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Extensions of Ćirić and Wardowski type fixed point theorems in D-generalized metric spaces

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Abstract
In this paper, we study an interesting generalization of standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces due to the recent work of Jleli and Samet. Here we modify the result for Ćirić quasi-contraction-type mappings and also prove the same result by taking D-admissible mappings. Moreover, we establish fixed point theorems for two well-known nonlinear contractions like rational contraction mappings and Wardowski type contraction mappings. Several important results in the literature can be derived from our results. Suitable examples are presented to substantiate our obtained results.

1 Introduction

Metric fixed point theory plays a crucial role in the field of functional analysis. It was first introduced by the great Polish mathematician Banach [1]. Over the years, due to its significance and application in different fields of science, a lot of generalizations have been done in different directions by several authors; see, for example, [2–11] and references therein. Recently, Jleli and Samet [12] introduced a very interesting generalization of metric spaces from which we can easily derive different known structures, namely standard metric spaces, b-metric spaces, dislocated metric spaces, *et cetera*. Also, they established a new version of several well-known fixed point theorems. Before proceeding further, we recall the definition of a generalized metric space.

Let X be a nonempty set, and $D : X \times X \to [0,\infty]$ be a mapping. For every $x \in X$, we define the set $C(D,X,x)$ as follows:

$$C(D,X,x) = \Big\{ (x_n) \subset X : \lim_{n\to\infty} D(x_n,x) = 0 \Big\}. \tag{1.1}$$

Definition 1.1 [12] Let X be a nonempty set, and $D : X \times X \to [0,\infty]$ be a mapping. Then (X,D) is said to be a generalized metric space if the following conditions are satisfied:

(D1) $\forall x,y \in X$, $D(x,y) = 0 \Rightarrow x = y$;
(D2) $\forall x,y \in X$, $D(x,y) = D(y,x)$;
(D3) there exists $c > 0$ such that for all $(x,y) \in X \times X$ and $(x_n) \in C(D,X,x)$,

$$D(x,y) \le c \limsup_{n\to\infty} D(x_n,y). \tag{1.2}$$

Throughout this article, we call such a space (X,D) a *D-generalized metric space*. The class of such metric spaces is always larger than the class of standard metric spaces, b-metric spaces, dislocated metric spaces, dislocated b-metric spaces, et cetera. For details, interested readers are referred to [12].

The purpose of this paper is to modify the Ćirić quasi-contractions. In this paper we introduce D-admissible mappings and establish the fixed point theorem for Ćirić quasi-contractions with the help of D-admissible mappings. This article includes an example of a D-generalized metric space to show that a sequence in this setting may be convergent without being a Cauchy sequence. We also investigate the existence and uniqueness of a fixed point for the mappings satisfying nonlinear rational contraction and Wardowski type F-contraction, where the function F is taken from a more general class of functions than that known in the existing literature.

We organize the paper as follows. Section 2 contains some useful notions and important results that will be needed in the paper. In Section 3, we exhibit an example to show that the Theorem 4.3 in [12] does not give the guarantee of the existence of a fixed point for any arbitrary value of $k \in (0,1)$. Accordingly, we present a modified version of Theorem 4.3 in [12]. Also, we establish the same result for D-admissible Ćirić quasi-contraction mappings. Moreover, we also prove a fixed point theorem for rational contraction type mappings. Finally, in the last section, we present a new version of fixed point theorem due to Wardowski [13].

2 Auxiliary notions and results

We use the standard notation and terminology of functional analysis. For the organization of the paper, we recall the following:

Definition 2.1 [12] Let (X,D) be a D-generalized metric space. Then a sequence (x_n) in X is said to be:

(i) *convergent* to $x \iff (x_n) \in C(D,X,x)$;

(ii) *Cauchy* $\iff \lim_{n,m\to\infty} D(x_n,x_{n+m}) = 0$.

Remark 2.2 In the D-generalized metric space (X,D), the following results hold:

(i) the limit of a convergent sequence is unique (see Jleli and Samet [12]);

(ii) a convergent sequence may not be Cauchy.

We construct an example of a D-generalized metric space and show that a convergent sequence may not be Cauchy in this structure.

Example 2.3 Let $X = \mathbb{R}^+ \cup \{0,\infty\}$, and let $D : X \times X \to [0,\infty]$ be defined as follows:

$$D(x,y) = \begin{cases} x+y & \text{if at least one of } x \text{ or } y \text{ is } 0, \\ 1+x+y & \text{otherwise.} \end{cases}$$

Now we check the axioms of a D-generalized metric space:

(i) $D(x,y) = 0 \Rightarrow$ either $x+y=0$ or $1+x+y=0$. Now $x+y=0 \Rightarrow x=y=0$ and $1+x+y=0 \Rightarrow x=-1-y$, which is impossible. So $D(x,y)=0 \Rightarrow x=y$.

