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Multipled fixed point theorems in cone metric spaces

Hallowed Olaoluwa* and Jonhson Olaleru

*Correspondence:
olu20_05@hotmail.com
Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria

Abstract

In this paper, the concept of common multipled fixed points of w-compatible nonlinear contractive maps in cone metric spaces, which are generalizations of common coupled fixed points, is introduced. A new concept of product cone metric spaces is used to establish the existence and uniqueness of their multipled fixed points. Our results generalize several results in the literature including those of Olaleru (Fixed Point Theory Appl. 2009:657914, 2009), Sabetghadam *et al.* (Fixed Point Theory Appl. 2009:125426, 2009) and Abbas *et al.* (Appl. Math. Comput. 217:195-202, 2010; Appl. Math. Comput. 216:80-86, 2010). In addition, the methodology of proof in this manuscript shows that some fixed point results in cone metric spaces are equivalent to multipled fixed point results and *vice versa*.

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1 Introduction

Over the years, fixed point theory has evolved from one-dimensional fixed point in metric spaces to multi-dimensional fixed points in metric-type spaces. Among such spaces are cone metric spaces and partially ordered metric spaces whose structures are induced implicitly or explicitly by partial orderings.

Huang and Zhang [1] top the chronological list of authors who have established fixed point results in cone metric spaces. They proved some fixed point theorems for single maps in cone metric spaces. Their results were followed by several other fixed point results on existence and uniqueness of fixed points for single contractive maps and common fixed point results of pairs of maps, sets of three self-maps and quartets of maps (see [2–8]). Theorems with nonuniqueness of fixed points of Ciric-type maps were later proved by Karapinar [9] in cone metric spaces.

Sabetghadam *et al.* in [10] introduced the concept of coupled fixed points in cone metric spaces and established some coupled fixed point theorems in cone metric spaces. Lakshmikantham and Ciric [11] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces. Later, Abbas *et al.* [12] extended their results to cone metric spaces. Karapinar [13] proved some coupled fixed point theorems for nonlinear contractions in cone metric spaces, Shatanawi *et al.* [14] proved some coupled fixed point theorems in ordered cone metric spaces with a c-distance, and Kadelburg and Radenović [15] established coupled fixed point results under TVS-cone metric and w-cone-distance.

Berinde in [16] and [17], and Amini-Harandi [18] discussed coupled and tripled fixed points of mixed monotone maps in partially ordered metric spaces, while Karapinar in [19] studied quadruple fixed point theorems for weak Φ-contractions.

Recently, Samet and Vetro [20] generalized the notion of coupled fixed points to fixed points of N-order and proved existence results for single maps in complete metric spaces. Their results came as an extension of the theory of fixed points to finite dimensions. Roldán *et al.* [21] then added more variation to the definition of multi-dimensional fixed points in partially ordered complete metric spaces by exploiting the notions of permutations and partitions. Their definition, though more general than that of Samet and Vetro [20], can hardly be used to establish fixed point theorems beyond the Banach contraction principle for two maps.

The main purpose of this article is to prove results on common fixed points of N-order of some contractive w-compatible mappings in cone metric spaces for as many as four maps, with generalized inequalities of Ciric type. Furthermore, we prove that multipled fixed point results can, in fact, generate fixed point results of lower dimensions.

We recall some notions in the concept of cone metric spaces.

Definition 1.1 [3] Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;
(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$;
(c) $P \cap (-P) = \{\theta\}$.

Given a cone P, a partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ stands for the interior of P. Also, we will use $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$.

The cone P in a normed space E is called normal whenever there is a real number $k > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P. Janković *et al.* in [22] proved that only fixed point results in non-normal cones improve the existing fixed point results in metric spaces.

Definition 1.2 [1] Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \longrightarrow E$ satisfies:

(d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is obviously more general than that of a metric space.

Definition 1.3 [1] Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is

(c_1) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n, m \geq k$, $d(x_n, x_m) \ll c$;

(c_2) a convergent sequence if for every $c \in E$ with $\theta \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n \geq k$, $d(x_n, x) \ll c$. Such x is called limit of the sequence $\{x_n\}$.

Note that every convergent sequence in a cone metric space X is a Cauchy sequence. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Example 1.4 Let $E = \mathbb{R}^n$, $P = \{(x_1, x_2, \ldots, x_n) \in E : x_i \geq 0\ \forall i = 1, 2, \ldots, n\}$, $X = \mathbb{R}$, $d : X \times X \to E$ such that $d(x, y) = \alpha|x - y|$, where $\alpha \in P$ is a constant. Then (X, d) is a cone metric space.

Example 1.5 [23] Let $E = l^p$ $(1 \leq p < \infty)$, $P = \{\{x_n\}_{n\geq 1} \geq 0$ for all $n\}$, (X, ρ) be a metric space, and $d : X \times X \to E$ be defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq 1}$. Then (X, d) is a cone metric space.

Example 1.6 [12] Let $X = [0, \infty)$, $E = C^1_{\mathbb{R}}[0, 1]$ and $P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0, 1]\}$. The mapping $d : X \times X \to E$ defined by $d(x, y) = |x - y|\varphi$, where $\varphi(t) = e^t$, gives (X, d) the structure of a complete cone metric space. The cone P is non-normal.

Olaleru in [23] and later Abbas *et al.* in [24] proved the existence of the fixed points of single maps and the common fixed points of pairs and quartets of w-compatible maps satisfying some general contractive conditions. Below are some of their main results.

Theorem 1.7 [23] *Let (X, d) be a complete cone metric space and $f : X \to X$ be a mapping such that*

$$d(fx, fy) \preceq a_1 d(fx, x) + a_2 d(fy, y) + a_3 d(fy, x) + a_4 d(fx, y) + a_5 d(y, x)$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Then the mapping f has a unique fixed point. Moreover, for any $x \in X$, the sequence $\{f^n(x)\}$ converges to the fixed point.

