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Convergence analysis for the equilibrium problems with numerical results

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Abstract

In this paper, we propose an iterative scheme modified from the work of Ceng *et al.* (Nonlinear Anal. Hybrid Syst. 4:743-754, 2010) and Plubtieng and Punpaeng (J. Math. Anal. Appl. 336(1):455-469, 2007) to prove the strong convergence theorem for approximating a common element of the set of fixed points of nonspreading mappings and a finite family of the set of solutions of the equilibrium problem. Using this result, we obtain the strong convergence theorem for a finite family of nonspreading mappings and a finite family of the set of solutions of equilibrium problem. Moreover, in order to compare numerical results between the combination of the equilibrium problem and the classical equilibrium problem, some examples are given in one- and two-dimensional spaces of real numbers.

Keywords: nonspreading mapping; quasi-nonexpansive mapping; the combination of equilibrium problem

1 Introduction

Throughout this paper, let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. We denote weak convergence and strong convergence by the notations '$\rightharpoonup$' and '$\rightarrow$', respectively. We use $\mathbb{R}$ to denote the set of real numbers and $\mathrm{Fix}(\mathcal{T})$ to represent the set of fixed points of $\mathcal{T}$, where $\mathcal{T}$ is a mapping from C into itself.

In 2008, Kohsaka and Takahashi [1] introduced *the nonspreading mapping* $\mathcal{T}$ in Hilbert space H as follows:

$$2\|\mathcal{T}u - \mathcal{T}v\|^2 \le \|\mathcal{T}u - v\|^2 + \|u - \mathcal{T}v\|^2, \quad \forall u, v \in C. \tag{1.1}$$

In 2009, it was shown by Iemoto and Takahashi [2] that (1.1) is equivalent to the following equation:

$$\|\mathcal{T}u - \mathcal{T}v\|^2 \le \|u - v\|^2 + 2\langle u - \mathcal{T}u, v - \mathcal{T}v \rangle, \quad \text{for all } u, v \in C.$$

Many researchers proved the strong convergence theorem for a nonspreading mapping and some related mappings in Hilbert space; see for example [3–6].

Let $B : C \to H$. *The variational inequality problem* is to find a point $u \in C$ satisfying the following inequality:

$$\langle Bu, v - u \rangle \geq 0, \tag{1.2}$$

for all $v \in C$. Moreover, VI(C, B) is used to denote the set of solutions of (1.2).

Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction. *The classical equilibrium problem* for Φ is to find $u \in C$ satisfying the following inequality:

$$\Phi(u, v) \geq 0, \quad \forall v \in C. \tag{1.3}$$

We use EP(Φ) to represent the set of solution of (1.3).

Let the bifunction Φ satisfy the following conditions for solving the equilibrium problem.

(A1) $\Phi(u, u) = 0$ for all $u \in C$;

(A2) Φ is monotone, *i.e.*, $\Phi(u, v) + \Phi(v, u) \leq 0$ for all $u, v \in C$;

(A3) for each $u, v, w \in C$,

$$\lim_{t \to 0^+} \Phi\big(tw + (1 - t)u, v\big) \leq \Phi(u, v);$$

(A4) for each $u \in C$, $v \mapsto \Phi(u, v)$ is convex and lower semicontinuous.

In 1994, Blum and Oettli [7] showed that the classical equilibrium problem (1.3) covers monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, and certain fixed point problems.

Let $\Psi = \{\Phi_i\}_{i=1,2,\ldots,N}$ be a finite family of bifunctions from $C \times C$ to $\mathbb{R}$. The system of equilibrium problem for Ψ is to determine common equilibrium points for $\Psi = \{\Phi_i\}_{i=1,2,\ldots,N}$, that is, the set

$$\mathrm{EP}(\Psi) = \big\{ u \in C : \Phi_i(u, v) \geq 0, \forall v \in C, \forall i \in 1, 2, \ldots, N \big\}. \tag{1.4}$$

The problem (1.4) extends (1.3) to a system of such problems covering various forms of feasibility problems [8]. Several iterative algorithms are proposed to solve the equilibrium problems and a finite family of equilibrium problems; see, for instance, [8–14].