(ii) It is clear that for all $x,y \in X$, $D(x,y) = D(y,x)$.

(iii) If (x_n) is a sequence converging to a point $x \in X$, then for every $x, y \in X$, we can always find a number $c > 0$ such that $D(x,y) \leq c \limsup_{n\to\infty} D(x_n, y)$. Note that for all $x \in X$, $C(D,X,x) = \emptyset$ except the point 0. So for any sequence (x_n) converging to 0 and $y \in X$, we can find $c > 0$ such that

$$D(y,0) = y \leq cy = c \limsup_{n\to\infty} D(x_n, y).$$

Therefore, all conditions (D1)-(D3) are satisfied. So (X,D) is a D-generalized metric space.

Now, in this structure, we show that every convergent sequence may not be a Cauchy sequence. Let us consider the sequence (x_n) where $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} D(x_n, 0) = \lim_{n\to\infty}\left(\frac{1}{n} + 0\right) = 0 \qquad \Rightarrow \qquad (x_n) \text{ converges to } 0.$$

But,

$$\lim_{n,m\to\infty} D(x_n, x_{n+m}) = \lim_{n,m\to\infty}\left(1 + \frac{1}{n} + \frac{1}{n+m}\right) \neq 0.$$

This shows that (x_n) is a convergent sequence but not a Cauchy sequence.

Note 2.4 The authors of [12] show that every metric space, dislocated metric space, b-metric space, or modular metric space is a D-generalized metric space. Here, our example establishes that D-generalization is a proper generalization of all these spaces since every convergent sequence in a metric space, dislocated metric space, or b-metric space must be a Cauchy sequence, and every modular convergent sequence is a modular Cauchy sequence in a modular metric space.

Definition 2.5 [12] Let (X,D) be a D-generalized metric space, and $T : X \to X$ be a mapping. For any $k \in (0,1)$, T is said to be a k-contraction if

$$D\big(T(x), T(y)\big) \leq kD(x,y) \quad \forall x, y \in X. \tag{2.1}$$

Definition 2.6 [12] For every $x_0 \in X$, we define

$$\delta(D,T,x_0) = \sup\big\{D\big(T^i x_0, T^j x_0\big) : i,j \in \mathbb{N}\big\}.$$

The following theorem is an extension of the Banach contraction principle.

Theorem 2.7 [12] *Suppose that (X,D) is a complete D-generalized metric space and T is a self mapping defined on X. If*

(i) *T is a k-contraction for some $k \in (0,1)$,*

(ii) *$\exists x_0 \in X$ such that $\delta(D,T,x_0) < \infty$,*

then $\{T^n(x_0)\}$ converges to some $w \in X$, a fixed point of T. If w' is another fixed point of T with $D(w,w') < \infty$, then $w = w'$.

They also proved that the Banach contraction principle in the setting of different abstract spaces is nothing but an immediate consequence of this theorem in the corresponding structure. Continuing in this way, they extended another important fixed point theorem for Ćirić quasi-contraction type mappings in D-generalized metric spaces, which again, generalizes the theorems concerning the Ćirić quasi-contraction type mappings in different topological spaces. In this regard, we recall the definition of a k-quasi-contraction.

Definition 2.8 [12] Let (X,D) be a D-generalized metric space, and $T: X \to X$ be a self-mapping. For any $k \in (0,1)$, T is a k-quasi-contraction if for all $x, y \in X$,

$$D(Tx, Ty) \leq kM(x,y),$$

where $M(x,y) = \max\{D(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\}$.

Proposition 2.9 [12] *Suppose that T is a k-quasi-contraction for some $k \in (0,1)$. Then any fixed point $w \in X$ of T satisfies*

$$D(w,w) < \infty \quad \Rightarrow \quad D(w,w) = 0.$$

Theorem 2.10 [12] *Suppose that (X,D) is a complete D-generalized metric space and T is a self-mapping defined on X. If*

(i) *T is a k-quasicontraction for some $k \in (0,1)$,*

(ii) *$\exists x_0 \in X$ such that $\delta(D,T,x_0) < \infty$,*

then $\{T^n(x_0)\}$ converges to some $w \in X$. If $D(x_0, T(w)) < \infty$ and $D(w, T(w)) < \infty$, then w is a fixed point of T. If w' is another fixed point of T with $D(w,w') < \infty$ and $D(w',w') < \infty$, then $w = w'$.

Observe that this theorem does not give the guarantee of the existence of a fixed point of the mapping T for any arbitrary value of $k \in (0,1)$. Indeed, the existence of a fixed point is guaranteed only when $k \in (0,1) \cap (0, \frac{1}{c})$, where, $c > 0$ is the least number for which (D3)-property is satisfied in Definition 1.1. We illustrate this by presenting an example in the next section.

