Guan et al. *Fixed Point Theory and Applications* (2015) 2015:214
DOI 10.1186/s13663-015-0457-4

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH
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Non-convex hybrid algorithm for a family of countable quasi-Lipschitz mappings and application

Jinyu Guan[1], Yanxia Tang[1], Pengcheng Ma[1], Yongchun Xu[1*] and Yongfu Su[2]

*Correspondence:
hbxuyongchun@163.com
[1]Department of Mathematics, Hebei North University, Zhangjiakou, 075000, China
Full list of author information is available at the end of the article

Abstract
The purpose of this article is to establish a kind of non-convex hybrid iteration algorithms and to prove relevant strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. Meanwhile, the main result is applied to get the common fixed points of finite family of quasi-asymptotically nonexpansive mappings. It is worth pointing out that a non-convex hybrid iteration algorithm is first presented in this article, a new technique is applied in our process of proof. Finally, an example is given which is a uniformly closed asymptotically family of countable quasi-Lipschitz mappings. The results presented in this article are interesting extensions of some current results.

MSC: 47H05; 47H09; 47H10

Keywords: nonexpansive mapping; hybrid algorithm; Cauchy sequence; closed quasi-nonexpansive

1 Introduction

Construction of fixed points of nonexpansive mappings (and asymptotically nonexpansive mappings) is an important subject in the theory of nonexpansive mappings and finds application in a number of applied areas. Recently, a great deal of literature on iteration algorithms for approximating fixed points of nonexpansive mappings has been published since one has a variety of applications in inverse problem, image recovery, and signal processing; see [1–8]. Mann's iteration process [1] is often used to approximate a fixed point of the operators, but it has only weak convergence (see [3] for an example). However, strong convergence is often much more desirable than weak convergence in many problems that arise in infinite dimensional spaces (see [7] and references therein). So, attempts have been made to modify Mann's iteration process so that strong convergence is guaranteed (see [9–24] and references therein).

In 2003, Nakajo and Takahashi [25] proposed a modification of Mann iteration method for a single nonexpansive mapping in a Hilbert space. In 2006, Kim and Xu [26] proposed a modification of Mann iteration method for asymptotically nonexpansive mapping T in a Hilbert space. They also proposed a modification of the Mann iteration method for asymptotically nonexpansive semigroup in a Hilbert space. In 2006, Martinez-Yanes and Xu [27] proposed a modification of the Ishikawa iteration method for nonexpansive mapping in a Hilbert space. Martinez-Yanes and Xu [27] proposed also a modification of the Halpern iteration method for nonexpansive mapping in a Hilbert space. In 2008, Su and Qin [28] proposed first a monotone hybrid iteration method for nonexpansive mapping in a Hilbert space. In 2015, Dong and Lu [29] proposed a new iteration method for nonexpansive mapping in a Hilbert space. In 2015, Liu *et al.* [30] proposed a new iteration method for a finite family of quasi-asymptotically pseudocontractive mappings in a Hilbert spaces.

Throughout this paper, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Let C be a nonempty, closed, and convex subset of H, we denote by $P_C(\cdot)$ the metric projection onto C. It is well known that $z = P_C(x)$ is equivalent to that $z \in C$ and $\langle z-y, x-z\rangle \geq 0$ for every $y \in C$. Recall that $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. It is well known that $F(T)$ is closed and convex. A mapping $T : C \to C$ is said to be quasi-Lipschitz, if the following conditions hold:

(1) the fixed point set $F(T)$ is nonempty;

(2) $\|Tx - p\| \leq L\|x - p\|$ for all $x \in C$, $p \in F(T)$,

where $1 \leq L < +\infty$ is a constant. T is said to be quasi-nonexpansive, if $L = 1$.

Recall that a mapping $T : C \to C$ is said to be closed if $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. A mapping $T : C \to C$ is said to be weak closed if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. It is obvious that a weak closed mapping must be a closed mapping, the inverse is not true.

Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $\{T_n\}$ be sequence of mappings from C into itself with a nonempty common fixed point set F. $\{T_n\}$ is said to be uniformly closed if for any convergent sequence $\{z_n\} \subset C$ such that $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

The purpose of this article is to establish a kind of non-convex hybrid iteration algorithms and to prove relevant strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. Meanwhile, the main result was applied to get the common fixed points of finite family of quasi-asymptotically nonexpansive mappings. It is worth pointing out that a non-convex hybrid iteration algorithm was first presented in this article, a new technique has been applied in our process of proof. Finally, an example has been given which is a uniformly closed asymptotically family of countable quasi-Lipschitz mappings. The results presented in this article are interesting extensions of some current results.

2 Main results

The following lemma is well known and is useful for our conclusions.

Lemma 2.1 *Let C be a nonempty, closed, and convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if we have the relation*

$$\langle x - z, z - y\rangle \geq 0$$

for all $y \in C$.

Definition 2.2 Let H be a Hilbert space, let C be a closed convex subset of E, and let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself, $\{T_n\}$ is said to be asymptotically, if $\lim_{n\to\infty} L_n = 1$.

Lemma 2.3 *Let H be a Hilbert space, let C be a closed convex subset of E, and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Then the common fixed point set F is closed and convex.*

Proof Let $p_n \in F$ and $p_n \to p$ as $n \to \infty$, we have

$$\|T_n p_n - p_n\| = 0 \to 0, \quad p_n \to p$$

as $n \to \infty$. Since $\{T_n\}$ is uniformly closed, we know that $p \in F$, therefore F is closed. Next we show that F is also convex. For any $x, y \in F$, let $z = tx + (1-t)y$ for any $t \in (0,1)$, we have

$$\begin{aligned}
\|T_n z - z\|^2 &= \langle T_n z - z, T_n z - z\rangle \\
&= \|T_n z\|^2 - 2\langle T_n z, z\rangle + \|z\|^2 \\
&= \|T_n z\|^2 - 2\bigl\langle T_n z, tx + (1-t)y\bigr\rangle + \|z\|^2 \\
&= \|T_n z\|^2 - 2t\langle T_n z, x\rangle + 2(1-t)\langle T_n z, y\rangle + \|z\|^2 \\
&= t\|T_n z - x\|^2 + (1-t)\|T_n z - y\|^2 - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&\leq tL_n^2\|z - x\|^2 + (1-t)L_n^2\|z - y\|^2 - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&= t\|z - x\|^2 + (1-t)\|z - y\|^2 - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&\quad + t\bigl(L_n^2 - 1\bigr)\|z - x\|^2 + (1-t)\bigl(L_n^2 - 1\bigr)\|y - x\|^2 \\
&= \|z\|^2 - 2\langle z, z\rangle + \|z\|^2 \\
&\quad + t\bigl(L_n^2 - 1\bigr)\|z - x\|^2 + (1-t)\bigl(L_n^2 - 1\bigr)\|y - x\|^2 \to 0
\end{aligned}$$

as $n \to \infty$. Since $z \to z$, and $\{T_n\}$ is uniformly closed, $z \in F$. Therefore F is convex. This completes the proof. □

The following conclusion is well known.

Lemma 2.4 *Let C be a closed convex subset of a Hilbert space H, for any given $x_0 \in H$, we have*

$$p = P_C x_0 \quad \Leftrightarrow \quad \langle p - z, x_0 - p\rangle \geq 0, \quad \forall z \in C.$$

Theorem 2.5 *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\} : C \to C$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0,1)$. Then $\{x_n\}$ generated by*

$$\begin{cases}
x_0 \in C = Q_0 \quad \text{chosen arbitrarily}, \\
y_n = (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\|\} \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{\overline{\mathrm{co}}\, C_n \cap Q_n} x_0,
\end{cases}$$

converges strongly to $P_F x_0$, *where* $\overline{\mathrm{co}}\, C_n$ *denotes the closed convex closure of* C_n *for all* $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof We split the proof into seven steps.

Step 1. It is obvious that $\overline{\mathrm{co}}\, C_n$, Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{\mathrm{co}}\, C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned} \|y_n - p\| &= \big\|(1 - \alpha_n)x_n + \alpha_n T_n x_n - p\big\| \\ &= \big\|\alpha_n(x_n - p) + (1 - \alpha_n)(T_n x_n - p)\big\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n L_n \|x_n - p\| \\ &= \big(1 + (L_n - 1)\alpha_n\big)\|x_n - z\| \end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. Therefore, $F \cap A \subset \overline{\mathrm{co}}\, C_n$ for all $n \geq 0$.