Example 1.1 Let $\Psi = \{\Phi_i\}_{i=1,2,\ldots,N}$ be a finite family of bifunctions from $C \times C$ to $\mathbb{R}$, where the bifunctions Φ_i are defined by

$$\Phi_i(u, v) = i(v - u)(v + 2u - 3), \quad \text{for every } u, v \in \mathbb{R}.$$

For each $i = 1, 2, \ldots, N$, it is obvious that the $\Phi_i(x, y)$ satisfy (A1)-(A4). Then we obtain

$$\mathrm{EP}(\Psi) = \bigcap_{i=1}^{N} \mathrm{EP}(\Phi_i) = \{1\}.$$

In 2010, Peng *et al.* [15] proposed the following iterative algorithm for solving a family of infinite nonexpansive mappings and a finite family of equilibrium problems in Hilbert

space:

$$\begin{cases} z_1 = z \in H, \\ u_n = T_{\beta_n}^{F_m} T_{\beta_n}^{F_{m-1}} \cdots T_{\beta_n}^{F_2} T_{\beta_n}^{F_1} z_n, \\ v_n = P_C(I - s_n A)u_n, \\ z_{n+1} = \alpha_n \gamma f(W_n z_n) + (I - \alpha_n B)W_n P_C(I - r_n A)v_n, \quad \forall n \in \mathbb{N}. \end{cases} \quad (1.5)$$

Under some appropriate conditions, they proved that $\{z_n\}$, $\{v_n\}$, and $\{u_n\}$ converge strongly to $q = P_\Omega(\gamma f + (I - B))(q)$, where $\Omega = \bigcap_{i=1}^\infty \operatorname{Fix}(S_i) \cap \operatorname{VI}(C, A) \cap \bigcap_{k=1}^m \operatorname{EP}(F_k)$ and f is a contractive mapping on H.

Over the past few years, many researchers have started working on the methods for finding a common solution of a finite family of equilibrium problems in Hilbert space; see, for instance, [16–18].

In 2013, Suwannaut and Kangtunyakarn [12] introduced *the combination of equilibrium problem* which is to find $u \in C$ such that

$$\left(\sum_{i=1}^N a_i \Phi_i\right)(u, v) \geq 0, \quad \forall v \in C, \quad (1.6)$$

where $\Phi_i : C \times C \to \mathbb{R}$ are bifunctions and $a_i \in (0, 1)$ with $\sum_{i=1}^N a_i = 1$, for every $i = 1, 2, \ldots, N$. The set of solutions (1.6) is denoted by $\operatorname{EP}(\sum_{i=1}^N a_i \Phi_i)$.

If $\Phi_i = \Phi$, for all $i = 1, 2, \ldots, N$, then the combination of equilibrium problem (1.6) reduces to the classical equilibrium problem (1.3).

Moreover, they obtain Lemma 2.10 as shown in the next section.

Example 1.2 For every $i = 1, 2, 3$, let the bifunctions $\Phi_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, be given by

$$\Phi_1(u, v) = (v - u)(v + u - 2),$$

$$\Phi_2(u, v) = (v - u)(3v + 5u - 8),$$

$$\Phi_3(u, v) = (v - u)(9v + 12u - 21), \quad \forall u, v \in \mathbb{R}.$$

For all $i = 1, 2, 3$, it is obvious that the $\Phi_i(u, v)$ satisfy (A1)-(A4). Let $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{9}$ and $a_3 = \frac{7}{18}$, thus we have

$$\sum_{i=1}^3 a_i \Phi_i(u, v) = \frac{1}{18}(v - u)(78v + 103u - 181).$$

This implies that

$$\operatorname{EP}\left(\sum_{i=1}^3 a_i \Phi_i\right) = \bigcap_{i=1}^3 \operatorname{EP}(\Phi_i) = \{1\}.$$

Remark 1.3 For all $i = 1, 2, \ldots, N$, let the mapping $\mathcal{A}_i : C \to H$ be defined by $\Phi_i(u, v) = \langle \mathcal{A}_i u, v - u \rangle$ for all $u, v \in C$. For each $i = 1, 2, \ldots, N$, if $\Phi_i(u, v) = \langle \mathcal{A}_i u, v - u \rangle \geq 0$ for all

$u, v \in C$, and $i = 1, 2, \ldots, N$, then $EP(\Phi_i) = VI(C, \mathcal{A}_i)$. Hence we have

$$EP\left(\sum_{i=1}^{N} a_i \Phi_i\right) = \bigcap_{i=1}^{N} EP(\Phi_i) = \bigcap_{i=1}^{N} VI(C, \mathcal{A}_i).$$

After we have studied research related to equilibrium problems, we obtain the following question.

Question Is it possible to prove strong convergence theorem for a finite family of equilibrium problem using different method from the result of Peng *et al.* [15], Piri [17] and references therein?

Inspired and motivated by the work of Iemoto and Takahashi [2], Suwannaut and Kangtunyakarn [12] and related research, we propose an iterative scheme modified from the work of Plubtieng and Punpaeng [19] and Ceng *et al.* [11] to prove the strong convergence theorem for approximating a common element of the set of fixed points of a nonspreading mapping and a finite family of the set of solutions of equilibrium problems using Lemma 2.10 and a different method from the work of Peng *et al.* [15] and Piri [17] and references therein. Moreover, some examples are given in order to compare the numerical results between the combination of the equilibrium problem and the classical equilibrium problem.

