ *is* g*-closed and* g *is injective on* $g(X)$ *and* $\mathcal{S}$*-continuous.* The completeness of X guarantees that there is $u \in X$ satisfying $\{gx_n\} \to u$. As g is $\mathcal{S}$-continuous and $gx_n \,\mathcal{S}\, gx_m$ for all $n < m$, then $\{ggx_m\} \to gu$. Moreover, the $(O,\mathcal{S})$-compatibility of (T,g) leads to

$$\lim_{n\to\infty} d(gu, Tgx_n) = \lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0.$$

Hence $\{Tgx_n\} \to gu$. We are going to show that $\{Tgx_n\}$ has a subsequence converging to Tu, and this will also finish the proof (because, in such a case, $Tu = gu$). Let us consider $M = \{n \in \mathbb{N} : Tgx_n = Tu\}$. If M is not finite, then there is a partial subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $Tgx_{n(k)} = Tu$ for all $k \in \mathbb{N}$. As $\{Tgx_n\} \to gu$, then $Tu = gu$, and the proof is finished. On the contrary, assume that M is finite. In such a case, there exists $n_0 \in \mathbb{N}$ such that $Tgx_n \neq Tu$ for all $n \geq n_0$. For simplicity, we will assume that

$$Tgx_n \neq Tu \quad \text{for all } n \in \mathbb{N}.$$

Since X is $(d,\mathcal{S})$-regular, $\{gx_n\} \to u$ and $gx_n \,\mathcal{S}\, gx_m$ for all $n < m$, then $gx_n \,\mathcal{S}\, u$ for all $n \in \mathbb{N}$. Moreover, as $\mathcal{S}$ is g-closed, then $ggx_n \,\mathcal{S}\, gu$ for all $n \in \mathbb{N}$. We must consider two cases depending on $M' = \{n \in \mathbb{N} : gx_n = u\}$.

Subcase (e.1). *Assume that M' is finite.* In this case, there exists $n_0 \in \mathbb{N}$ such that $gx_n \neq u$ for all $n \geq n_0$. Then $ggx_n \neq gu$ for all $n \geq n_0$ because g is injective. Hence $ggx_n \,\mathcal{S}^*\, gu$ for all $n \geq n_0$. Let $\{a_n\}$ and $\{b_n\}$ be the sequences of positive real numbers given by

$$a_n = d(Tgx_n, Tu) > 0 \quad \text{and} \quad b_n = d(ggx_n, gu) > 0 \quad \text{for all } n \geq n_0.$$

Clearly, $\{b_n\} \to 0$. Applying the contractivity condition (5),

$$\varrho(a_n, b_n) = \varrho\big(d(Tgx_n, Tu), d(ggx_n, gu)\big) > 0 \quad \text{for all } n \geq n_0.$$

By (ϱ_3), we deduce that $\{a_n\} \to 0$, which implies that $\{Tgx_n\} \to Tu$.

Subcase (e.2). *Assume that M' is not finite.* In this case, there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $gx_{n(k)} = u$ for all $k \in \mathbb{N}$. Hence $\{Tgx_{n(k)}\} \to Tu$. □

If $x \,\mathcal{S}\, y$ for all $x, y \in X$, we derive the following consequence.

Corollary 38 *Given two maps $T, g : X \to X$ in a metric space (X, d), assume that conditions* (A) *and* (B) *hold.*

(A) *There is on X a (T, g)-Picard-Jungck sequence.*
(B) *There exists an R-function $\varrho \in R_D$ such that (ϱ_3) holds,* $\operatorname{ran}(d) \subseteq D$ *and*

$$\varrho\big(d(Tx, Ty), d(gx, gy)\big) > 0 \quad \textit{for all } x, y \in X \textit{ such that } gx \neq gy.$$

In addition to this, suppose, at least, one of the following assumptions holds.
(c) $(g(X), d)$ *is complete*;
(d) (X, d) *is complete and $g(X)$ is closed*;
(e) (X, d) *is complete, g is injective on $g(X)$ and continuous, and the pair (T, g) is O-compatible.*

Then T and g have, at least, a coincidence point.

In the following result, we denote by $\prec$ a transitive binary relation (for instance, a preorder or a partial order), which is not necessarily reflexive.