On the other hand, in 2012, Wardowski [13] introduced the notion of an F-contraction, which is perceived to be one of the most general nonlinear contractions in the literature. After that, a lot of research works have been done concerning F-contractions; see, for example, [14–17]. Wardowski introduced the F-contractions as follows.

Definition 2.11 [13] Let (X,d) be a metric space, and $T: X \to X$ be a self-mapping. The function T is said to be an F-contraction mapping if there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y)\bigr),$$

where F belongs to a family of functions $\mathbf{F}$ from $\mathbb{R}_+$ to $\mathbb{R}$ having the following properties:

(F1) F is a strictly increasing function on $\mathbb{R}_+$;

(F2) For each sequence (α_n) of positive numbers, $\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) $\exists k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Lemma 2.12 [14] *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function, and (α_n) be a sequence of positive real numbers. Then the following assertions hold*:

(1) *If $F(\alpha_n) \to -\infty$, then $\alpha_n \to 0$;*
(2) *If $\inf F = -\infty$ and $\alpha_n \to 0$ then $F(\alpha_n) \to -\infty$.*

Taking into account this lemma, Piri and Kumam [14] considered a new set $\mathcal{F}$ of functions $F : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(F1′) F is a strictly increasing function on $\mathbb{R}_+$;
(F2′) $\inf F = -\infty$;
(F3′) F is continuous.

They proved some new fixed point results concerning F-contractions.

Here, we consider $\overline{\mathbb{R}}_+ = (0,\infty]$. We use the standard arithmetic operations on $\overline{\mathbb{R}}_+$ and suppose that $a \leq \infty$ for all $a \in \overline{\mathbb{R}}_+$. Now, we consider a new family $\mathfrak{F}$ of functions having the following properties:

(F1″) F is a strictly increasing function, that is, for $x, y \in \overline{\mathbb{R}}_+$ such that $x < y$, $F(x) < F(y)$;
(F2″) $\inf F = -\infty$.

Example 2.13 We consider the function $F : \overline{\mathbb{R}}_+ \to \mathbb{R}$ defined by

$$F(\alpha) = \begin{cases} -\frac{1}{\alpha}, & 0 < \alpha \leq 1; \\ \alpha + \frac{1}{\alpha}, & \alpha > 1; \\ \infty, & \alpha = \infty. \end{cases}$$

Note that $F \in \mathfrak{F}$, whereas F belongs neither to $\mathbf{F}$ nor to $\mathcal{F}$ since F does not satisfy conditions (F3) and (F3′). Therefore, $\mathbf{F} \subset \mathfrak{F}$ and $\mathcal{F} \subset \mathfrak{F}$, but the converse is not true.

Considering the new family $\mathfrak{F}$ of functions, we prove the result of Wardowski in the setting of the newly defined complete D-generalized metric space.

3 Ćirić quasi-contraction

We start this section by presenting an example.

Example 3.1 Let $X = [0,1]$. We define the distance function D on X as follows:

$$D(x,y) = \begin{cases} x + y & \text{if at least one of } x \text{ or } y \text{ is } 0, \\ \frac{x+y}{3} & \text{otherwise.} \end{cases}$$

First, we check the axioms of a D-generalized metric space.

(i) It is clear that $D(x,y) = 0 \Rightarrow x = y$.
(ii) $D(x,y) = D(y,x)$ for all $x,y \in X$.
(iii) For all $x \neq 0$, we have $C(D,X,x) = \emptyset$. If $x = 0$, then we can always find a sequence (x_n) converging to 0. So for any $y \in X$, there exists a number $c \geq 3$ such that $D(0,y) = y \leq c\frac{y}{3} = c \limsup_{n\to\infty} D(x_n,y)$. Furthermore, if (x_n) is a zero sequence, then for all $c \geq 1$, we have $D(0,y) = y \leq cy = c \limsup_{n\to\infty} D(x_n,y)$.

Therefore, all conditions (D1)-(D3) are satisfied. So (X, D) is a D-generalized metric space. Now we define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{2} & \text{for } x \neq 0, \\ 1 & \text{whenever } x = 0. \end{cases}$$

Now, we show that this mapping satisfies Ćirić quasi-contraction condition.