2 Preliminaries
We now recall the following definition and well-known lemmas.

Definition 2.1

(i) $\mathcal{A}$ is strongly positive operator on H if there exists a constant $\beta > 0$ such that

$$\langle \mathcal{A}u, u \rangle \geq \beta \|u\|^2, \quad \forall u \in H.$$

(ii) $\mathcal{T}$ is a nonexpansive mapping if

$$\|\mathcal{T}u - \mathcal{T}v\| \leq \|u - v\|, \quad \forall u, v \in C.$$

(iii) For every $u \in H$, there is a unique nearest point $P_C u$ in C such that

$$\|u - P_C u\| \leq \|u - v\|, \quad \forall v \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 2.1 ([20]) *For a given $w \in H$ and $u \in C$,*

$$u = P_C w \quad \Leftrightarrow \quad \langle u - w, v - u \rangle \geq 0, \quad \forall v \in C.$$

Furthermore, P_C is a nonexpansive mapping.

Lemma 2.2 ([21]) *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence* $\{u_n\} \subset H$ *with* $u_n \rightharpoonup u$, *the inequality*

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ *with* $v \neq u$.

Lemma 2.3 ([22]) *Let* $\{u_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$u_{n+1} \le (1 - \beta_n)u_n + \eta_n, \quad \forall n \ge 0,$$

where α_n *is a sequence in* $(0,1)$ *and* $\{\eta_n\}$ *is a sequence such that*
  (1) $\sum_{n=1}^{\infty} \beta_n = \infty$,
  (2) $\limsup_{n\to\infty} \frac{\eta_n}{\beta_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\eta_n| < \infty$.
Then $\lim_{n\to\infty} u_n = 0$.

Lemma 2.4 ([4]) *Let H be a real Hilbert space. Then the following results hold*:
  (i) *For all* $u, v \in H$ *and* $t \in [0,1]$,

$$\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - t(1-t)\|u-v\|^2,$$

  (ii) $\|u + v\|^2 \le \|u\|^2 + 2\langle v, u+v\rangle$, *for each* $u, v \in H$.

Lemma 2.5 ([20]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let* $\mathcal{A}$ *be a mapping of C into H. Then, for* $\alpha > 0$,

$$\mathrm{Fix}\big(P_C(I - \alpha\mathcal{A})\big) = \mathrm{VI}(C, \mathcal{A}),$$

where P_C *is the metric projection of H onto C.*

Lemma 2.6 ([23]) *Assume* $\mathcal{A}$ *is a strongly positive linear bounded operator on a Hilbert space H with coefficient* $\beta > 0$ *and* $0 < \delta < \|\mathcal{A}\|^{-1}$. *Then* $\|I - \delta\mathcal{A}\| \le 1 - \delta\beta$.

Lemma 2.7 ([2]) *Let C be a nonempty closed convex subset of H. Then a mapping* $\mathcal{T} : C \to C$ *is nonspreading if and only if*

$$\|\mathcal{T}u - \mathcal{T}v\|^2 \le \|u - v\|^2 + 2\langle u - \mathcal{T}u, v - \mathcal{T}v\rangle, \quad \text{for all } u, v \in C.$$

Remark 2.8 If $\mathcal{T}$ is a nonexpansive mapping and $\langle u - \mathcal{T}u, v - \mathcal{T}v\rangle \ge 0$, for every $u, v \in C$, then $\mathcal{T}$ is a nonspreading mapping.

Lemma 2.9 *Let C be a nonempty closed convex subset of a real Hilbert space H and let* $\mathcal{T} : C \to C$ *be a nonspreading mapping with* $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. *Then we have the following statements*:
  (i) $\mathrm{Fix}(\mathcal{T}) = \mathrm{VI}(C, I - \mathcal{T})$;
  (ii) *for every* $u \in C$ *and* $v \in \mathrm{Fix}(T)$,

$$\big\|P_C\big(I - \lambda(I - \mathcal{T})\big)u - v\big\| \le \|u - v\|, \quad \text{where } \lambda \in (0,1).$$

Proof To prove (i), let $x^* \in \text{Fix}(\mathcal{T})$. Then $x^* = \mathcal{T}x^*$. Since

$$\langle v - x^*, (I - \mathcal{T})x^* \rangle = 0, \quad \forall v \in C,$$

we have $x^* \in \text{VI}(C, I - \mathcal{T})$, from which it follows that $\text{Fix}(T) \subseteq \text{VI}(C, I - \mathcal{T})$.

Next, we show $\text{VI}(C, I - \mathcal{T}) \subseteq \text{Fix}(\mathcal{T})$.