Corollary 39 *Let (X, d) be a metric space endowed with a binary relation $\prec$ such that X (or $g(X)$) is $(d, \prec)$-nondecreasing-regular, and let $T, g : X \to X$ be two mappings. Assume that conditions* (A) *and* (B) *hold.*

(A) *$TX \subseteq g(X)$, T is $(g, \prec)$-nondecreasing and there exists a point $x_0 \in X$ such that $gx_0 \prec Tx_0$.*

(B) *There exists an R-function* $\varrho \in R_D$ *such that* (ϱ_3) *holds*, $\operatorname{ran}(d) \subseteq D$, *and*

$$\varrho\big(d(Tx,Ty),d(gx,gy)\big) > 0 \quad \text{for all } x,y \in X \text{ such that } gx\,\mathcal{S}^*\,gy.$$

(C) *The binary relation* $\prec$ *is transitive (or g-transitive, or* (T,g)*-transitive).*

In addition to this, suppose, at least, one of the following assumptions holds.
(c) $(g(X),d)$ *is complete;*
(d) (X,d) *is complete and* $g(X)$ *is closed;*
(e) (X,d) *is complete, the pair* (T,g) *is* $(O,\prec)$*-compatible, g is* $\prec$*-nondecreasing, and g is injective on* $g(X)$ *and* $\mathcal{S}$*-continuous.*
Then T and g have, at least, a coincidence point.

Furthermore, if $g = I_X$ is the identity mapping on X, then we derive the following fixed point result.

Corollary 40 *Let* (X,d) *be a complete,* $(d,\mathcal{S})$*-regular metric space endowed with a binary relation* $\mathcal{S}$ *and let* $T : X \to X$ *be a mapping. Assume that conditions* (A) *and* (B) *hold.*

(A) *There is on X a Picard sequence* $\{x_{n+1} = Tx_n\}$ *such that* $x_n\,\mathcal{S}\,x_m$ *for all* $n,m \in \mathbb{N}$ *such that* $n < m$.
(B) *There exists an R-function* $\varrho \in R_D$ *such that* (ϱ_3) *holds*, $\operatorname{ran}(d) \subseteq D$, *and*

$$\varrho\big(d(Tx,Ty),d(x,y)\big) > 0 \quad \text{for all } x,y \in X \text{ such that } x\,\mathcal{S}^*\,y.$$

Then T has, at least, a fixed point.

4.3 Coincidence point theorems under $\mathcal{S}$-regularity and condition (C_3)

If we assume that the contractivity condition is more restrictive, we can avoid the injectivity of g in condition (e) in Theorem 37.

Theorem 41 *Let* $\mathcal{S}$ *be a binary relation on a metric space* (X,d) *and let* $T,g : X \to X$ *be two mappings. Assume that conditions* (A) *and* (B) *hold.*

(A) *There is on X a* $(T,g,\mathcal{S})$*-Picard-Jungck sequence.*
(B) *There exists an R-function* $\varrho \in R_D$ *such that* (ϱ_4) *holds*, $\operatorname{ran}(d) \subseteq D$, *and*

$$\varrho\big(d(Tx,Ty),d(gx,gy)\big) > 0 \quad \text{for each } x,y \in X \text{ such that } x \neq y \text{ and } gx\,\mathcal{S}\,gy. \tag{7}$$

(e′) (X,d) *is complete,* $\mathcal{S}$ *is g-closed,* (T,g) *is* $(O,\mathcal{S})$*-compatible, g is* $\mathcal{S}$*-continuous and X is* $(d,\mathcal{S})$*-regular.*

Then T and g have, at least, a coincidence point.

Proof Since (7) ⇒ (4), the proof of Theorem 33 can be followed, point by point, to deduce that $\{gx_n\}$ is Cauchy. The completeness of (X,d) implies the existence of $u \in X$ such that $\{gx_n\} \to u$. Since $gx_n\,\mathcal{S}\,gx_m$ for each $n < m$ and g is $\mathcal{S}$-continuous, then $\{ggx_m\} \to gu$. Furthermore, from the $(O,\mathcal{S})$-compatibility of the pair (T,g),

$$\lim_{n\to\infty} d(gu, Tgx_n) = \lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0.$$

Hence $\{Tgx_n\} \to gu$. Let us consider $M = \{n \in \mathbb{N} : Tgx_n = Tu\}$. If M is not finite, then there is a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $Tgx_{n(k)} = Tu$ for all $k \in \mathbb{N}$. As $\{Tgx_n\} \to gu$, then $Tu = gu$, and the proof is finished. On the other case, if M is finite, there is $n_0 \in \mathbb{N}$ such that $Tgx_n \neq Tu$ for each $n \geq n_0$. For simplicity, we will assume that

$$Tgx_n \neq Tu \quad \text{for all } n \in \mathbb{N}.$$

The proof will be finished if we prove that $\{Tgx_n\} \to Tu$. Indeed, the $(d, \mathcal{S})$-regularity of X yields $\{gx_n\} \to u$ and $gx_n \mathcal{S} gx_m$ for each $n < m$, so $gx_n \mathcal{S} u$ for all $n \in \mathbb{N}$. In addition, as $\mathcal{S}$ is g-closed, $ggx_n \mathcal{S} gu$ for each $n \in \mathbb{N}$. Let $\{a_n\}$ and $\{b_n\}$ be the sequences of nonnegative real numbers given by

$$a_n = d(Tgx_n, Tu) \quad \text{and} \quad b_n = d(ggx_n, gu) \quad \text{for all } n \in \mathbb{N}.$$