Case I: For all $x, y \in (0, 1]$, we have

$$D(Tx, Ty) = \frac{x+y}{6}$$

and

$$\begin{aligned} M(x,y) &= \max\left\{D(x,y), D\left(x, \frac{x}{2}\right), D\left(y, \frac{y}{2}\right), D\left(x, \frac{y}{2}\right), D\left(y, \frac{x}{2}\right)\right\} \\ &= \max\left\{\frac{x+y}{3}, \frac{x}{3}, \frac{y}{3}, \frac{x+\frac{y}{2}}{3}, \frac{y+\frac{x}{2}}{3}\right\}. \end{aligned}$$

Clearly, $\frac{x+y}{3} \leq M(x,y)$, and hence, for all $k \in [\frac{1}{2}, 1)$, we get

$$D(Tx, Ty) \leq kM(x, y).$$

Case II: Let $x = 0$, and y be any arbitrary point. Then

$$D(T0, Ty) = D\left(1, \frac{y}{2}\right) = \frac{1 + \frac{y}{2}}{3}$$

and

$$\begin{aligned} M(0,y) &= \max\left\{D(0,y), D(0,1), D\left(y, \frac{y}{2}\right), D\left(0, \frac{y}{2}\right), D(y,1)\right\} \\ &= \max\left\{y, 1, \frac{y}{2}, \frac{y}{2}, \frac{1+y}{3}\right\}. \end{aligned}$$

Let us consider $y = 1$. Then $D(T0, T1) = \frac{1}{2}$ and $M(0,1) = 1$. Therefore, for all $k \in [\frac{1}{2}, 1)$, we get

$$D(T0, T1) \leq kM(0, 1).$$

Similarly, for all $y \in [0, 1]$, we can find some $k \in (0, 1)$ such that

$$D(T0, Ty) \leq kM(0, y).$$

Considering these two cases, we can conclude that, for all $x, y \in X$,

$$D(Tx, Ty) \leq kM(x, y)$$

for some $k \in (0,1)$, that is, T satisfies the k-quasi-contraction condition. Let us set $x_0 = 1$. Then it is clear that $\delta(D,T,x_0) < \infty$. Now, $Tx_0 = \frac{1}{2}, T^2x_0 = \frac{1}{2^2}, \dots, T^nx_0 = \frac{1}{2^n}$, and so on. Clearly, (T^nx_0) is a Cauchy sequence and converges to $w = 0$. Comparing with the conditions of Theorem 2.10, we have

$$D(x_0, Tw) = D(1, T0) = \frac{2}{3} < \infty$$

and

$$D(w, Tw) = D(0, T0) = 1 < \infty.$$

Thus, all the conditions of Theorem 2.10 are satisfied, but still $w = 0$ is not a fixed point of T since $T0 = 1$. Also, note that the mapping T does not have any fixed points.

Such a problem occurs due to the choice of arbitrary value of $k \in (0,1)$. We can avoid this problem by taking $k \in (0,1) \cap (0, \frac{1}{c})$, where c is the least positive number for which (D3)-property is satisfied. Here we give a modified version of Theorem 2.10.

Theorem 3.2 *Suppose that (X,D) is a complete D-generalized metric space and T is a self-mapping defined on X. If*

(i) *T is a k-quasi-contraction for some $k \in (0,1) \cap (0, \frac{1}{c})$,*

(ii) *$\exists x_0 \in X$ such that $\delta(D,T,x_0) < \infty$,*

then $\{T^n(x_0)\}$ converges to some $w \in X$. If $D(x_0, T(w)) < \infty$ and $D(w, T(w)) < \infty$, then w is a fixed point of T. If w' is another fixed point of T with $D(w,w') < \infty$ and $D(w',w') < \infty$, then $w = w'$.

Proof Proof of the first part of this theorem follows from that of Theorem 4.3 in [12]. Here, we just give a corrected version of the last part of the proof. Using property (D3) and $D(w, T(w)) < \infty$, we have

$$\begin{aligned} D(w, Tw) &\le c \limsup_{n\to\infty} D\big(T^{n+1}x_0, Tw\big) \le kcD(w, Tw) \\ &\Rightarrow \quad (1 - kc)D(w, Tw) \le 0 \\ &\Rightarrow \quad D(w, Tw) = 0 \quad \text{as } (1 - kc) > 0 \\ &\Rightarrow \quad w = Tw, \end{aligned}$$

that is, w is a fixed point of T. □

Next, we introduce the concept of a D-admissible mapping.

Definition 3.3 Let (X,D) be a D-generalized metric space, and T be a self-mapping on X. Then T is said to be a D-admissible mapping if for all $x, y \in X$,

$$D(x,y) < \infty \quad \Rightarrow \quad D(Tx, Ty) < \infty.$$

Lemma 3.4 *Suppose that (X,D) is a D-generalized metric space and T is a D-admissible mapping on X. Then for every sequence (x_n) converging to a point $w \in X$, we have $D(w,Tw) < \infty$.*

Proof Since $x_n \to w$ as $n \to \infty \Rightarrow \lim_{n\to\infty} D(x_n, w) = 0$, we can find a positive integer n_0 such that $D(x_n, w) < \infty$ for all $n > n_0$. Again, since T is a D-admissible mapping, we must have $D(Tx_n, Tw) < \infty$ for all $n > n_0$, that is, $\limsup_{n\to\infty} D(Tx_n, Tw) < \infty$. Using the (D3)-property, we have,

$$D(w,Tw) \leq c \limsup_{n\to\infty} D(Tx_n, Tw) \quad \Rightarrow \quad D(w,Tw) < \infty. \qquad \square$$

Using the concept of a D-admissible mapping, we can establish the fixed point result for Ćirić quasi-contraction mappings in a different way.