Let $\tilde{x} \in \text{VI}(C, I - \mathcal{T})$. This implies that

$$\langle v - \tilde{x}, (I - \mathcal{T})\tilde{x} \rangle \geq 0, \quad \forall v \in C. \tag{2.1}$$

Let $x^* \in \text{Fix}(\mathcal{T})$. Then, by Lemma 2.7, we obtain

$$\left\| \mathcal{T}\tilde{x} - \mathcal{T}x^* \right\|^2 \leq \left\| \tilde{x} - x^* \right\|^2 + 2\langle \tilde{x} - \mathcal{T}\tilde{x}, x^* - \mathcal{T}x^* \rangle = \left\| \tilde{x} - x^* \right\|^2. \tag{2.2}$$

Observe that

$$\begin{aligned}
\left\| \mathcal{T}\tilde{x} - x^* \right\|^2 &= \left\| \tilde{x} - x^* - (I - \mathcal{T})\tilde{x} \right\|^2 \\
&= \left\| \tilde{x} - x^* \right\|^2 - 2\langle \tilde{x} - x^*, (I - \mathcal{T})\tilde{x} \rangle + \left\| (I - \mathcal{T})\tilde{x} \right\|^2.
\end{aligned} \tag{2.3}$$

From (2.1), (2.2), and (2.3), we get

$$\left\| (I - \mathcal{T})\tilde{x} \right\|^2 \leq 2\langle \tilde{x} - x^*, (I - \mathcal{T})\tilde{x} \rangle \leq 0,$$

which yields $\tilde{x} \in \text{Fix}(\mathcal{T})$. Therefore $\text{VI}(C, I - \mathcal{T}) \subseteq \text{Fix}(\mathcal{T})$.

To prove (ii), let $u \in C$ and $v \in \text{Fix}(\mathcal{T})$. Since $\mathcal{T}$ is a nonspreading mapping and we have Lemma 2.7, we get

$$\| \mathcal{T}u - \mathcal{T}v \|^2 \leq \| u - v \|^2 + 2\langle u - \mathcal{T}u, v - \mathcal{T}v \rangle = \| u - v \|^2. \tag{2.4}$$

Thus we have

$$\begin{aligned}
\| \mathcal{T}u - v \|^2 &= \left\| u - v - (I - \mathcal{T})u \right\|^2 \\
&= \| u - v \|^2 - 2\langle u - v, (I - \mathcal{T})u \rangle + \left\| (I - \mathcal{T})u \right\|^2.
\end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$\left\| (I - \mathcal{T})u \right\|^2 \leq 2\langle u - v, (I - \mathcal{T})u \rangle. \tag{2.6}$$

From (i) and Lemma 2.5, we have

$$v \in \text{Fix}(\mathcal{T}) = \text{VI}(C, I - \mathcal{T}) = \text{Fix}\left(P_C \left(I - \lambda(I - \mathcal{T}) \right) \right). \tag{2.7}$$

By the nonexpansiveness of P_C, (2.6), and (2.7), we get

$$\begin{aligned}
\left\| P_C \left(I - \lambda(I - \mathcal{T}) \right)u - v \right\|^2 &= \left\| P_C \left(I - \lambda(I - \mathcal{T}) \right)u - P_C \left(I - \lambda(I - \mathcal{T}) \right)v \right\|^2 \\
&\leq \left\| \left(I - \lambda(I - \mathcal{T}) \right)u - \left(I - \lambda(I - \mathcal{T}) \right)v \right\|^2
\end{aligned}$$

$$= \left\| (u - v) - \lambda\big((I - \mathcal{T})u - (I - \mathcal{T})v\big) \right\|^2$$

$$= \left\| (u - v) - \lambda(I - \mathcal{T})u \right\|^2$$

$$= \|u - v\|^2 - 2\lambda\langle u - v, (I - \mathcal{T})u \rangle + \lambda^2 \left\| (I - \mathcal{T})u \right\|^2$$

$$\leq \|u - v\|^2 - \lambda \left\| (I - \mathcal{T})u \right\|^2 + \lambda^2 \left\| (I - \mathcal{T})u \right\|^2$$

$$= \|u - v\|^2 - \lambda(1 - \lambda) \left\| (I - \mathcal{T})u \right\|^2$$

$$\leq \|u - v\|^2,$$

which implies that $\|P_C(I - \lambda(I - \mathcal{T}))u - v\| \leq \|u - v\|$. □

Lemma 2.10 ([12]) *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $\Phi_i : C \times C \to \mathbb{R}$ be bifunctions satisfying* (A1)-(A4) *with $\bigcap_{i=1}^{N} EP(\Phi_i) \neq \emptyset$. Then*

$$EP\left(\sum_{i=1}^{N} a_i \Phi_i\right) = \bigcap_{i=1}^{N} EP(\Phi_i),$$

where $a_i \in (0, 1)$ for every $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.11 ([7]) *Let C be a nonempty closed convex subset of H and let Φ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $t > 0$ and $u \in H$. Then there exists $w \in C$ such that*

$$\Phi(w, v) + \frac{1}{t}\langle v - w, w - u \rangle \geq 0, \quad \forall v \in C.$$