Theorem 1.8 [23] *Let (X, d) be a complete cone metric space and let $f, g : X \to X$ be mappings such that*

$$d(fx, fy) \preceq a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fy, gx) + a_4 d(fx, gy) + a_5 d(gy, gx)$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

Suppose that f and g are weakly compatible and $f(X) \subseteq g(X)$ such that $f(X)$ or $g(X)$ is a complete subspace of X. Then the mappings f and g have a unique common fixed point. Moreover, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $g(x_n) = f(x_{n-1})$ for all n, converges to the fixed point.

Theorem 1.9 [24] *Let f, g, S and T be self-mappings of a cone metric space X (with the cone P having a nonempty interior) such that $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$d(fx, gy) \preceq pd(Sx, Ty) + qd(fx, Sx) + rd(gy, Ty) + t\big[d(fx, Ty) + d(gy, Sx)\big]$$

for all $x, y \in X$, where $p, q, r, t \in [0, 1)$ satisfy $p + q + r + 2t < 1$.

If one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and $\{g,T\}$ have a unique point of coincidence in X. Moreover if $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point which is the limit of the sequences $\{u_n\}$ and $\{x_n\}$ defined by:

$$\begin{cases} u_{2n-1} := fx_{2n-2} = Tx_{2n-1}, \\ u_{2n} := gx_{2n-1} = Sx_{2n}. \end{cases}$$

We now show that results on common multipled fixed points (and of course common coupled fixed points) for single maps, pairs of maps and four maps in cone metric spaces can be derived from known results (*e.g.*, Theorems 1.7-1.9) on fixed points of contractive maps in cone metric spaces. In order to show this, we make use of the concept of product cone metric spaces.

2 Multipled fixed points and w-compatibility

Definition 2.1 (see [20]) Let X be a nonempty set. An element $x = (x_1, x_2, \dots, x_m) \in X^m$, $m \geq 2$, is said to be a fixed point of m-order of a mapping $F : X^m \to X$ if

$$\begin{cases} F(x_1, x_2, \dots, x_{m-1}, x_m) = x_1; \\ F(x_2, x_3, \dots, x_m, x_1) = x_2; \\ F(x_3, x_4, \dots, x_m, x_1, x_2) = x_3; \\ \vdots \\ F(x_m, x_1, x_2, \dots, x_{m-1}) = x_m. \end{cases} \tag{2.1}$$

Observe that (2.1) can be written as

$$F\bigl(t_i(x)\bigr) = x_i, \quad \text{for all } i \in \{1, 2, \dots, m\}, \tag{2.2}$$

where t_i is the ith line of the circular matrix of x,

$$t(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} & x_m \\ x_2 & x_3 & \cdots & x_{m-1} & x_m & x_1 \\ x_3 & x_4 & \cdots & x_m & x_1 & x_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_m & x_1 & \cdots & x_{m-3} & x_{m-2} & x_{m-1} \end{pmatrix}. \tag{2.3}$$

x is called a fixed point, a coupled fixed point, a tripled fixed point, and a quadruple fixed point of F if $m = 1, 2, 3, 4$, respectively. All through this paper, we will refer to *'fixed points of m-order'* simply as *'multipled fixed points'*; the integer m is defined as the *multiplicity* or the *order* of the multipled fixed point x.

Now, we introduce the following definitions of multipled coincidence points, common multipled fixed points and w-compatibility for maps defined in finite-dimensional product spaces.

Definition 2.2 An element $x = (x_1, x_2, \dots, x_m) \in X^m$ is said to be a coincidence point of m-order (or a multipled coincidence point) of mappings $F : X^m \to X$ and $g : X \to X$ if

$$\begin{cases} F(x_1, x_2, \dots, x_{m-1}, x_m) = g(x_1); \\ F(x_2, x_3, \dots, x_m, x_1) = g(x_2); \\ F(x_3, x_4, \dots, x_m, x_1, x_2) = g(x_3); \\ \vdots \\ F(x_m, x_1, x_2, \dots, x_{m-1}) = g(x_m); \end{cases} \tag{2.4}$$

or simply if

$$F\bigl(t_i(x)\bigr) = g(x_i) := u_i \quad \text{for all } i \in \{1, 2, \dots, m\}, \tag{2.5}$$

where t_i is the ith line of the circular matrix of x, $t(x)$, as defined previously in (2.3). The element $u = (u_1, \dots, u_m)$ is called the multipled point of coincidence of F and g.

If all the coordinates x_i of such an element x are fixed points of g, x is called a common multipled fixed point of $F : X^m \to X$ and $g : X \to X$. In such a case, the equalities below are satisfied:

$$\begin{cases} F(x_1, x_2, \dots, x_{m-1}, x_m) = g(x_1) = x_1; \\ F(x_2, x_3, \dots, x_m, x_1) = g(x_2) = x_2; \\ F(x_3, x_4, \dots, x_m, x_1, x_2) = g(x_3) = x_3; \\ \vdots \\ F(x_m, x_1, x_2, \dots, x_{n-1}) = g(x_m) = x_m; \end{cases} \tag{2.6}$$

i.e., $F(t_i(x)) = g(x_i) = x_i$ for all $i \in \{1, 2, \dots, m\}$, with $t(x)$, the circular matrix of x as previously defined.

When $m = 1, 2, 3, 4$, x is a common fixed point, a common coupled fixed point, a common tripled fixed point and a common quadruple fixed point, respectively.