Theorem 3.5 *Suppose that (X,D) is a complete D-generalized metric space and T is a D-admissible self-mapping defined on X. If*

(i) *T is a k-quasi-contraction for some $k \in (0,1) \cap (0, \frac{1}{c})$,*

(ii) *$\exists x_0 \in X$ such that $\delta(D,T,x_0) < \infty$,*

then $\{T^n(x_0)\}$ converges to some $w \in X$, and this w is a fixed point of T. If w' is another fixed point of T with $D(w,w') < \infty$ and $D(w',w') < \infty$, then $w = w'$.

Proof Since T is k-quasi-contraction, for all $n \geq 1$ and $i,j \in \mathbb{N}$, we have

$$\begin{aligned} D\bigl(T^{n+i}x_0, T^{n+j}x_0\bigr) \leq k \max\bigl\{ & D\bigl(T^{n-1+i}x_0, T^{n-1+j}x_0\bigr), D\bigl(T^{n-1+i}x_0, T^{n+i}x_0\bigr), \\ & D\bigl(T^{n-1+j}x_0, T^{n+j}x_0\bigr), D\bigl(T^{n-1+i}x_0, T^{n+j}x_0\bigr), \\ & D\bigl(T^{n+i}x_0, T^{n-1+j}x_0\bigr)\bigr\}. \end{aligned} \tag{3.1}$$

Using Definition 2.6, we obtain

$$\begin{aligned} \delta\bigl(D,T,T^n(x_0)\bigr) &\leq k\delta\bigl(D,T,T^{n-1}(x_0)\bigr) \\ &\leq k^2\delta\bigl(D,T,T^{n-2}(x_0)\bigr) \\ &\vdots \\ &\leq k^n\delta(D,T,x_0). \end{aligned} \tag{3.2}$$

From this inequality, for all $n,m \in \mathbb{N}$, we get

$$\begin{aligned} D\bigl(T^n x_0, T^{n+m}x_0\bigr) = D\bigl(T\bigl(T^{n-1}x_0\bigr), T^{m+1}\bigl(T^{n-1}x_0\bigr)\bigr) &\leq \delta\bigl(D,T,T^{n-1}(x_0)\bigr) \\ &\leq k^{n-1}\delta(D,T,x_0). \end{aligned}$$

Since $k \in (0,1)$, we have

$$\lim_{n\to\infty} D\bigl(T^n x_0, T^{n+m}x_0\bigr) = 0, \quad \forall m,$$

which implies that $(T^n(x_0))$ is a Cauchy sequence. Since (X,D) is complete, we must have some $w \in X$ such that $(T^n(x_0))$ is convergent to w. We prove that w is a fixed point of T. Now,

$$\begin{aligned} D\bigl(T^{n+1}x_0, Tw\bigr) &\leq k \max\bigl\{D\bigl(T^n x_0, w\bigr), D\bigl(T^{n+1}x_0, T^n x_0\bigr), D(w, Tw), \\ &\qquad D\bigl(T^n x_0, Tw\bigr), D\bigl(T^{n+1}x_0, w\bigr)\bigr\} \\ \Rightarrow \quad \limsup_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr) &\leq \max\Bigl\{0, D(w, Tw), \limsup_{n\to\infty} D\bigl(T^n x_0, Tw\bigr)\Bigr\}. \end{aligned} \tag{3.3}$$

If $\limsup_{n\to\infty} D(T^n x_0, Tw)$ is the maximum then from inequality (3.3), then we have

$$\limsup_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr) \leq k \limsup_{n\to\infty} D\bigl(T^n x_0, Tw\bigr).$$

Since T is a D-admissible mapping, we must have $\limsup_{n\to\infty} D(T^n x_0, Tw) < \infty$, which implies that the last inequality is impossible since $k \in (0,1)$. Therefore, we must have

$$\limsup_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr) \leq kD(w, Tw).$$

Using property (D3) and Lemma 3.4, we have

$$\begin{aligned} &D(w, Tw) \leq c \limsup_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr) \leq kcD(w, Tw) \\ &\quad\Rightarrow \quad (1-kc)D(w, Tw) \leq 0 \\ &\quad\Rightarrow \quad D(w, Tw) = 0 \quad \text{as } (1-kc) > 0 \\ &\quad\Rightarrow \quad w = Tw, \end{aligned}$$

that is, w is a fixed point of T.