Lemma 2.12 ([8]) *Assume that $\Phi : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $t > 0$, define a mapping $S_t : H \to C$ as follows*:

$$S_t(x) = \left\{ w \in C : \Phi(w, v) + \frac{1}{t}\langle v - w, w - u \rangle \geq 0, \forall v \in C \right\},$$

for all $u \in H$. Then the following hold:
  (i) *S_t is single-valued*;
  (ii) *S_t is firmly nonexpansive, i.e., for each $u, v \in H$,*

$$\left\| S_t(u) - S_t(v) \right\|^2 \leq \langle S_t(u) - S_t(u), u - v \rangle;$$

  (iii) *$\mathrm{Fix}(S_t) = EP(\Phi)$*;
  (iv) *$EP(\Phi)$ is closed and convex.*

Remark 2.13 ([12]) From Lemma 2.10, it is easy to see that $\sum_{i=1}^{N} a_i \Phi_i$ satisfies (A1)-(A4). By using Lemma 2.12, we obtain

$$\mathrm{Fix}(S_t) = EP\left(\sum_{i=1}^{N} a_i \Phi_i\right) = \bigcap_{i=1}^{N} EP(\Phi_i),$$

where $a_i \in (0, 1)$, for each $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} a_i = 1$.

3 Strong convergence theorem

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{F}$ be an α-contractive mapping on H and let $\mathcal{A}$ be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\mathcal{T} : C \to C$ be a nonspreading mapping. For every $i = 1, 2, \ldots, N$, let $\Phi_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) with $\Omega := \mathrm{Fix}(\mathcal{T}) \cap \bigcap_{i=1}^{N} \mathrm{EP}(\Phi_i) \neq \emptyset$. Let $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ be sequences generated by $\mathcal{Z}_1 \in H$ and*

$$\begin{cases} \sum_{i=1}^{N} a_i \Phi_i(\mathcal{V}_n, y) + \frac{1}{\varphi_n}\langle y - \mathcal{V}_n, \mathcal{V}_n - \mathcal{Z}_n \rangle \geq 0, \quad \forall y \in C, \\ \mathcal{Y}_n = \theta_n P_C \mathcal{Z}_n + (1 - \theta_n)\mathcal{V}_n, \\ \mathcal{Z}_{n+1} = \delta_n \gamma \mathcal{F}(\mathcal{Z}_n) + (I - \delta_n \mathcal{A})P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\delta_n\}, \{\theta_n\}, \{\varphi_n\}, \{\psi_n\} \subset (0, 1)$, $0 < a_i < 1$, for all $i = 1, 2, \ldots, N$. Suppose the conditions (i)-(vi) hold.

(i) $\lim_{n \to \infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;

(ii) $0 < \tau \leq \theta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;

(iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, *for some* $\epsilon, \eta > 0$;

(v) $\sum_{n=1}^{N} a_i = 1$;

(vi) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$, $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$, $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ converge strongly to $q = P_\Omega(I - \mathcal{A} + \gamma \mathcal{F})q$.

Proof The proof of this theorem is divided into five steps.

Step 1. Claim that $\{\mathcal{Z}_n\}$ is a bounded sequence.

Since $\delta_n \to 0$ as $n \to \infty$, without loss of generality, we assume $\delta_n < \frac{1}{\|\mathcal{A}\|}$, for every $n \in \mathbb{N}$. Since $\sum_{i=1}^{N} a_i \Phi_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^{N} a_i \Phi_i(\mathcal{V}_n, y) + \frac{1}{\varphi_n}\langle y - \mathcal{V}_n, \mathcal{V}_n - \mathcal{Z}_n \rangle \geq 0, \quad \forall y \in C,$$

by Lemma 2.12 and Remark 2.13, we have $\mathcal{V}_n = T_{\varphi_n}\mathcal{Z}_n$ and $\mathrm{Fix}(T_{\varphi_n}) = \bigcap_{i=1}^{N} \mathrm{EP}(\Phi_i)$.