Clearly, $\{b_n\} \to 0$. Applying the contractivity condition (7),

$$\varrho(a_n, b_n) = \varrho\big(d(Tgx_n, Tu), d(ggx_n, gu)\big) > 0 \quad \text{for all } n \in \mathbb{N}.$$

By (ϱ_4), we deduce that $\{a_n\} \to 0$, which implies that $\{Tgx_n\} \to Tu$. □

The reader may particularize the previous results to the cases: (1) $x \mathcal{S} y$ for all $x, y \in X$; (2) $\prec$ is a transitive binary relation on X; (3) g is the identity mapping on X.

5 Common fixed point theorems under $(R, \mathcal{S})$-contractivity conditions

In this section, we study when the existence of a coincidence point can help us to derive the existence and uniqueness of coincidence (or common fixed) points. Before that, we describe an assumption that we will use in the main results of this section. Given $x, y \in \operatorname{Coin}(T, g)$, we will say that *assumption* $(\mathcal{A}_{xy})$ holds if the following property is verified:

$(\mathcal{A}_{xy})$ there is a (T, g)-Picard-Jungck sequence $\{z_n\} \subseteq X$ such that, for all $n \in \mathbb{N}$, gz_n is $\mathcal{S}$-comparable, at the same time, to gx and to gy.

The following result shows that this condition can be guaranteed under some usual properties.

Lemma 42 *If $x, y \in \operatorname{Coin}(T, g)$, $TX \subseteq g(X)$, T is $(g, \mathcal{S})$-nondecreasing and there exists a point $z_0 \in X$ such that $gz_0 \mathcal{S} Tz_0$ and gz_0 are $\mathcal{S}$-comparable, at the same time, to gx and to gy, then property $(\mathcal{A}_{xy})$ holds.*

Proof Suppose, for instance, that $gx \mathcal{S} gz_0$ and $gy \mathcal{S} gz_0$ (the order of the arguments is not important). Let $\{z_n\}$ be a (T, g)-Picard-Jungck sequence on X based on z_0 (it exists by Proposition 27). As T is $(g, \mathcal{S})$-nondecreasing, then $Tx \mathcal{S} Tz_0$ and $Ty \mathcal{S} Tz_0$, which means that $gx \mathcal{S} gz_1$ and $gy \mathcal{S} gz_1$. Repeating this argument by induction, property $(\mathcal{A}_{xy})$ holds. □

In the following result, we take advantage of property $(\mathcal{A}_{xy})$ in order to give a first step about the uniqueness of the coincidence point.

Lemma 43 *Under the hypotheses of Theorem* 33 *(or Theorem* 37*), let $x, y \in \operatorname{Coin}(T, g)$ be two coincidence points of T and g. In addition to this, suppose, at least, one of the following* *assumptions holds.*

(p) *gx and gy are $\mathcal{S}$-comparable.*

(q) *Property* $(\mathcal{A}_{xy})$ *holds and* $\mathcal{S}$ *is* (T,g)*-compatible.*

Then $gx = gy$.

Proof Taking into account that (5) $\Rightarrow$ (4), we can use the contractivity condition (4).

(a) Reasoning by contradiction, assume that $gx \neq gy$. As gx and gy are $\mathcal{S}$-comparable, we can suppose, without loss of generality, that $gx\,\mathcal{S}\,gy$. As $Tx = gx$ and $Ty = gy$, we observe that $gx\,\mathcal{S}^*\,gy$ and $Tx\,\mathcal{S}^*\,Ty$. By (4),

$$\varrho\big(d(gx,gy),d(gx,gy)\big) = \varrho\big(d(Tx,Ty),d(gx,gy)\big) > 0,$$

which contradicts Proposition 5.

(b) Let $\{z_n\} \subseteq X$ be a Picard-Jungck sequence such that, for all $n \in \mathbb{N}$, gz_n is $\mathcal{S}$-comparable, at the same time, to gx and to gy. We are going to show that, in any case, $\{gz_n\} \to gx$ and $\{gz_n\} \to gy$, so we will deduce $gx = gy$. We only reason using gx, but the same arguments are valid for gy. By hypothesis, $gx\,\mathcal{S}\,gz_n$ or $gz_n\,\mathcal{S}\,gx$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $gx = gz_{n_0}$, the (T,g)-compatibility of $\mathcal{S}$ implies that $Tx = Tz_{n_0}$. Then $gx = Tx = Tz_{n_0} = gz_{n_0+1}$. Repeating this argument, $gx = gz_n$ for all $n \geq n_0$, and, therefore, $\{gz_n\} \to gx$. In other case, if $gz_n \neq gx$ for each $n \in \mathbb{N}$, $Tz_n = gx_{n+1} \neq gx = Tx$ for all $n \in \mathbb{N}$, so $gz_n\,\mathcal{S}^*\,gx$ and $Tz_n\,\mathcal{S}^*\,Tx$ (or, in the opposite case, $gx\,\mathcal{S}^*\,gz_n$ and $Tx\,\mathcal{S}^*\,Tz_n$). Using the contractivity condition (4), it follows that, for all $n \in \mathbb{N}$,