Step 2. We show that $F \cap A \subset \overline{\mathrm{co}}\, C_n \cap Q_n$ for all $n \geq 0$. It suffices to show that $F \cap A \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$, we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{\mathrm{co}}\, C_n \cap Q_n$, from Lemma 2.1, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \quad \forall z \in \overline{\mathrm{co}}\, C_n \cap Q_n$$

as $F \cap A \subset \overline{\mathrm{co}}\, C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{\mathrm{co}}\, C_n \cap Q_n$ holds for all $n \geq 0$.

Step 3. We prove $\{x_n\}$ is bounded. Since F is a nonempty, closed, and convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{\mathrm{co}}\, C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{\mathrm{co}}\, C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{\mathrm{co}}\, C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

Step 4. We show that $\{x_n\}$ converges strongly to a point of C (we show that $\{x_n\}$ is a Cauchy sequence). As $x_{n+1} = P_{\overline{\mathrm{co}}\, C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Lemma 2.4), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0\rangle \leq 0$ and hence

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &= \big\|(x_{n+m} - x_0) - (x_n - x_0)\big\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \quad n \to \infty \end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a Cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

Step 5. We show that $y_n \to q$, as $n \to \infty$. Let

$$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + 4(L_n - 1)(L_n + 1)\}.$$

From the definition of D_n, we have

$$\begin{aligned} D_n &= \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle + (L_n - 1)(L_n + 1)2\} \\ &= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 + 4(L_n - 1)(L_n + 1)\} \\ &= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + 4(L_n - 1)(L_n + 1)\}. \end{aligned}$$

This implies that D_n is closed and convex, for all $n \geq 0$. Next, we show that

$$C_n \subset D_n, \quad n \geq 0.$$

In fact, for any $z \in C_n$, we have

$$\begin{aligned} \|y_n - z\|^2 &\leq \bigl(1 + (L_n - 1)\alpha_n\bigr)^2 \|x_n - z\|^2 \\ &= \|x_n - z\|^2 + \bigl[2(L_n - 1)\alpha_n + (L_n - 1)^2\alpha_n^2\bigr]\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + \bigl[2(L_n - 1) + (L_n - 1)^2\bigr]\|x_n - z\|^2 \\ &= \|x_n - z\|^2 + (L_n - 1)(L_n + 1)\|x_n - z\|^2. \end{aligned}$$

From

$$C_n = \{z \in C : \|y_n - z\| \leq \bigl(1 + (L_n - 1)\alpha_n\bigr)\|x_n - z\|\} \cap A, \quad n \geq 0,$$

we have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{\mathrm{co}}\, C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{\mathrm{co}}\, C_{n-1}$, we know that

$$\begin{aligned} \|y_n - z\|^2 &\leq \|x_n - z\|^2 + (L_n - 1)(L_n + 1)\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + 4(L_n - 1)(L_n + 1). \end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Since D_n is convex, we have $\overline{\mathrm{co}}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + 4(L_n - 1)(L_n + 1) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

Step 6. We show that $q \in F$. From the definition of y_n, we have

$$\alpha_n\|T_n x_n - x_n\| = \|y_n - x_n\| \to 0$$

as $n \to \infty$. Since $\alpha_n \in (a,1] \subset [0,1]$, from the above limit we have

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\|z_0 - x_0\|^2 = \|z_0 - x_n + x_n - x_0\|^2 = \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. □

Next, we give an example of C_n not involving a convex subset.