From Lemma 2.5 and Lemma 2.9(i), we obtain

$$\mathrm{Fix}(\mathcal{T}) = \mathrm{Fix}\big(P_C(I - \psi_n(I - \mathcal{T}))\big).$$

Let $z \in \Omega$. By the nonexpansiveness of P_C and T_{φ_n}, we have

$$\|\mathcal{Y}_n - z\| \leq \theta_n \|P_C \mathcal{Z}_n - z\| + (1 - \theta_n)\|T_{\varphi_n}\mathcal{Z}_n - z\| \leq \|\mathcal{Z}_n - z\|. \tag{3.2}$$

From Lemma 2.6, Lemma 2.9(ii), and (3.2), we obtain

$$\|\mathcal{Z}_{n+1} - z\|$$
$$\leq \delta_n \|\gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}z\| + \|I - \delta_n \mathcal{A}\| \|P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n - z\|$$
$$\leq \delta_n \gamma \|\mathcal{F}(\mathcal{Z}_n) - \mathcal{F}(z)\| + \delta_n \|\gamma \mathcal{F}(z) - \mathcal{A}z\| + (1 - \delta_n \bar{\gamma})\|\mathcal{Y}_n - z\|$$
$$\leq \delta_n \gamma \alpha \|\mathcal{Z}_n - z\| + \delta_n \|\gamma \mathcal{F}(z) - \mathcal{A}z\| + (1 - \delta_n \bar{\gamma})\|\mathcal{Z}_n - z\|$$

$$= \big(1 - \delta_n(\bar{\gamma} - \gamma\alpha)\big)\|\mathcal{Z}_n - z\| + \delta_n\|\gamma\mathcal{F}(z) - \mathcal{A}z\|$$

$$\leq \max\left\{\|\mathcal{Z}_1 - z\|, \frac{\|\gamma\mathcal{F}(z) - \mathcal{A}z\|}{\bar{\gamma} - \gamma\alpha}\right\}.$$

By induction, we obtain $\|\mathcal{Z}_n - z\| \leq \max\{\|\mathcal{Z}_1 - z\|, \frac{\|\gamma\mathcal{F}(z) - \mathcal{A}z\|}{\bar{\gamma} - \gamma\alpha}\}$, $\forall n \in \mathbb{N}$. It shows that $\{\mathcal{Z}_n\}$ is bounded and so are $\{\mathcal{V}_n\}$ and $\{\mathcal{Y}_n\}$.

Step 2. Show that $\lim_{n\to\infty}\|\mathcal{Z}_{n+1} - \mathcal{Z}_n\| = 0$.

By the definition of $\mathcal{Z}_n$ and Lemma 2.6, we obtain

$$\|\mathcal{Z}_{n+1} - \mathcal{Z}_n\|$$
$$\leq \delta_n\gamma\|\mathcal{F}(\mathcal{Z}_n) - \mathcal{F}(\mathcal{Z}_{n-1})\| + \gamma|\delta_n - \delta_{n-1}|\|\mathcal{F}(\mathcal{Z}_{n-1})\|$$
$$\quad + \|I - \delta_n\mathcal{A}\|\|P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n - P_C\big(I - \psi_{n-1}(I - \mathcal{T})\big)\mathcal{Y}_{n-1}\|$$
$$\quad + \big\|(I - \delta_n\mathcal{A})P_C\big(I - \psi_{n-1}(I - \mathcal{T})\big)\mathcal{Y}_{n-1}$$
$$\quad - (I - \delta_{n-1}A)P_C\big(I - \psi_{n-1}(I - \mathcal{T})\big)\mathcal{Y}_{n-1}\big\|$$
$$\leq \delta_n\gamma\alpha\|\mathcal{Z}_n - \mathcal{Z}_{n-1}\| + \gamma|\delta_n - \delta_{n-1}|\|\mathcal{F}(\mathcal{Z}_{n-1})\|$$
$$\quad + (1 - \delta_n\bar{\gamma})\big\|\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n - \big(I - \psi_{n-1}(I - \mathcal{T})\big)\mathcal{Y}_{n-1}\big\|$$
$$\quad + |\delta_n - \delta_{n-1}|\big\|\mathcal{A}P_C\big(I - \psi_{n-1}(I - \mathcal{T})\big)\mathcal{Y}_{n-1}\big\|$$
$$\leq \delta_n\gamma\alpha\|\mathcal{Z}_n - \mathcal{Z}_{n-1}\| + \gamma|\delta_n - \delta_{n-1}|\big\|\mathcal{F}(\mathcal{Z}_{n-1})\big\| + (1 - \delta_n\bar{\gamma})\big[\theta_n\|\mathcal{Z}_n - \mathcal{Z}_{n-1}\|$$
$$\quad + |\theta_n - \theta_{n-1}|\|P_C\mathcal{Z}_{n-1}\| + (1 - \theta_n)\|\mathcal{V}_n - \mathcal{V}_{n-1}\| + |\theta_n - \theta_{n-1}|\|\mathcal{V}_{n-1}\|$$
$$\quad + \psi_n\big\|(I - \mathcal{T})\mathcal{Y}_n - (I - \mathcal{T})\mathcal{Y}_{n-1}\big\| + |\psi_n - \psi_{n-1}|\big\|(I - \mathcal{T})\mathcal{Y}_{n-1}\big\|\big]$$
$$\quad + |\delta_n - \delta_{n-1}|\big\|\mathcal{A}P_C\big(I - \psi_{n-1}(I - \mathcal{T})\big)\mathcal{Y}_{n-1}\big\|. \tag{3.3}$$