Definition 2.3 The mappings $F : X^m \to X$ and $g : X \to X$ are called w-compatible if $g(F(x_1, x_2, \dots, x_m)) = F(gx_1, gx_2, \dots, gx_m)$ whenever

$$\begin{cases} F(x_1, x_2, \dots, x_{m-1}, x_m) = g(x_1); \\ F(x_2, x_3, \dots, x_m, x_1) = g(x_2); \\ F(x_3, x_4, \dots, x_m, x_1, x_2) = g(x_3); \\ \vdots \\ F(x_m, x_1, x_2, \dots, x_{m-1}) = g(x_m). \end{cases}$$

There is a natural relationship between fixed points of m-order in X and fixed points in X^m. Consider the mappings $F : X^m \to X$ and $g : X \to X$ and their *'associate'* mappings $\tilde{F} : X^m \to X^m$ and $\tilde{g} : X^m \to X^m$ defined for all $x = (x_1, x_2, \dots, x_m) \in X^m$ by

$$\begin{cases} \tilde{F}(x) = (F(t_1(x)), F(t_2(x)), \dots, F(t_m(x))), \\ \tilde{g}(x) = (gx_1, gx_2, \dots, gx_m), \end{cases} \tag{2.7}$$

where $t_i(x)$ is the ith line of the circular matrix $t(x)$ of x as defined in (2.3). It is obvious that a point $x \in X^m$ is a multipled fixed point of F if and only if it is a fixed point of $\tilde{F}$.

Recall that:

(i) $x = (x_1, \ldots, x_m) \in X^m$ is a fixed point of $\tilde{F}$, a coincidence point of $\tilde{F}$ and $\tilde{g}$, or a common fixed point of $\tilde{F}$ and $\tilde{g}$ if $x = \tilde{F}(x)$, $\tilde{F}(x) = \tilde{g}(x)$ or $x = \tilde{F}(x) = \tilde{g}(x)$, respectively;

(ii) $\tilde{F}$ and $\tilde{g}$ are w-compatible if $\tilde{F}(\tilde{g}(x)) = \tilde{g}(\tilde{F}(x))$ whenever $\tilde{F}(x) = \tilde{g}(x)$.

Example 2.4 Let $X = \mathbb{R}$ and let $F : X^m \to X$ be defined for all $x = (x_1, x_2, \ldots, x_m)$ by $F(x) = 2x_1 + x_2 + x_3 + \cdots + x_m - 1$. The system $F(t_i(x)) = x_i$ $\forall i \in \{1, \ldots, m\}$ is satisfied by all x such that $\sum_{j=1}^m x_j = 1$. In particular, $(\frac{1}{m}, \ldots, \frac{1}{m})$ and $(1, 0, \ldots, 0)$ are both multipled fixed points of F. Observe that they are also fixed points of the mapping defined by

$$\tilde{F}(x) = \bigl(F\bigl(t_1(x)\bigr), F\bigl(t_2(x)\bigr), \ldots, F\bigl(t_m(x)\bigr)\bigr) = \left(2x_i + \sum_{j \neq i} x_j - 1\right)_{1 \leq i \leq m}.$$

We also state the following proposition.

Proposition 2.5

(i) *An element $x = (x_1, x_2, \ldots, x_m) \in X^m$ is a multipled coincidence point* (*or a common multipled fixed point*) *of $F : X^m \to X$ and $g : X \to X$ if and only if $x = (x_1, x_2, \ldots, x_m)$ is a coincidence point* (*or a common fixed point*) *of the associate mappings $\tilde{F} : X^m \to X^m$ and $\tilde{g} : X^m \to X^m$ defined in* (2.7).

(ii) *The maps F and g are w-compatible if and only if $\tilde{F}$ and $\tilde{g}$ are w-compatible in X^m.*

It is interesting to note the form of multipled fixed points, multipled coincidence points or common multipled fixed points, when they are unique. Rearranging equalities (2.1), (2.4) or (2.6), we can state the following.

Proposition 2.6 *If $x = (x_1, x_2, \ldots, x_m) \in X^m$ is a multipled coincidence* (*or common multipled fixed*) *point of F and g, then the elements $t_i(x)$, $1 \leq i \leq m$* (*where $t(x)$ is the circular matrix of x*) *are also multipled coincidence* (*or common multipled fixed*) *points of F and g. Hence, if x is a unique multipled coincidence* (*or common multipled fixed*) *point of F and g, then $x = t_i(x)$ for all $i \in \{2, 3, \ldots, m\}$. Thus $x_1 = x_2 = \cdots = x_m$.*

Example 2.7 Let $X = \mathbb{R}$ and let $F : X^m \to X$ be defined by $F(x) = 1 - m + \sum_{j=1}^m x_j$.
$F(t_i(x)) = x_i$ $\forall i \in \{1, \ldots, m\} \Longleftrightarrow \sum_{j \neq i} x_j = m - 1$. The determinant of the system

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = (-1)^{m-1}(m-1) \neq 0,$$

hence the system has a unique solution, $(1, \ldots, 1)$, which is the unique multipled fixed point of F.

3 Multipled fixed point theorems in cone metric spaces

We start the section by introducing and defining some notions on fixed point results in product cone metric spaces.

Definition 3.1 Let (X_i, d_i), $i \in \{1, 2, \dots, m\}$, be m cone metric spaces with respect to cones P_i in a Banach space E such that $P_i \cap (-P_j) = \{\theta\}$ for all i, j. The set $Z = \prod_{i=1}^{i=m} X_i$, together with $d : Z \times Z \to E$ defined by $d(x, y) = \sum_{i=1}^{m} d_i(x_i, y_i)$, where $x = (x_1, x_2, \dots, x_m)$, $y = (y_1, y_2, \dots, y_m)$, is a cone metric space with respect to the cone $P = \sum_{i=1}^{m} P_i$. Z is called a product cone metric space.

When $X_i = X$ for each $i \in \{1, 2, \dots, m\}$, where (X, d) is a cone metric space with respect to the cone $P \subset E$, we define the cone product metric space X^m with respect to P by considering the cone metric $D : X^m \times X^m \to E$ such that

$$D(x, y) = \sum_{i=1}^{m} d(x_i, y_i), \tag{3.1}$$

where $x = (x_i)_{1 \leq i \leq m}$, $y = (y_i)_{1 \leq i \leq m}$.

The proof of the following follows immediately.