$$\varrho\big(d(gz_{n+1},gx),d(gz_n,gx)\big) = \varrho\big(d(Tz_n,Tx),d(gz_n,gx)\big) > 0.$$

If $\{a_n = d(gz_n,gx)\}$, condition (ϱ_1) guarantees that $\{a_n\} \to 0$, so $\{gz_n\} \to gx$. Similarly, $\{gz_n\} \to gy$, so $gx = gy$. □

Theorem 44 *Under the hypotheses of Theorem* 33 (*or Theorem* 37), *suppose that, for all distinct coincidence points* $x,y \in \operatorname{Coin}(T,g)$, *at least, one of the following conditions holds.*

(a) *gx and gy are $\mathcal{S}$-comparable.*

(b) *Property* $(\mathcal{A}_{xy})$ *holds and* $\mathcal{S}$ *is* (T,g)*-compatible.*

Hence T and g have a unique point of coincidence.

If we additionally assume that g (*or T*) *is injective on* $\operatorname{Coin}(T,g)$, *then T and g have a unique coincidence point.*

Proof By Theorem 33 (or Theorem 37), the set of all coincidence points of T and g is nonempty, so T and g have, at least, a point of coincidence. Let ω and ω' be two points of coincidence of T and g. By definition, there are two coincidence points $x,y \in \operatorname{Coin}(T,g)$ such that $\omega = Tx = gx$ and $\omega' = Ty = gy$. Thus, it follows from Lemma 43 that $\omega = gx = gy = \omega'$, so T and g have a unique point of coincidence.

Additionally, assume that g (or T) is injective on $\operatorname{Coin}(T,g)$, and let $x,y \in \operatorname{Coin}(T,g)$ be two arbitrary coincidence points of T and g. In order to prove that $x = y$, assume that $x \neq y$. By Lemma 43, $Tx = gx = gy = Ty$. And as g (or T) is injective on $\operatorname{Coin}(T,g)$, then $x = y$, which contradicts the fact that $x \neq y$. Thus, $x = y$ and T and g have a unique coincidence point. □

Theorem 45 *Under the hypotheses of Theorem* 44, *assume that T and g are weakly compatible* (*or commuting*). *Then T and g have a unique common fixed point*.

Proof Let $x_0 \in X$ be a coincidence point of T and g, and let define $\omega = gx_0$. Since $Tx_0 = gx_0$ and T and g are weakly compatible, $Tgx_0 = gTx_0$, so $T\omega = Tgx_0 = gTx_0 = g\omega$. Then ω is another coincidence point of T and g. By Theorem 43, $gx_0 = g\omega$, so $\omega = gx_0 = g\omega$. In particular, $\omega = g\omega = T\omega$, so ω is a common fixed point of T and g.

The uniqueness of the common fixed point follows from the fact that any common fixed point is a point of coincidence, and Theorem 44 guarantees that there exists a unique point of coincidence. □

6 A new kind of coincidence point theorems involving *R*-functions

In the past, the Dutta and Choudhury contractivity condition

$$\widetilde{\psi}\big(d(Tx,Ty)\big) \leq \widetilde{\psi}\big(d(gx,gy)\big) - \widetilde{\phi}\big(d(gx,gy)\big) \quad \text{for all } x,y \in X$$

has been widely studied. If $\phi = \widetilde{\psi}$ and $\psi = \widetilde{\psi} - \widetilde{\phi}$, this condition can be equivalently expressed as

$$\phi\big(d(Tx,Ty)\big) \leq \psi\big(d(gx,gy)\big) \quad \text{for all } x,y \in X,$$

where $\psi < \phi$ in $(0,\infty)$. For instance, if

$$\psi(t) = t \quad \text{and} \quad \phi(t) = \frac{t}{1+t^2} \quad \text{for all } t \in [0,\infty),$$

then the previous contractivity condition is equivalent to

$$d(Tx,Ty) \leq d(gx,gy) - \frac{d(gx,gy)}{1+d(gx,gy)^2} = \frac{d(gx,gy)^3}{1+d(gx,gy)^2} \quad \text{for all } x,y \in X.$$

In this section we are going to show that R-functions permit us to include some terms in this contractivity conditions depending both on $d(gx,gy)$ and on $d(Tx,Ty)$ in order to obtain more general inequalities that also guarantee the existence and uniqueness of the coincidence point.