Example 2.6 Let $H = R^2$, $T_n : R^2 \to R^2$ be a sequence of mappings defined by

$$T_n : (t_1, t_2) \mapsto \left(t_1, \frac{1}{8}t_2\right), \quad \forall (t_1, t_2) \in R^2, \forall n \geq 0.$$

It is obvious that $\{T_n\}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings with the common fixed point set $F = \{(t_1, 0) : t_1 \in (-\infty, +\infty)\}$. Take $x_0 = (4, 0)$, $\alpha_0 = \frac{6}{7}$, we have

$$y_0 = \frac{1}{7}x_0 + \frac{6}{7}T_0 x_0 = \left(4 \times \frac{1}{7} + \frac{4}{8} \times \frac{6}{7}, 0\right) = (1, 0).$$

Take $1 + (L_0 - 1)\alpha_0 = \sqrt{\frac{5}{2}}$, we have

$$C_0 = \left\{z \in R^2 : \|y_0 - z\| \leq \sqrt{\frac{5}{2}}\|x_0 - z\|\right\}.$$

It is easy to show that $z_1 = (1, 3), z_2 = (-1, 3) \in C_0$. But

$$z' = \frac{1}{2}z_1 + \frac{1}{2}z_2 = (0, 3) \overline{\in} C_0,$$

since $\|y_0 - z'\| = 2$, $\|x_0 - z'\| = 1$. Therefore C_0 is not convex.

Corollary 2.7 *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a closed quasi-nonexpansive mapping from C into itself. Assume that $\alpha_n \in (a,1]$ holds for some $a \in (0,1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0 \quad \textit{chosen arbitrarily}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is closed and convex, for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.7. □

Since a nonexpansive mapping must be a closed quasi-nonexpansive mapping, from Corollary 2.7, we obtain the following result.

Corollary 2.8 *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping from C into itself. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0 \quad \text{chosen arbitrarily}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

3 Application to family of quasi-asymptotically nonexpansive mappings

In this section, we will apply the above result to study the following finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C, p \in F,$$

where F denotes the common fixed point set of $\{T_n\}_{n=0}^{N-1}$, $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N - 1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is said to be uniformly L-Lipschitz, if

$$\|T_i^j x - T_i^j y\| \leq L\|x - y\|, \quad \forall x, y \in C$$

for all $i = 0, 1, 2, \ldots, N - 1$, $j \geq 1$, where $L \geq 1$.

Theorem 3.1 *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}_{n=0}^{N-1} : C \to C$ be a uniformly L-Lipschitz finite family of asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0 \quad \text{chosen arbitrarily}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (k_{i(n),j(n)} - 1)\alpha_n)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{\mathrm{co}}\, C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{\mathrm{co}}\, C_n$ denotes the closed convex closure of C_n for all $n \geq 1$, $n = (j(n) - 1)N + i(n)$ for all $n \geq 0$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof It is sufficient to prove the following two conclusions.

Conclusion 1 $\{T_{i(n)}^{j(n)}\}_{n=0}^{\infty}$ *is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself.*

Conclusion 2 $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, *where $F(T)$ denotes the fixed point set of the mapping T.*

Proof of Conclusion 1 Let

$$\|T_{i(n)}^{j(n)}x_n - x_n\| \to 0, \quad x_n \to p$$

as $n \to \infty$. Observe that

$$\begin{aligned}
\|T_{i(n)}x_n - x_n\| &\le \|T_{i(n)}^{j(n)}x_n - x_n\| + \|T_{i(n)}^{j(n)}x_n - T_{i(n)}x_n\| \\
&\le \|T_{i(n)}^{j(n)}x_n - x_n\| + L\|T_{i(n)}^{j(n)-1}x_n - x_n\| \\
&\le \|T_{i(n)}^{j(n)}x_n - x_n\| + L\|T_{i(n)}^{j(n-N)}x_n - T_{i(n)}^{j(n-N)}x_{n-N}\| \\
&\quad + L\|T_{i(n-N)}^{j(n-N)}x_{n-N} - x_{n-N}\| + L\|x_{n-N} - x_n\| \\
&\le \|T_{i(n)}^{j(n)}x_n - x_n\| + (L + L^2)\|x_{n-N} - x_n\| \\
&\quad + L\|T_{i(n-N)}^{j(n-N)}x_{n-N} - x_{n-N}\|
\end{aligned}$$

from which it turns out that $\|T_{i(n)}x_n - x_n\| \to 0$ as $n \to \infty$. This implies there exists subsequence $\{n_k\} \subset \{x_n\}$ such that