Proposition 3.2 *Let (X, d) be a cone metric space with respect to d and (X^m, D) the product cone metric space.*

- (p_1) *A sequence $\{x_n\} = \{(x_n^1, x_n^2, \dots, x_n^m)\}$ converges to $x = (x^1, x^2, \dots, x^m)$ if and only if the sequences $\{x_n^i\}$ converge to x^i for all $i \in \{1, 2, \dots, m\}$.*
- (p_2) *A sequence $\{x_n\} = \{(x_n^1, x_n^2, \dots, x_n^m)\}$ is a Cauchy sequence in X^m if and only if the sequences $\{x_n^i\}$ are Cauchy sequences in X for all $i \in \{1, 2, \dots, m\}$.*
- (p_3) *(X^m, D) is complete if and only if (X, d) is complete.*

Example 3.3 Let $X_1 = [0, 1]$, $X_2 = \mathbb{R}^n$ ($n \in \mathbb{N}$), $E = \mathbb{R}^2$, $P = \mathbb{R}_+^2$, $d_1 : X_1 \times X_1 \to E$ and $d_2 : X_2 \times X_2 \to E$ such that $d_1(u, v) = (|u - v|, 2|u - v|)$ and $d_2(x, y) = (0, \sqrt{\sum_{i=1}^{n} |x_i - y_i|})$, $x = (x_i)_{1 \leq i \leq n}$, $y = (y_i)_{1 \leq i \leq n}$. (X_1, d_1) and (X_2, d_2) are cone metric spaces. The set $Z = X_1 \times X_2 = [0, 1] \times \mathbb{R}^n$ with $d : Z \times Z \to E$ defined as

$$d(u, v) = \left(|u_1 - v_1|, 2|u_1 - v|_1 + \sqrt{\sum_{i=1}^{n} |u_{i+1} - v_{i+1}|} \right)$$

is the product cone metric space of X_1 and X_2.

It should be noted that Theorems 1.7-1.9 can be extended to product cone metric spaces endowed with metric D as defined in (3.1).

Now, we state the following theorem for a single map.

Theorem 3.4 *Let (X, d) be a complete cone metric space, $F : X^m \to X$ be a mapping satisfying, for some $j \in \{1, \dots, m\}$,*

$$\begin{aligned} d\big(F(t_j x), F(t_j u)\big) \preceq{} & a_1 d(Ft_j x, x_j) + a_2 d(Ft_j u, u_j) + a_3 d(Ft_j u, x_j) \\ & + a_4 d(Ft_j x, u_j) + \sum_{i=1}^{m} b_i d(t_{ji} u, t_{ji} x) \end{aligned} \tag{3.2}$$

for all $x = (x_1, x_2, \dots, x_m)$, $u = (u_1, u_2, \dots, u_m) \in X^m$, where $t_{ji} x$ is the element in the jth line and ith column of $t(x)$, and $a_1, a_2, a_3, a_4, b_1, \dots, b_m \in (0, 1)$ satisfy $a_1 + a_2 + a_3 + a_4 + \sum_{i=1}^{m} b_i <$

1. *Then the mapping F has a unique multipled fixed point $(x^*,\dots,x^*) \in X^m$. Moreover, for any $(x^1,x^2,\dots,x^m) \in X^m$, the sequences $\{x_n^1\},\{x_n^2\},\dots,\{x_n^m\} \subset X$ defined by*

$$\begin{cases} x_{n+1}^1 = F(t_1 x_n) = F(x_n^1,x_n^2,\dots,x_n^m); \\ x_{n+1}^2 = F(t_2 x_n) = F(x_n^2,\dots,x_n^m,x_n^1); \\ \vdots \\ x_{n+1}^m = F(t_m x_n) = F(x_n^m,x_n^1,\dots,x_n^{m-1}), \end{cases} \tag{3.3}$$

converge to $x^* \in X$ for each $i \in \{1,2,\dots,m\}$.

Proof Due to the symmetry of d, we can assume without loss of generality that $a_1 = a_2$ and $a_3 = a_4$. Hence we have

$$\begin{aligned} d\bigl(F(t_j x),F(t_j u)\bigr) \preceq{}& a_1 d(Ft_j x,x_j) + a_1 d(Ft_j u,u_j) + a_3 d(Ft_j u,x_j) \\ &+ a_3 d(Ft_j x,u_j) + \sum_{i=1}^m b_i d(t_{ji} u,t_{ji} x) \end{aligned}$$

and by simple interchanges

$$\begin{aligned} d\bigl(F(t_k x),F(t_k u)\bigr) \preceq{}& a_1 d(Ft_k x,x_k) + a_1 d(Ft_k u,u_k) + a_3 d(Ft_k u,x_k) \\ &+ a_3 d(Ft_k x,u_k) + \sum_{i=1}^m b_i d(t_{ki} u,t_{ki} x) \end{aligned}$$

for all $k \in \{1,\dots,m\}$.

Summing the m-inequalities, we have

$$\begin{aligned} \sum_{i=1}^m d\bigl(F(t_i x),F(t_i u)\bigr) \preceq{}& a_1 \sum_{i=1}^m d(Ft_i x,x_i) + a_1 \sum_{i=1}^m d(Ft_i u,u_i) \\ &+ a_3 \sum_{i=1}^m d(Ft_i u,x_i) + a_3 \sum_{i=1}^m d(Ft_i x,u_i) \\ &+ \left(\sum_{i=1}^m b_i\right) \sum_{i=1}^m d(u_i,x_i). \end{aligned}$$

In view of (2.7) and (3.1), we get

$$\begin{aligned} D(\tilde{F}x,\tilde{F}u) \preceq{}& a_1 D(\tilde{F}x,x) + a_1 D(\tilde{F}u,u) + a_3 D(\tilde{F}u,x) + a_3 D(\tilde{F}x,u) \\ &+ \left(\sum_{i=1}^m b_i\right) D(u,x). \end{aligned}$$

Applying Theorem 1.7 to product cone metric spaces, it follows that $\tilde{F}$ has a unique fixed point, which is the unique multipled fixed point of F. Since, the multipled fixed point is unique, it is of the form $(x^*,x^*,\dots,x^*)$ with $x^* \in X$ (from Proposition 2.6).