Definition 46 We will denote by Ω the set of all (ψ,ϕ,φ), where $\psi,\phi,\varphi : [0,\infty) \to [0,\infty)$ are functions, such that the following properties hold.

(Ω_1) If $\{a_n\} \subset (0,\infty)$ is a sequence such that $\phi(a_{n+1}) < \psi(a_n) + \varphi(a_n a_{n+1})$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.

(Ω_2) If $\{a_n\}, \{b_n\} \subset (0,\infty)$ are two sequences converging to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\phi(a_n) < \psi(b_n) + \varphi(a_n b_n)$ for all $n \in \mathbb{N}$, then $L = 0$.

In condition (Ω_1), the term $\varphi(a_n a_{n+1})$ can be replaced in different ways, depending on the researcher's interest.

Example 47 If ψ and ϕ are altering distance functions such that $\psi < \phi$, and $\varphi(t) = 0$ for all $t \in [0,\infty)$, then $(\psi,\phi,\varphi) \in \Omega$.

Example 48 If ψ is an altering distance function, ϕ is a lower semi-continuous function such that $\psi < \phi$ and $\phi^{-1}(\{0\}) = \{0\}$, and $\varphi(t) = 0$ for all $t \in [0,\infty)$, then $(\psi,\phi,\varphi) \in \Omega$.

The following properties are given in order to show Example 51.

Proposition 49 *If $a \in [0,\infty)$ and $b \in (0,\infty)$ are real numbers such that*

$$a \leq \frac{b^3}{1+b^2} + \frac{\sqrt{ab}}{2(1+ab)}, \tag{8}$$

then $a < b$.

Proof If $a = 0$, it is obvious because $b > 0$. Assume that $a > 0$. The case $a = b$ is impossible because

$$b = a \leq \frac{b^3}{1+b^2} + \frac{\sqrt{ab}}{2(1+ab)} = \frac{b^3}{1+b^2} + \frac{b}{2(1+b^2)} = \frac{b(2b^2+1)}{2(b^2+1)}$$

is equivalent to $2 \leq 1$, which is false. Hence, $a \neq b$. In order to prove that $a < b$, assume that $b < a$ and we will get a contradiction. Let us consider the function $f : [0,\infty) \to \mathbb{R}$ given, for all $t \in [0,\infty)$, by

$$f(t) = t - \left(\frac{b^3}{1+b^2} + \frac{\sqrt{tb}}{2(1+tb)}\right) = t - \frac{\sqrt{b}}{2} \cdot \frac{\sqrt{t}}{1+tb} - \frac{b^3}{1+b^2}.$$

Notice that

$$f(b) = b - \frac{b^3}{1+b^2} - \frac{b}{2(1+b^2)} = \frac{b}{2(1+b^2)} > 0.$$

Inequality (8) means that $f(a) \leq 0$. By simple calculation,

$$f'(t) = 1 + \frac{\sqrt{b}}{4} \cdot \frac{tb-1}{\sqrt{t}(1+tb)^2} \quad \text{for all } t > 0.$$

If $t \geq b$, then

$$\begin{aligned} f'(t) &= 1 + \frac{tb\sqrt{b}}{4\sqrt{t}(1+tb)^2} - \frac{\sqrt{b}}{4\sqrt{t}(1+tb)^2} \\ &> 1 - \frac{\sqrt{b}}{4\sqrt{t}(1+tb)^2} = \frac{4\sqrt{t}(1+tb)^2 - \sqrt{b}}{4\sqrt{t}(1+tb)^2} \\ &\geq \frac{4\sqrt{t} - \sqrt{b}}{4\sqrt{t}(1+tb)^2} \geq \frac{4\sqrt{b} - \sqrt{b}}{4\sqrt{t}(1+tb)^2} = \frac{3\sqrt{b}}{4\sqrt{t}(1+tb)^2} > 0. \end{aligned}$$

As a consequence, $f'(t) > 0$ for all $t \geq b$ implies that f is strictly increasing in $[b,\infty)$. In particular, if we assume that $b < a$, then $f(b) < f(a)$, but this is a contradiction because $f(b) > 0$ and $f(a) \leq 0$. □

Proposition 50 *For all $x \geq 0$, there exists a unique solution s_x of the equation*

$$s_x = \frac{x^3}{1+x^2} + \frac{\sqrt{s_x x}}{2(1+s_x x)}. \tag{9}$$

This solution verifies $s_x \leq x$ and, if $x > 0$, then $0 < s_x < x$. Furthermore, $x - 1 \leq s_x < x$ and $\lim_{x\to\infty} \frac{s_x}{x} = 1$.