Let w' be another fixed point of T with $D(w,w') < \infty$ and $D(w',w') < \infty$. So by Proposition 2.9 we must have $D(w',w') = 0$. Again, by the property of k-quasi-contraction of T,

$$\begin{aligned} &D\bigl(w, w'\bigr) = D\bigl(Tw, Tw'\bigr) \leq kD\bigl(w, w'\bigr) \\ &\quad\Rightarrow \quad D\bigl(w, w'\bigr) = 0 \\ &\quad\Rightarrow \quad w = w' \end{aligned}$$

since $k \in (0,1)$ and $D(w,w') < \infty$. Hence, the proof is completed. □

Definition 3.6 Let (X,D) be a complete D-generalized metric space, and T be a self-mapping on X. T is said to satisfy the rational inequality if

$$D(Tx, Ty) \leq k \max\left\{D(x,y), \frac{D(x, Tx)}{1 + D(x, Tx)}, \frac{D(y, Ty)}{1 + D(y, Ty)}, \frac{D(x, Ty) + D(y, Tx)}{2}\right\}$$

for some $k \in (0,1)$.

Theorem 3.7 *Suppose that T is a D-admissible mapping defined on a complete D-generalized metric space (X,D) satisfying the rational inequality. If there exists $x_0 \in X$ such that $\delta(D,T,x_0) < \infty$, then the sequence $(T^n x_0)$ converges to some point $w \in X$, and this w is a fixed point of T. Again, if w' is another fixed point of T with $D(w,w') < \infty$ and $D(w',w') < \infty$, then $w = w'$.*

Proof For all $i,j,n \in \mathbb{N}$, we have

$$\begin{aligned}
&D\bigl(T^{n+i}x_0, T^{n+j}x_0\bigr)\\
&\quad\le k\max\biggl\{D\bigl(T^{n-1+i}x_0, T^{n-1+j}x_0\bigr), \frac{D(T^{n-1+i}x_0, T^{n+i}x_0)}{1+D(T^{n-1+i}x_0, T^{n+i}x_0)},\\
&\qquad\frac{D(T^{n-1+j}x_0, T^{n+j}x_0)}{1+D(T^{n-1+j}x_0, T^{n+j}x_0)}, \frac{D(T^{n-1+i}x_0, T^{n+j}x_0)+D(T^{n+i}x_0, T^{n-1+j}x_0)}{2}\biggr\}\\
&\quad\le k\max\biggl\{D\bigl(T^{n-1+i}x_0, T^{n-1+j}x_0\bigr), D\bigl(T^{n-1+i}x_0, T^{n+i}x_0\bigr),\\
&\qquad D\bigl(T^{n-1+j}x_0, T^{n+j}x_0\bigr), \frac{D(T^{n-1+i}x_0, T^{n+j}x_0)+D(T^{n+i}x_0, T^{n-1+j}x_0)}{2}\biggr\}\\
&\quad\le k\max\biggl\{D\bigl(T^{i}\bigl(T^{n-1}x_0\bigr), T^{j}\bigl(T^{n-1}x_0\bigr)\bigr), D\bigl(T^{i}\bigl(T^{n-1}x_0\bigr), T^{i+1}\bigl(T^{n-1}x_0\bigr)\bigr),\\
&\qquad D\bigl(T^{j}\bigl(T^{n-1}x_0\bigr), T^{j+1}\bigl(T^{n-1}x_0\bigr)\bigr),\\
&\qquad\frac{D(T^{i}(T^{n-1}x_0), T^{j+1}(T^{n-1}x_0))+D(T^{i+1}(T^{n-1}x_0), T^{j}(T^{n-1}x_0))}{2}\biggr\}\\
&\Rightarrow\quad \delta\bigl(D,T,T^n(x_0)\bigr)\le k\delta\bigl(D,T,T^{n-1}(x_0)\bigr)\\
&\qquad\qquad\qquad\qquad\vdots\\
&\qquad\qquad\qquad\qquad\le k^n\delta(D,T,x_0).
\end{aligned}\tag{3.4}$$