Also, from Theorem 1.7, for any $(x^1, x^2, \dots, x^m) \in X^m$, the sequence $\tilde{F}^n(x^1, x^2, \dots, x^m)$ converges to the fixed point $(x^*, x^*, \dots, x^*)$. If we set

$$\left(x_n^1, x_n^2, \dots, x_n^m\right) = \tilde{F}^n\left(x^1, x^2, \dots, x^m\right),$$

then the sequences $\{x_n^1\}, \{x_n^2\}, \dots, \{x_n^m\} \subset X$ converge each to $x^* \in X$ and we have

$$\begin{aligned}
\left(x_{n+1}^1, x_{n+1}^2, \dots, x_{n+1}^m\right) &= \tilde{F}\left(\tilde{F}^n\left(x^1, x^2, \dots, x^m\right)\right) = \tilde{F}\left(x_n^1, x_n^2, \dots, x_n^m\right) \\
&= \left(F(t_1 x_n), F(t_2 x_n), \dots, F(t_m x_n)\right) \\
&\iff \begin{cases} x_{n+1}^1 = F(t_1 x_n) = F(x_n^1, x_n^2, \dots, x_n^m); \\ x_{n+1}^2 = F(t_2 x_n) = F(x_n^2, \dots, x_n^m, x_n^1); \\ \vdots \\ x_{n+1}^m = F(t_m x_n) = F(x_n^m, x_n^1, \dots, x_n^{m-1}) \end{cases}
\end{aligned}$$

as defined in (3.3). □

Theorem 3.5 *Let (X, d) be a cone metric space, $F : X^m \to X$ and $g : X \to X$ be mappings satisfying, for some $j \in \{1, \dots, m\}$,*

$$\begin{aligned}
d(Ft_j x, Ft_j u) \preceq{} & a_1 d(Ft_j x, gx_j) + a_2 d(Ft_j u, gu_j) + a_3 d(Ft_j u, gx_j) \\
& + a_4 d(Ft_j x, gu_j) + \sum_{i=1}^m b_i d(gt_{ji} u, gt_{ji} x)
\end{aligned} \tag{3.4}$$

for all $x = (x_1, x_2, \dots, x_m), u = (u_1, u_2, \dots, u_m) \in X^m$, where $a_1, a_2, a_3, a_4, b_1, \dots, b_m \in [0, 1)$ and $a_1 + a_2 + a_3 + a_4 + \sum_{i=1}^m b_i < 1$. If

(i) *F and g are such that $F(X^m) \subset g(X)$,*
(ii) *$F(X^m)$ or $g(X)$ is a complete subset of X, and*
(iii) *F and g are w-compatible,*

then F and g have a unique common multipled fixed point $(x^*, \dots, x^*) \in X^m$. Moreover, for any $x_0 = (x_0^1, x_0^2, \dots, x_0^m) \in X^m$, the sequence $\{x_n\} = \{(x_n^1, x_n^2, \dots, x_n^m)\}$ in X^m defined by $gx_n^i = Ft_i x_{n-1}$ for all $i \in \{1, 2, \dots, m\}$, i.e.,

$$\begin{cases} gx_{n+1}^1 = F(t_1 x_n) = F(x_n^1, x_n^2, \dots, x_n^m); \\ gx_{n+1}^2 = F(t_2 x_n) = F(x_n^2, \dots, x_n^m, x_n^1); \\ \vdots \\ gx_{n+1}^m = F(t_m x_n) = F(x_n^m, x_n^1, \dots, x_n^{m-1}), \end{cases} \tag{3.5}$$

converges to $(x^*, \dots, x^*)$.

Proof Due to the symmetry of d, we can assume without loss of generality that $a_1 = a_2$ and $a_3 = a_4$. Hence we have

$$\begin{aligned}
d\left(F(t_j x), F(t_j u)\right) \preceq{} & a_1 d(Ft_j x, gx_j) + a_1 d(Ft_j u, gu_j) + a_3 d(Ft_j u, gx_j) \\
& + a_3 d(Ft_j x, gu_j) + \sum_{i=1}^m b_i d(gt_{ji} u, gt_{ji} x)
\end{aligned}$$

and by simple interchanges

$$\begin{aligned} d\big(F(t_kx),F(t_ku)\big) &\preceq a_1d(Ft_kx,gx_k)+a_1d(Ft_ku,gu_k)+a_3d(Ft_ku,gx_k)\\ &\quad+a_3d(Ft_kx,gu_k)+\sum_{i=1}^m b_id(gt_{ki}u,gt_{ki}x)\end{aligned}$$

for all $k\in\{1,\dots,m\}$.

Summing the m-inequalities, we have

$$\begin{aligned}\sum_{i=1}^m d\big(F(t_ix),F(t_iu)\big) &\preceq a_1\sum_{i=1}^m d(Ft_ix,gx_i)+a_1\sum_{i=1}^m d(Ft_iu,gu_i)\\ &\quad+a_3\sum_{i=1}^m d(Ft_iu,gx_i)+a_3\sum_{i=1}^m d(Ft_ix,gu_i)\\ &\quad+\left(\sum_{i=1}^m b_i\right)\sum_{i=1}^m d(gu_i,gx_i).\end{aligned}$$

In view of (2.7) and (3.1), we get

$$\begin{aligned}D(\tilde{F}x,\tilde{F}u) &\preceq a_1D(\tilde{F}x,\tilde{g}x)+a_1D(\tilde{F}u,\tilde{g}u)+a_3D(\tilde{F}u,\tilde{g}x)+a_3D(\tilde{F}x,\tilde{g}u)\\ &\quad+\left(\sum_{i=1}^m b_i\right)D(\tilde{g}u,\tilde{g}x).\end{aligned}$$