Proof If $x = 0$, the unique solution of equation (9) is $s_0 = 0$. Henceforth, assume that $x > 0$ is fixed.

Step 1. *Existence of solution*. Let us consider the function $f : [0,\infty) \to \mathbb{R}$ given, for all $t \in [0,\infty)$, by

$$f(t) = t - \frac{x}{2} \cdot \frac{\sqrt{t}}{1+t}.$$

Clearly, f is continuous in $[0,\infty)$, $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$. Hence, the image of f contains the interval $[0,\infty)$. As $x^4/(1+x^2) \in [0,\infty)$, there exists $p_x \in (0,\infty)$ such that $f(p_x) = x^4/(1+x^2)$. Let $s_x = p_x/x \in (0,\infty)$. Then

$$s_x x - \frac{x\sqrt{s_x x}}{2(1+s_x x)} = p_x - \frac{x\sqrt{p_x}}{2(1+p_x)} = f(p_x) = \frac{x^4}{1+x^2},$$

which implies that s_x is a solution of equation (9).

By Proposition 49, any solution s_x of equation (9) satisfies $0 < s_x < x$. As we have just seen, to find a solution s_x of equation (9) is equivalent to find a solution $p_x = s_x x$ of equation $f(p) = x^4/(1+x^2)$.

Step 2. *Uniqueness of the solution*. Notice that, for all $t > 0$,

$$f'(t) = 1 - \frac{x(1-t)}{4\sqrt{t}(t+1)^2} = \frac{4\sqrt{t}(t+1)^2 - x(1-t)}{4\sqrt{t}(t+1)^2} = \frac{4\sqrt{t}(t+1)^2 + x(t-1)}{4\sqrt{t}(t+1)^2}.$$

If $t \geq 1$, then $f'(t) > 0$. If $0 < t < 1$, then

$$f'(t) = 0 \quad \Leftrightarrow \quad 4\sqrt{t}(t+1)^2 = x(1-t) \quad \Leftrightarrow \quad \frac{4\sqrt{t}(t+1)^2}{1-t} = x.$$

Let us consider the function $h : (0,1) \to \mathbb{R}$ given, for all $t \in (0,1)$, by

$$h(t) = \frac{4\sqrt{t}(t+1)^2}{1-t}.$$

Since

$$h'(t) = \frac{2(t+1)(1+6t-3t^2)}{\sqrt{t}(t-1)^2} > 0 \quad \text{for all } t \in (0,1),$$

the function h is strictly increasing in $(0,1)$, and the equation

$$h(t) = \frac{4\sqrt{t}(t+1)^2}{1-t} = x$$

has, at most, a unique solution $t_x \in (0,1)$, that is, t_x is the unique zero of f'. Taking into account that $f(0) = 0$ and $\lim_{t\to 0^+} f'(t) = -\infty$, the function f is strictly decreasing in $(0,t_x)$, taking negative values, and it is strictly increasing in (t_x,∞). In particular, the equation $f(p) = x^4/(1+x^2)$ must have a unique solution, so equation (9) can only have a unique solution.

Step 3. $x-1 \le s_x$. If $x \le 1$, then $x-1 \le 0 < s_x$. Assume that $x > 1$. Then

$$\begin{aligned} x - s_x &= x - \left(\frac{x^3}{1+x^2} + \frac{\sqrt{s_x x}}{2(1+s_x x)} \right) = \frac{x}{1+x^2} - \frac{\sqrt{s_x x}}{2(1+s_x x)} \\ &= \frac{2x(1+s_x x) - (1+x^2)\sqrt{s_x x}}{2(1+s_x x)(1+x^2)}. \end{aligned}$$

Hence

$$\begin{aligned} x - s_x \le 1 \quad &\Leftrightarrow \quad \frac{2x(1+s_x x) - (1+x^2)\sqrt{s_x x}}{2(1+s_x x)(1+x^2)} \le 1 \\ &\Leftrightarrow \quad 2x(1+s_x x) - \bigl(1+x^2\bigr)\sqrt{s_x x} \le 2(1+s_x x)\bigl(1+x^2\bigr) \\ &\Leftrightarrow \quad 2x + 2x^2 s_x - \bigl(1+x^2\bigr)\sqrt{s_x x} \le 2s_x x^3 + 2x^2 + 2s_x x + 2 \\ &\Leftrightarrow \quad -\bigl(1+x^2\bigr)\sqrt{s_x x} \le 2s_x x^2(x-1) + 2x(x-1) + 2s_x x + 2. \end{aligned}$$

As $x > 1$, the previous inequality holds. □

Next we show a non-trivial example of functions ψ, ϕ, and φ such that $(\psi,\phi,\varphi) \in \Omega$.