$$\|T_i x_{n_k} - x_{n_k}\| \to 0, \quad i = 0, 1, 2, \ldots, N-1$$

as $k \to \infty$. That is, $p \in F = \bigcap_{n=0}^{N-1} F(T_n)$. Therefore, $p \in \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, hence $\{T_{i(n)}^{j(n)}\}$ is uniformly closed. On the other hand, we have

$$\|T_{i(n)}^{j(n)}x - p\| \le k_{i(n),j(n)}\|x - p\|, \quad \forall x \in C, p \in \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)}),$$

and $\lim_{n\to\infty} k_{i(n),j(n)} = 1$. So, $\{T_{i(n)}^{j(n)}\}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself with $L_n = k_{i(n),j(n)}$. □

Proof of Conclusion 2 It is obvious that

$$\bigcap_{n=0}^{N-1} F(T_n) \subset \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)}).$$

On the other hand, for any $p \in \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, let $n = 0, 1, 2, \ldots, N-1$, we obtain

$$p \in F(T_0), \qquad p \in F(T_1), \qquad p \in F(T_2), \qquad \ldots, \qquad p \in F(T_{n-1}),$$

which implies that

$$\bigcap_{n=0}^{N-1} F(T_n) \supset \bigcap_{n=0}^{\infty} F\bigl(T_{i(n)}^{j(n)}\bigr).$$

Hence

$$\bigcap_{n=0}^{N-1} F(T_n) = \bigcap_{n=0}^{\infty} F\bigl(T_{i(n)}^{j(n)}\bigr).$$

□

By using Theorem 2.5, the iterative sequence $\{x_n\}$ converges strongly to $P_{\bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})}x_0 = P_F x_0$. This completes the proof of Theorem 3.1. □

Corollary 3.2 *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty fixed point set F. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0 \quad \text{chosen arbitrarily}, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (k_n - 1)\alpha_n)\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{\mathrm{co}}\, C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $\overline{\mathrm{co}}\, C_n$ denotes the closed convex closure of C_n for all $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof Take $T_n \equiv T$ in Theorem 3.1, we obtain Corollary 3.2. □

Since a nonexpansive mapping must be a Lipschitz asymptotically quasi-nonexpansive mapping, from Corollary 3.2, we can obtain Corollary 2.8.

4 Example

Conclusion 4.1 *Let H be a Hilbert space, $\{x_n\}_{n=1}^{\infty} \subset H$ be a sequence such that it converges weakly to a non-zero element x_0 and $\|x_i - x_j\| \geq 1$ for any $i \neq j$. Define a sequence of mappings $T_n : H \to H$ as follows*:

$$T_n(x) = \begin{cases} L_n x_n & \text{if } x = x_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n \ (\forall n \geq 1), \end{cases}$$

where $\{L_n\}_{n=1}^{\infty}$ is a sequence of number such that $L_n > 1$, $\lim_{n\to\infty} L_n = 1$. Then $\{T_n\}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings with the common fixed point set $F = \{0\}$.

Proof It is obvious that $\{T_n\}$ has a unique common fixed point 0. Next, we prove that $\{T_n\}$ is uniformly closed. In fact, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$, there exists sufficiently large natural number N such that

$z_n \neq x_m$, for any $n, m > N$. Then $T_n z_n = -z_n$ for $n > N$, it follows from $\|z_n - T_n z_n\| \to 0$ that $2z_n \to 0$ and hence $z_0 \in F$. Finally, from the definition of $\{T_n\}$, we have

$$\|T_n x - 0\| = \|T_n x\| \leq \|L_n x\| = L_n \|x - 0\|, \quad \forall x \in H,$$

so that $\{T_n\}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings. □

Remark In the result of Liu *et al.* [30], the boundedness of C was assumed and the hybrid iterative process was complex. In our hybrid iterative process, C_n was constructed as a non-convex set can makes it more simple, meanwhile, the boundedness of C can be removed. Of course, a new technique has been applied in our process of proof.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Hebei North University, Zhangjiakou, 075000, China. [2]Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China.

Acknowledgements
This project is supported by major project of Hebei North University under grant (No. ZD201304).

Received: 28 July 2015 Accepted: 5 November 2015 Published online: 21 November 2015

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