Since $F(X^m)\subset g(X)$, $F(X^m)$ or $g(X)$ is complete in X and F and g are w-compatible, then $\tilde{F}(X^m)\subset\tilde{g}(X^m)$, $\tilde{F}(X^m)$ or $\tilde{g}(X^m)$ is complete in X^m and $\tilde{F}$ and $\tilde{g}$ are w-compatible. From Theorem 1.8 applied to product cone metric spaces, $\tilde{F}$ and $\tilde{g}$ have a unique common fixed point which is the common multipled fixed point of F and g. From Proposition 2.6, it is of the form $(x^*,x^*,\dots,x^*)$, $x^*\in X$. By Theorem 1.8, the sequence $\{x_n\}=\{x_n^1,\dots,x_n^m\}\subset X^m$ defined by $\tilde{g}(x_n^1,\dots,x_n^m)=\tilde{F}(x_{n-1}^1,\dots,x_{n-1}^m)$ for any $x_0=(x_0^1,\dots,x_0^m)\in X^m$ converges to $(x^*,\dots,x^*)$. Hence, the coordinate sequences $\{x_n^1\},\{x_n^2\},\dots,\{x_n^m\}$ converge each to x^* and satisfy (3.5) since

$$\begin{aligned}&\tilde{g}\big(x_n^1,\dots,x_n^m\big)=\tilde{F}\big(x_{n-1}^1,\dots,x_{n-1}^m\big)\\ &\quad\Longleftrightarrow\quad \big(gx_n^1,gx_n^2,\dots,gx_n^m\big)=(Ft_1x_{n-1},\dots,Ft_mx_{n-1})\\ &\quad\Longleftrightarrow\quad gx_n^i=Ft_ix_{n-1}\quad\forall i\in\{1,2,\dots,m\}.\end{aligned}$$

□

Theorem 3.6 *Let (X,d) be a cone metric space, $f:X^m\to X$, $g:X^m\to X$, $S:X\to X$ and $T:X\to X$ be four mappings such that $f(X^m)\subset T(X)$, $g(X^m)\subset S(X)$ and for some $j\in\{1,\dots,m\}$,*

$$\begin{aligned}d(ft_jx,gt_ju) &\preceq \sum_{i=1}^m p_id(St_{ji}x,Tt_{ji}u)+qd(ft_jx,Sx_j)+rd(gt_ju,Tu_j)\\ &\quad+t\big[d(ft_jx,Tu_j)+d(gt_ju,Sx_j)\big] \end{aligned}\tag{3.6}$$

for all $x=(x_1,x_2,\dots,x_m)$, $u=(u_1,u_2,\dots,u_m)\in X^m$, where p_i $(i=1,2,\dots,m)$, $q,r,t\in(0,1)$ and $\sum_{i=1}^m p_i+q+r+2t<1$.

If one of $f(X^m)$, $g(X^m)$, $S(X)$ or $T(X)$ is a complete subspace of X, then $\{f,S\}$ and $\{g,T\}$ have a unique m-tupled point of coincidence in X.

Moreover, if $\{f,S\}$ and $\{g,T\}$ are w-compatible, then f, g, S and T have a unique multipled common fixed point $(u,\dots,u)\in X^m$ and for every $(x_0^1,x_0^2,\dots,x_0^m)\in X^m$, the sequences $\{x_n\}=\{(x_n^1,x_n^2,\dots,x_n^m)\}\subset X^m$ and $\{u_n\}=\{(u_n^1,u_n^2,\dots,u_n^m)\}$ defined by

$$\begin{cases} u_{2n-1}^i := ft_i x_{2n-2} = Tx_{2n-1}^i, \\ u_{2n}^i := gt_i x_{2n-1} = Sx_{2n}^i \end{cases} \quad \forall i=1,2,\dots,m \tag{3.7}$$

converge both to $(u,u,\dots,u)$.

Proof From (3.6) and by simple interchanges, we have

$$\begin{aligned} d(ft_k x, gt_k u) \preceq{} & \sum_{i=1}^m p_i d(St_{ki}x, Tt_{ki}u) + qd(ft_k x, Sx_k) + rd(gt_k u, Tu_k) \\ & + t\bigl[d(ft_k x, Tu_k) + d(gt_k u, Sx_k)\bigr] \end{aligned}$$

for every $k\in\{1,\dots,m\}$.

Summing the m inequalities, we have

$$\begin{aligned} & \sum_{i=1}^m d\bigl(f(t_i x), g(t_i u)\bigr) \\ & \quad \preceq \left(\sum_{i=1}^m p_i\right) \sum_{i=1}^m d(Sx_i, Tu_i) + q\sum_{i=1}^m d(ft_i x, Sx_i) \\ & \qquad + r\sum_{i=1}^m d(gt_i u, Tu_i) + t\left[\sum_{i=1}^m d(ft_i x, Tu_i) + \sum_{i=1}^m d(gt_i u, Sx_i)\right]. \end{aligned}$$

In view of (2.7) and (3.1), we get

$$\begin{aligned} D(\tilde{f}x, \tilde{g}u) \preceq{} & \left(\sum_{i=1}^m p_i\right) D(\tilde{S}x, \tilde{T}u) + qD(\tilde{f}x, \tilde{S}x) + rD(\tilde{g}u, \tilde{T}u) \\ & + t\bigl[D(\tilde{f}x, \tilde{T}u) + D(\tilde{g}u, \tilde{S}x)\bigr], \end{aligned}$$

where $\tilde{f}$, $\tilde{g}$, $\tilde{S}$ and $\tilde{T}$ are defined for all $x=(x_i)_{1\le i\le m}\in X^m$ and $u=(u_i)_{1\le i\le m}\in X^m$ by

$$\begin{cases} \tilde{f}(x) = (ft_1x, ft_2x, \dots, ft_mx), \\ \tilde{g}(x) = (gt_1x, gt_2x, \dots, gt_mx), \\ \tilde{S}(u) = (Su_1, Su_2, \dots, Su_m), \\ \tilde{T}(u) = (Tu_1, Tu_2, \dots, Tu_m). \end{cases}$$

The contractive condition in Theorem 1.9 is satisfied for $\tilde{f}$, $\tilde{g}$, $\tilde{S}$ and $\tilde{T}$.