Using the same method as in [12] (Step 2 of Theorem 3.1), we have

$$\|\mathcal{V}_n - \mathcal{V}_{n-1}\| \leq \|\mathcal{Z}_n - \mathcal{Z}_{n-1}\| + \frac{1}{\epsilon}|\varphi_n - \varphi_{n-1}|\|\mathcal{V}_n - \mathcal{Z}_n\|. \tag{3.4}$$

Substitute (3.3) into (3.4) to get

$$\|\mathcal{Z}_{n+1} - \mathcal{Z}_n\|$$
$$\leq \delta_n\gamma\alpha\|\mathcal{Z}_n - \mathcal{Z}_{n-1}\| + \gamma|\delta_n - \delta_{n-1}|\big\|\mathcal{F}(\mathcal{Z}_{n-1})\big\| + (1 - \delta_n\bar{\gamma})\bigg[\|\mathcal{Z}_n - \mathcal{Z}_{n-1}\|$$
$$\quad + |\theta_n - \theta_{n-1}|\|P_C\mathcal{Z}_{n-1}\| + \frac{1 - \theta_n}{\epsilon}|\varphi_n - \varphi_{n-1}|\|\mathcal{V}_n - \mathcal{Z}_n\| + |\theta_n - \theta_{n-1}|\|\mathcal{V}_{n-1}\|$$
$$\quad + \psi_n\big\|(I - \mathcal{T})\mathcal{Y}_n - (I - \mathcal{T})\mathcal{Y}_{n-1}\big\| + |\psi_n - \psi_{n-1}|\big\|(I - \mathcal{T})\mathcal{Y}_{n-1}\big\|\bigg]$$
$$\quad + |\delta_n - \delta_{n-1}|\big\|\mathcal{A}P_C\big(I - \psi_{n-1}(I - \mathcal{T})\big)\mathcal{Y}_{n-1}\big\|$$
$$\leq \big(1 - \delta_n(\bar{\gamma} - \gamma\alpha)\big)\|\mathcal{Z}_n - \mathcal{Z}_{n-1}\| + (1 + \gamma)|\delta_n - \delta_{n-1}|K + 2|\theta_n - \theta_{n-1}|K$$
$$\quad + \frac{1}{\epsilon}|\varphi_n - \varphi_{n-1}|K + 2\psi_nK + |\psi_n - \psi_{n-1}|K, \tag{3.5}$$

where $K = \max_{n \in \mathbb{N}}\{\|\mathcal{V}_n\|, \|\mathcal{F}(\mathcal{Z}_n)\|, \|\mathcal{V}_n - \mathcal{Z}_n\|, \|P_C \mathcal{Z}_n\|, \|(I - \mathcal{T})\mathcal{Y}_n\|, \|\mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n\|\}$. From (3.5), the conditions (i), (iii), (v), and Lemma 2.3, we have

$$\lim_{n \to \infty} \|\mathcal{Z}_{n+1} - \mathcal{Z}_n\| = 0. \tag{3.6}$$

Step 3. Prove that $\lim_{n \to \infty} \|\mathcal{V}_n - \mathcal{Z}_n\| = \lim_{n \to \infty} \|P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Z}_n - \mathcal{Z}_n\| = 0$.

To claim this, let $z \in \Omega$. Since $\mathcal{V}_n = T_{\varphi_n}\mathcal{Z}_n$ and T_{φ_n} is a firmly nonexpansive mapping, we have

$$\|z - T_{\varphi_n}\mathcal{Z}_n\|^2 = \|T_{\varphi_n}z - T_{\varphi_n}\mathcal{Z}_n\|^2$$

$$\leq \langle T_{\varphi_n}z - T_{\varphi_n}\mathcal{Z}_n, z - \mathcal{Z}_n \rangle$$

$$= \frac{1}{2}\left(\|T_{\varphi_n}\mathcal{Z}_n - z\|^2 + \|\mathcal{Z}_n - z\|^2 - \|T_{\varphi_n}\mathcal{Z}_n - \mathcal{Z}_n\|^2\right),$$

from which it follows that

$$\|\mathcal{V}_n - z\|^2 \leq \|\mathcal{Z}_n - z\|^2 - \|\mathcal{V}_n - \mathcal{Z}_n\|^2. \tag{3.7}$$