Taking the limiting value of n, we get

$$k^n\delta(D,T,x_0) = 0 \quad \text{since } k\in(0,1).$$

As previously, for all $n,m \in \mathbb{N}$, we get

$$D\bigl(T^n(x_0), T^{n+m}x_0\bigr) = 0,$$

which shows that $T^n(x_0)$ is a Cauchy sequence. Let $T^n(x_0)$ converge to some $w \in X$ since (X,D) is complete. Let us show that this w is a fixed point of T. We have

$$\begin{aligned}
D\bigl(T^{n+1}x_0, Tw\bigr) &\le k\max\biggl\{D\bigl(T^n x_0, w\bigr), \frac{D(T^n x_0, T^{n+1}x_0)}{1+D(T^n x_0, T^{n+1}x_0)}, \frac{D(Tw,w)}{1+D(Tw,w)},\\
&\qquad\frac{D(T^n x_0, Tw)+D(w, T^{n+1}x_0)}{2}\biggr\}
\end{aligned}$$

$$\Rightarrow\quad \limsup_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr)\le k\max\biggl\{D(w,Tw), \frac{1}{2}\limsup_{n\to\infty} D\bigl(T^n x_0, Tw\bigr)\biggr\}. \tag{3.5}$$

From Equation (3.5) it is clear that

$$\limsup_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr) \leq kD(w, Tw).$$

Proceeding as before, we get that w is a fixed point of T. □

The following result is an immediate consequence of Theorem 3.7.

Corollary 3.8 *Let $T : X \to X$ be a D-admissible self mapping, and (X,D) be a complete D-generalized metric space. Suppose that the following conditions hold:*

(i) *for all $x, y \in X$, there exists $k \in (0,1)$ such that*

$$D(Tx, Ty) \leq k \max\left\{D(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(Tx,y)}{2}\right\};$$

(ii) *$\exists x_0 \in X$ such that $\delta(D,T,x_0) < \infty$.*

Then $(T^n(x_0))$ converges to some $w \in X$, and this w is a fixed point of T. Moreover, if w' is another fixed point of T with $D(w,w') < \infty$ and $D(w',w') < \infty$, then $w = w'$.

4 Results of *F*-contraction

This section is devoted to a fixed point theorem of Wardowski type contraction. It is worth mentioning that our proof of the following theorem is very precise and it is interesting to compare it with the existing proof in the literature. First, we introduce the definition of F-contraction. Note that in a metric space, $d(x,x) = 0$ for all x, so the condition $d(Tx,Ty) > 0 \Rightarrow d(x,y) > 0$, and hence the condition

$$d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y)\bigr)$$

for some $\tau > 0$, is appropriate. But in a D-generalized metric space, since $D(x,x) \neq 0$ for all x, $D(Tx,Ty) > 0$ may not give the guarantee that $D(x,y) > 0$. Hence, we modify the definition of F-contraction as follows.

Definition 4.1 A self-mapping T defined on X is said to be an F-contraction mapping if for all $x, y \in X$,

$$\begin{aligned} &D(x,y) > 0 \quad \text{and} \quad D(Tx,Ty) > 0 \\ &\quad \Rightarrow \quad \tau + F\bigl(D(Tx,Ty)\bigr) \leq F\bigl(D(x,y)\bigr) \end{aligned}$$

for some $\tau > 0$.

Now, we state the theorem that establishes the existence and uniqueness of a fixed point for the mappings satisfying the F-contraction principle.

Theorem 4.2 *Let (X,D) be a D-generalized metric space, and $T : X \to X$ be an F-contraction mapping with $F \in \mathfrak{F}$. Assume that the following conditions hold:*

(1) *(X,D) is complete;*

(2) *$\exists x_0 \in X$ such that $\delta(D,T,x_0) = c$ for some finite $c \neq 0$.*

Then T has a fixed point. If w' is another fixed point with $D(w,w') < \infty$, then $w = w'$.

Proof By the hypothesis of the theorem, there exists some $x_0 \in X$ such that $\delta(D,T,x_0) = c$ for some finite $c \neq 0$. Since T is an F-contraction, for all $n \geq 1$ and $i,j \in \mathbb{N}$, we get

$$\begin{aligned} F\bigl(D\bigl(T^{n+i}x_0, T^{n+j}x_0\bigr)\bigr) &\leq F\bigl(D\bigl(T^{n-1+i}x_0, T^{n-1+j}x_0\bigr)\bigr) - \tau \\ &\leq F\bigl(D\bigl(T^{n-2+i}x_0, T^{n-2+j}x_0\bigr)\bigr) - 2\tau \\ &\vdots \\ &\leq F\bigl(D\bigl(T^{i}x_0, T^{j}x_0\bigr)\bigr) - n\tau. \end{aligned} \tag{4.1}$$

From Definition 2.6 it is clear that, for all $i,j \in \mathbb{N}$,

$$D\bigl(T^{i}x_0, T^{j}x_0\bigr) \leq \delta(D,T,x_0).$$

Since F is a strictly increasing function, we must have

$$F\bigl(D\bigl(T^{i}x_0, T^{j}x_0\bigr)\bigr) \leq F\bigl(\delta(D,T,x_0)\bigr) \quad \forall i,j \in \mathbb{N}.$$