We have

$$\begin{cases} f(X^m)\subset T(X) \implies \tilde{f}(X^m)\subset \tilde{T}(X^m), \\ g(X^m)\subset S(X) \implies \tilde{g}(X^m)\subset \tilde{S}(X^m). \end{cases}$$

If one of $f(X^m)$, $g(X^m)$, $S(X)$ or $T(X)$ is complete, then $\tilde{f}(X^m)$, $\tilde{g}(X^m)$, $\tilde{S}(X^m)$ or $\tilde{T}(X^m)$ is complete in X^m, hence by Theorem 1.9 applied to the product cone metric space X^m, the pairs $\{\tilde{f},\tilde{S}\}$ and $\{\tilde{g},\tilde{T}\}$ have unique points of coincidence which are unique multiple coincidence points of $\{f,S\}$ and $\{g,T\}$.

If, in addition, $\{f,S\}$ and $\{g,T\}$ are w-compatible, then $\{\tilde{f},\tilde{S}\}$ and $\{\tilde{g},\tilde{T}\}$ are w-compatible. By Theorem 1.9, $\tilde{f}$, $\tilde{g}$, $\tilde{S}$ and $\tilde{T}$ have a unique common fixed point which is the unique multiple common fixed point of f, g, S and T. Because of the uniqueness, it is of the form $(u,\dots,u)$ for some $u \in X$.

Also, from Theorem 1.9, for any $(x_0^1,x_0^2,\dots,x_0^m) \in X^m$, the sequences $\{x_n\} = \{(x_n^i)_{1\le i\le m}\}$ and $\{u_n\} = \{(u_n^i)_{1\le i\le m}\}$ defined by

$$\begin{cases} u_{2n-1} := \tilde{f}x_{2n-2} = \tilde{T}x_{2n-1}, \\ u_{2n} := \tilde{g}x_{2n-1} = \tilde{S}x_{2n}, \end{cases} \tag{3.8}$$

converge to $(u,\dots,u) \in X^m$ and (3.8) is equivalent to

$$\begin{cases} u_{2n-1}^i := ft_ix_{2n-2} = Tx_{2n-1}^i, \\ u_{2n}^i := gt_ix_{2n-1} = Sx_{2n}^i \end{cases} \qquad \forall i = 1,2,\dots,m.$$

Hence the sequences $\{x_n\}$ and $\{u_n\}$ defined in (3.7) converge to $(u,u,\dots,u) \in X^m$. □

Remark 3.7 Theorem 3.4 extends Theorem 1.7 (Olaleru [23]), the results of Sabetghadam *et al.* [10] and Theorem 4.1.1 of Olaleru and Olaoluwa [25] to multipled fixed points. Theorem 3.5 is an extension of Theorem 2.4 of Abbas *et al.* [12] and Theorem 4.2.1 of Olaleru and Olaoluwa [25] to multipled fixed points. Theorem 3.6 generalizes Theorems 3.4 and 3.5 and extends the results of Abbas *et al.* [24] (Theorem 1.9) and Theorem 4.3.1 of Olaleru and Olaoluwa [25] to multipled fixed points. To the best of our knowledge, it is the most general theorem in the theory of multipled fixed points that deals with four maps.

Example 3.8 Let (X,d) be the cone metric space defined in Example 1.6. Consider the mappings $F : X^m \to X$ and $g : X \to X$ defined by $F(x_1,x_2,\dots,x_m) = \frac{1}{m}\sum_{i=1}^m x_i$ and $g(x) = mx$, where $m \ge 2$.

$F(X^m) = g(X) = X$ is complete. Also, F and g are w-compatible. The condition (3.4) of Theorem 3.5 is satisfied for $j = 1$ and $a_1 = a_2 = a_3 = a_4 = \frac{1}{10}$, $b_i = \frac{1}{2m}$, $\forall i \in \{1,\dots,n\}$. That is,

$$d(Fx,Fu) \preceq \frac{1}{10}\big[d(Fx,gx_1) + d(Fu,gu_1) + d(Ftu,gx_1) + d(Ft_x,gu_1)\big] + \sum_{i=1}^m \frac{1}{2m} d(gu_i,gx_i)$$

for all $x = (x_1,\dots,x_m)$, $u = (u_1,\dots,u_m) \in X^m$, and $\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \sum_{i=1}^m \frac{1}{2m} = \frac{9}{10} < 1$. Hence F and g have a unique common multipled fixed point $(x^*,\dots,x^*) \in X^m$. It is easy to notice that $x^* = 0$.

Example 3.9 Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \ge 0\}$, $X = \{2^{-p} : p \in \mathbb{N}\}$ and $d : X \times X \to X$ be defined by $d(2^{-p},2^{-r}) = (|p-r|, |2^{-p} - 2^{-r}|)$. (X,d) is a cone metric space. Let $f,g : X \times X \to X$ and $S,T : X \to X$ be defined by $f(x_1,x_2) = g(x_1,x_2) = x_1x_2$ and $Sx = Tx = x^3$. f, g, S, T satisfy the conditions of Theorem 3.6 with $m = 2$, $j \in \{1,2\}$, $p_1 = p_2 = \frac{1}{3}$ and $q = r = t = 0$. The couple $(1,1)$ is the unique common coupled fixed point of the four maps f, g, S and T.

4 Retrieving lower multiplicity fixed points

It would be interesting to show that fixed point of m-order results can be used to obtain parallel results on fixed points of a lower order k. That is, fixed point results (which are fixed point of 1-order results) can be generated from similar results on coupled, tripled or even quadruple fixed points. In the sequel, we prove Theorem 3.6 for order k, using the same theorem for a higher-order m.