Example 51 We claim that if

$$\phi(t) = t, \qquad \psi(t) = \frac{t^3}{1+t^2} \quad \text{and} \quad \varphi(t) = \frac{\sqrt{t}}{2(1+t)} \quad \text{for all } t \in [0,\infty),$$

then $(\psi,\phi,\varphi) \in \Omega$. To prove it, let $\{a_n\} \subset (0,\infty)$ be a sequence such that $\phi(a_{n+1}) < \psi(a_n) + \varphi(a_n a_{n+1})$ for all $n \in \mathbb{N}$, that is,

$$a_{n+1} \le \frac{a_n^3}{1+a_n^2} + \frac{\sqrt{a_n a_{n+1}}}{2(1+a_n a_{n+1})} \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

By Proposition 49, $a_{n+1} \le a_n$ for all $n \in \mathbb{N}$. Then $\{a_n\}$ is convergent, and there is $L \ge 0$ such that $\{a_n\} \to L$. Letting $n \to \infty$ in (10), we deduce that

$$L \le \frac{L^3}{1+L^2} + \frac{L}{2(1+L^2)} = \frac{2L^3+L}{2(1+L^2)}, \tag{11}$$

which implies that $L \le 0$. Hence, $L = 0$ and condition (Ω_1) holds.

Next, assume that $\{a_n\},\{b_n\} \subset (0,\infty)$ are two sequences converging to the same limit $L \ge 0$ and verifying that $L < a_n$ and $\phi(a_n) < \psi(b_n) + \varphi(a_n b_n)$ for all $n \in \mathbb{N}$. Then

$$a_n \le \frac{b_n^3}{1+b_n^2} + \frac{\sqrt{a_n b_n}}{2(1+a_n b_n)} \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in the previous inequality, we derive again inequality (11), which implies that $L \leq 0$, so $L = 0$ and condition (Ω_2) also holds. As a result, $(\psi, \phi, \varphi) \in \Omega$.

Functions in Ω permit us to introduce a new kind of R-functions.

Lemma 52 *If* $(\psi, \phi, \varphi) \in \Omega$, *then the function* $\varrho_{\psi,\phi,\varphi} : [0,\infty) \times [0,\infty) \to [0,\infty)$, *given by*

$$\varrho_{\psi,\phi,\varphi}(t,s) = \psi(s) - \phi(t) + \varphi(ts) \quad \textit{for all } t,s \in [0,\infty),$$

is an R-function.

Obviously, $\varrho_{\psi,\phi,\varphi}$ is an R-function that, in general, cannot be decomposed as $\psi(s) - \phi(t)$. Obviously, the same property can be studied in an appropriate subset $A \subseteq [0,\infty)$.

Proof It follows from

$$\phi(a_{n+1}) < \psi(a_n) + \varphi(a_n a_{n+1}) \quad \Leftrightarrow \quad \varrho_{\psi,\phi,\varphi}(a_{n+1}, a_n) > 0 \quad \text{and}$$
$$\phi(a_n) < \psi(b_n) + \varphi(a_n b_n) \quad \Leftrightarrow \quad \varrho_{\psi,\phi,\varphi}(a_n, b_n) > 0$$

for each $n \in \mathbb{N}$. Then (ϱ_1) is equivalent to (Ω_1) and (ϱ_2) is equivalent to (Ω_2) when $A = [0,\infty)$. □

Example 53 If ψ, ϕ, and φ are defined as in Example 51, then the corresponding R-function $\varrho_{\psi,\phi,\varphi}$ given in Lemma 52 verifies conditions (ϱ_3) and (ϱ_4). For instance, let us show condition (ϱ_4). Assume that $\{a_n\}, \{b_n\} \subset [0,\infty)$ are two sequences such that $\{b_n\} \to 0$ and $\varrho_{\psi,\phi,\varphi}(a_n, b_n) > 0$ for all $n \in \mathbb{N}$. This inequality is equivalent to

$$a_n \leq \frac{b_n^3}{1 + b_n^2} + \frac{\sqrt{a_n b_n}}{2(1 + a_n b_n)} \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 49, $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Therefore, $\{b_n\} \to 0$ implies $\{a_n\} \to 0$.

The following corollary is a particular case of Theorem 37 using the described functions ψ, ϕ, and φ. Notice that the new contractivity condition has not been studied in the past. This is only an example of how using R-functions in order to establish new contractivity conditions.