So from Equation (4.1) we obtain

$$F\bigl(D\bigl(T^{n+i}x_0, T^{n+j}x_0\bigr)\bigr) \leq F\bigl(\delta(D,T,x_0)\bigr) - n\tau.$$

Therefore, for all $n,m \in \mathbb{N}$,

$$\begin{aligned} F\bigl(D\bigl(T^{n}x_0, T^{n+m}x_0\bigr)\bigr) &= F\bigl(D\bigl(T\bigl(T^{n-1}x_0\bigr), T^{m+1}\bigl(T^{n-1}x_0\bigr)\bigr)\bigr) \\ &\leq F\bigl(\delta(D,T,x_0)\bigr) - (n-1)\tau. \end{aligned} \tag{4.2}$$

Taking $n \to \infty$ in both sides of inequality (4.2) and using property (F2$''$) and Lemma 2.12, we have

$$\begin{aligned} &\lim_{n\to\infty} F\bigl(D\bigl(T^{n}x_0, T^{n+m}x_0\bigr)\bigr) = -\infty \\ &\quad\Rightarrow\quad \lim_{n\to\infty} D\bigl(T^{n}x_0, T^{n+m}x_0\bigr) = 0 \quad \forall m \in \mathbb{N} \\ &\quad\Rightarrow\quad \bigl(T^{n}x_0\bigr) \text{ is a Cauchy sequence.} \end{aligned}$$

Since (X,D) is complete, $T^n x_0 \to w$ for some $w \in X$ as $n \to \infty$. We prove that this w is a fixed point of T. Since F is a strictly increasing function, we have

$$\begin{aligned} &F\bigl(D\bigl(T^{n+1}x_0, Tw\bigr)\bigr) \leq F\bigl(D\bigl(T^{n}x_0, w\bigr)\bigr) - \tau \\ &\quad\Rightarrow\quad F\bigl(D\bigl(T^{n+1}x_0, Tw\bigr)\bigr) \leq F\bigl(D\bigl(T^{n}x_0, w\bigr)\bigr) \quad \text{since } \tau > 0 \\ &\quad\Rightarrow\quad D\bigl(T^{n+1}x_0, Tw\bigr) \leq D\bigl(T^{n}x_0, w\bigr) \\ &\quad\Rightarrow\quad \lim_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr) \leq \lim_{n\to\infty} D\bigl(T^{n}x_0, w\bigr) \\ &\quad\Rightarrow\quad \lim_{n\to\infty} D\bigl(T^{n+1}x_0, Tw\bigr) = 0. \end{aligned} \tag{4.3}$$

Using property (D3) and Equation (4.3), we obtain

$$\begin{aligned} &D(Tw,w) \le c \limsup_{n\to\infty} D\bigl(Tw, T^{n+1}x_0\bigr) \quad \text{for some } c > 0 \\ &\quad\Rightarrow\quad D(Tw,w) = 0 \\ &\quad\Rightarrow\quad w = Tw \\ &\quad\Rightarrow\quad w \text{ is a fixed point of } T. \end{aligned}$$

If possible, let w' be another fixed point of T with $D(w,w') < \infty$. We show that $D(w,w') = 0$. If not, let $D(w,w') = k$ for some positive k. Then by the property of F-contraction of T we have

$$\begin{aligned} &\tau + F\bigl(D\bigl(Tw,Tw'\bigr)\bigr) \le F\bigl(D\bigl(w,w'\bigr)\bigr) \\ &\quad\Rightarrow\quad \tau \le 0 \to \text{contradiction} \\ &\quad\Rightarrow\quad D\bigl(w,w'\bigr) = 0 \\ &\quad\Rightarrow\quad w = w'. \end{aligned}$$

So, w is a unique fixed point of T. □

Notice that every standard metric space is a D-generalized metric space. Consequently, the result of Wardowski [13] can be presented as an immediate consequence of Theorem 4.2. We state this as follows.

Corollary 4.3 *Let (X,d) be a complete metric space, and T be a self-mapping defined on X such that*

$$d(Tx,Ty) > 0 \quad\Rightarrow\quad \tau + F\bigl(d(Tx,Ty)\bigr) \le F\bigl(d(x,y)\bigr) \quad \textit{for some } \tau > 0,$$

where $F \in \mathfrak{F}$. Now if there exists $x_0 \in X$ such that

$$\delta(d,T,x_0) = \sup\bigl\{d\bigl(T^i x_0, T^i x_0\bigr) : i,j \in \mathbb{N}\bigr\} = c$$

for some finite $c \neq 0$, then T has a fixed point. Also, if w' is another fixed point with $D(w,w') < \infty$, then $w = w'$.

Obviously, it is notable that the domain space of a function F is much wider than that of the existing literature since F does not satisfy conditions (F3) and (F3′) mentioned in Section 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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