Let X be a cone metric space and (X^i, D_i) be a product cone metric space, $i \in \mathbb{N}$. Consider the mappings $S, T : X \to X$ and $f, g : X^k \to X$. Also consider the maps $\bar{f}, \bar{g} : X^m \to X$ defined by $\bar{f}(\bar{x}) = f(x)$ and $\bar{g}(\bar{x}) = g(x)$ for all $\bar{x} = (x_1, \dots, x_k, \dots, x_m) \in X^m$, where $x = (x_1, \dots, x_k) \in X^k$ (k and m are integers such that $k < m$).

Obviously, $\bar{f}(X^m) = f(X^k)$ and $\bar{g}(X^m) = g(X^k)$; hence the conditions $\bar{f}(X^m) \subset T(X)$ and $\bar{g}(X^m) \subset S(X)$ imply that $f(X^k) \subset T(X)$ and $g(X^k) \subset S(X)$, respectively. Also, if $\bar{f}(X^m)$ or $\bar{g}(X^m)$ is complete, then $f(X^k)$ or $g(X^k)$ is complete, respectively.

Now consider inequality (3.6) in Theorem 3.6 for maps $f, g : X^k \to X$ and $S, T : X \to X$, *i.e.*,

$$\begin{aligned} d(ft_jx, gt_ju) \preceq{}& \sum_{i=1}^{k} p_i d(St_{ji}x, Tt_{ji}u) + qd(ft_jx, Sx_j) + rd(gt_ju, Tu_j) \\ &+ t\bigl[d(ft_jx, Tu_j) + d(gt_ju, Sx_j)\bigr] \end{aligned} \tag{4.1}$$

for all $x = (x_1, x_2, \dots, x_k), u = (u_1, u_2, \dots, u_k) \in X^k$, where $p_1, \dots, p_k, q, r, t \in [0,1)$ and $q + r + 2t + \sum_{i=1}^{k} p_i < 1$.

It is obvious that for any $x_{k+1}, \dots, x_m, u_{k+1}, \dots, u_m \in X$, the mappings $\bar{f}, \bar{g} : X^m \to X$ and $S, T : X \to X$ are such that

$$\begin{aligned} d(ft_j\bar{x}, gt_j\bar{u}) \preceq{}& \sum_{i=1}^{m} p_i d(St_{ji}\bar{x}, Tt_{ji}\bar{u}) + qd(\bar{f}t_j\bar{x}, S\bar{x}_j) + rd(\bar{g}t_j\bar{u}, T\bar{u}_j) \\ &+ t\bigl[d(\bar{f}t_j\bar{x}, T\bar{u}_j) + d(\bar{g}t_j\bar{u}, S\bar{x}_j)\bigr] \end{aligned} \tag{4.2}$$

for all $\bar{x} = (x_1, x_2, \dots, x_m), \bar{u} = (u_1, u_2, \dots, u_m) \in X^m$, where $q, r, t, p_1, \dots, p_k \in [0,1)$, $p_{k+1} = \cdots = p_m = 0$ and $q + r + 2t + \sum_{i=1}^{m} p_i = q + r + 2t + \sum_{i=1}^{k} p_i < 1$. If f, g, S, T are such that $f(X^k) \subset T(X)$, $g(X^k) \subset S(X)$ and either $f(X^k)$, $g(X^k)$, $S(X)$ or $T(X)$ is complete, then $\bar{f}(X^m) \subset T(X)$, $\bar{g}(X^m) \subset S(X)$ and either $\bar{f}(X^m)$, $\bar{g}(X^m)$, $S(X)$ or $T(X)$ is complete. Under these conditions, $\{\bar{f}, S\}$ and $\{\bar{g}, T\}$ have unique points of coincidence of order m. From Proposition 2.6, they are of the form $(u, \dots, u) \in X^m$ and $(v, \dots, v) \in X^m$.

If, in addition, $\{f, S\}$ and $\{g, T\}$ are w-compatible, they commute at their respective coincidence points $(u, \dots, u), (v, \dots, v) \in X^k$ and so, obviously, $\{\bar{f}, S\}$ and $\{\bar{g}, T\}$ also commute at their respective coincidence points $(u, \dots, u), (v, \dots, v) \in X^m$. The coincidence points being unique, $\{\bar{f}, S\}$ and $\{\bar{g}, T\}$ are w-compatible; from Theorem 3.6, $\bar{f}$, $\bar{g}$, S and T have a unique common fixed point of order m, $(z, \dots, z) \in X^m$. It obviously follows that f, g, S, T have a unique common fixed point $(z, \dots, z) \in X^k$ of k-order.

The choice of Theorem 3.6 does not lessen the generality given that the notion of w-compatibility is used. The same method can be used for Theorem 3.4 or Theorem 3.5. We summarize by stating the following theorem.

Theorem 4.1 *Theorems of existence of multipled fixed points of maps satisfying some contractive conditions can be considered as applications of corresponding results of fixed points of higher multiplicity. In particular, the fixed point results are applications of coupled fixed point results or tripled fixed point results, coupled fixed point results are applications of tripled fixed point results, etc.*

Remark 4.2 Our methodology of proof shows that some results in cone metric spaces (CMSs) of fixed point (FP) of higher order can be obtained through corresponding fixed point results via product cone metric spaces (PCMSs). Theorem 4.1 is sort of the converse. We can sketch the following diagram in terms of applications or corollaries:

$$\text{FP results in CMSs} \quad \overset{\text{(via PCMSs)}}{\Longrightarrow} \quad \text{FP of higher order results,}$$

$$\text{FP of higher order results} \quad \overset{\text{(via Theorem 4.1)}}{\Longrightarrow} \quad \text{FP of lower order in CMSs,}$$

$$\text{FP of lower order in CMSs} \quad \overset{\text{(via Theorem 4.1)}}{\Longrightarrow} \quad \text{FP results in CMSs}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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