Corollary 54 *Let* $\mathcal{S}$ *be a binary relation on a metric space* (X,d) *such that* X (*or* $g(X)$) *is* $(d,\mathcal{S})$-*regular, and let* $T,g : X \to X$ *be two mappings. Assume that conditions* (A) *and* (B) *hold.*

(A) *There is on* X *a* $(T,g,\mathcal{S})$-*Picard-Jungck sequence.*
(B) *For all* $x,y \in X$ *such that* $gx\,\mathcal{S}^*\,gy$, *we have*

$$d(Tx,Ty) \leq \frac{d(gx,gy)^3}{1 + d(gx,gy)^2} + \frac{\sqrt{d(Tx,Ty)d(gx,gy)}}{2(1 + d(Tx,Ty)d(gx,gy))}. \tag{12}$$

In addition to this, suppose, at least, one of the following assumptions holds.

(c) $(g(X), d)$ *is complete*;
(d) (X, d) *is complete and* $g(X)$ *is closed*;
(e) (X, d) *is complete*, g *is injective on* $g(X)$ *and* $\mathcal{S}$*-continuous*, $\mathcal{S}$ *is* g*-closed, and the pair* (T, g) *is* $(O, \mathcal{S})$*-compatible*.

Then T *and* g *have, at least, a coincidence point*.

Proof It follows from Theorem 37 using the R-function $\varrho_{\psi,\phi,\varphi}$ described in Lemma 52 and the triple (ψ, ϕ, φ) of Example 51. □

Example 55 Let $X = [0,1] \cup \{3^n, 2 \cdot 3^n, s_{3^n}\}_{n=1}^{\infty}$ where s_{3^n} denotes the unique solution of equation (9) for $x = 3^n$ for some $n \in \mathbb{N}$. Then X is a complete subset of $\mathbb{R}$ endowed with the Euclidean metric. Let us consider on X the binary relation $\mathcal{S}$ given by

$$x \mathcal{S} y \quad \Leftrightarrow \quad \bigl(x, y \in [0,1] \text{ or } (x,y) = \bigl(3^n, 2 \cdot 3^n\bigr) \text{ for some } n \in \mathbb{N}\bigr).$$

Let T and g be the self-maps on X given by

$$gx = x \quad \text{and} \quad Tx = \begin{cases} 0, & \text{if } x \in [0,1] \cup \{3^n + 1, 2 \cdot 3^n, s_{3^n}\}_{n=1}^{\infty}, \\ s_{3^n}, & \text{if } x \in \{3^n\}_{n=1}^{\infty}. \end{cases}$$

Let $x, y \in X$ be such that $gx \mathcal{S}^* gy$, that is, $x \mathcal{S}^* y$. If $x, y \in [0,1]$, then $d(Tx, Ty) = d(0,0) = 0$, so (12) holds. Assume that $(x, y) = (3^n, 2 \cdot 3^n)$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} &d(Tx, Ty) = d\bigl(T\bigl(3^n\bigr), T\bigl(2 \cdot 3^n\bigr)\bigr) = d(s_{3^n}, 0) = s_{3^n}, \\ &d(gx, gy) = d\bigl(3^n, 2 \cdot 3^n\bigr) = 3^n. \end{aligned}$$

In particular,

$$\begin{aligned} &d(Tx, Ty) \leq \frac{d(gx,gy)^3}{1 + d(gx,gy)^2} + \frac{\sqrt{d(Tx,Ty)d(gx,gy)}}{2(1 + d(Tx,Ty)d(gx,gy))} \\ &\quad \Leftrightarrow \quad s_{3^n} \leq \frac{(3^n)^3}{1 + (3^n)^2} + \frac{\sqrt{s_{3^n} \cdot 3^n}}{2(1 + s_{3^n} \cdot 3^n)}, \end{aligned}$$

which is true because s_{3^n} is the unique solution of equation (9) for $x = 3^n$. Hence, the equality is reached. As a consequence, (12) holds. As X is $(d, \mathcal{S})$-regular, Corollary 54 is applicable.

We cannot use the contractivity condition

$$d(Tx, Ty) \leq d(gx, gy) - \frac{d(gx,gy)}{1 + d(gx,gy)^2} = \frac{d(gx,gy)^3}{1 + d(gx,gy)^2}$$

in the previous example because the equality

$$d(Tx, Ty) = \frac{d(gx,gy)^3}{1 + d(gx,gy)^2} + \frac{\sqrt{d(Tx,Ty)d(gx,gy)}}{2(1 + d(Tx,Ty)d(gx,gy))}$$

is reached for some points of the space. Therefore, Corollary 8 and the Dutta and Choudhury Theorem 9 are not applicable. Also notice that

$$\lim_{n\to\infty}\frac{d(T(3^n),T(2\cdot 3^n))}{d(g(3^n),g(2\cdot 3^n))}=\lim_{n\to\infty}\frac{s_{3^n}}{3^n}=1,$$

so we cannot apply any contractivity condition of the type $d(Tx,Ty)\le\lambda d(gx,gy)$ for some $\lambda\in[0,1)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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