By the definition of $\mathcal{Z}_n$, Lemma 2.6, Lemma 2.9(ii), and (3.7), we get

$$\|\mathcal{Z}_{n+1} - z\|^2$$

$$= \left\|\delta_n\left(\gamma\mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n\right) + \left(P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n - z\right)\right\|^2$$

$$\leq \left\|P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n - z\right\|^2$$

$$\quad + 2\delta_n\langle\gamma\mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n, \mathcal{Z}_{n+1} - z\rangle$$

$$\leq \|\mathcal{Y}_n - z\|^2 + 2\delta_n\left\|\gamma\mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\|$$

$$\leq \theta_n\|P_C\mathcal{Z}_n - z\|^2 + (1 - \theta_n)\|\mathcal{V}_n - z\|^2$$

$$\quad + 2\delta_n\left\|\gamma\mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\|$$

$$\leq \theta_n\|\mathcal{Z}_n - z\|^2 + (1 - \theta_n)\left(\|\mathcal{Z}_n - z\|^2 - \|\mathcal{V}_n - \mathcal{Z}_n\|^2\right)$$

$$\quad + 2\delta_n\left\|\gamma\mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\|$$

$$= \|\mathcal{Z}_n - z\|^2 - (1 - \theta_n)\|\mathcal{V}_n - \mathcal{Z}_n\|^2$$

$$\quad + 2\delta_n\left\|\gamma\mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\|,$$

which implies that

$$(1 - \theta_n)\|\mathcal{V}_n - \mathcal{Z}_n\|^2 \leq \left(\|\mathcal{Z}_n - z\| + \|\mathcal{Z}_{n+1} - z\|\right)\|\mathcal{Z}_{n+1} - \mathcal{Z}_n\|$$

$$+ 2\delta_n\left\|\gamma\mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C(I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\|.$$

From (3.6), the conditions (i) and (ii), this yields

$$\lim_{n \to \infty} \|\mathcal{V}_n - \mathcal{Z}_n\| = 0. \tag{3.8}$$

By Lemma 2.6 and Lemma 2.9(ii), we get

$$
\begin{aligned}
\|\mathcal{Z}_{n+1} - z\|^2 \\
\leq{}& \left\|P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n - z\right\|^2 \\
&+ 2\delta_n\big\langle \gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n, \mathcal{Z}_{n+1} - z\big\rangle \\
\leq{}& \|\mathcal{Y}_n - z\|^2 + 2\delta_n\left\|\gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\| \\
={}& \theta_n\|P_C\mathcal{Z}_n - z\|^2 + (1 - \theta_n)\|\mathcal{V}_n - z\|^2 - \theta_n(1 - \theta_n)\|P_C\mathcal{Z}_n - \mathcal{V}_n\|^2 \\
&+ 2\delta_n\left\|\gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\| \\
\leq{}& \|\mathcal{Z}_n - z\|^2 - \theta_n(1 - \theta_n)\|P_C\mathcal{Z}_n - \mathcal{V}_n\|^2 \\
&+ 2\delta_n\left\|\gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\|,
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\theta_n(1 - \theta_n)\|P_C\mathcal{Z}_n - \mathcal{V}_n\|^2 \leq{}& \big(\|\mathcal{Z}_n - z\| + \|\mathcal{Z}_{n+1} - z\|\big)\|\mathcal{Z}_{n+1} - \mathcal{Z}_n\| \\
&+ 2\delta_n\left\|\gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n\right\|\|\mathcal{Z}_{n+1} - z\|.
\end{aligned}
$$

From (3.6), the conditions (i) and (ii), this implies that

$$
\lim_{n \to \infty}\|P_C\mathcal{Z}_n - \mathcal{V}_n\| = 0. \tag{3.9}
$$

Since

$$
\|P_C\mathcal{Z}_n - \mathcal{Z}_n\| \leq \|P_C\mathcal{Z}_n - \mathcal{V}_n\| + \|\mathcal{V}_n - \mathcal{Z}_n\|,
$$

using (3.8) and (3.9), we have

$$
\lim_{n \to \infty}\|P_C\mathcal{Z}_n - \mathcal{Z}_n\| = 0. \tag{3.10}
$$

Since

$$
\|\mathcal{Y}_n - \mathcal{Z}_n\| \leq \theta_n\|P_C\mathcal{Z}_n - \mathcal{Z}_n\| + (1 - \theta_n)\|\mathcal{V}_n - \mathcal{Z}_n\|,
$$

by (3.8) and (3.10), thus we obtain

$$
\lim_{n \to \infty}\|\mathcal{Y}_n - \mathcal{Z}_n\| = 0. \tag{3.11}
$$

Observe that

$$
\begin{aligned}
\big\|\mathcal{Z}_n - P_C\big(I &- \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n\big\| \\
&\leq \|\mathcal{Z}_n - \mathcal{Z}_{n+1}\| + \big\|\mathcal{Z}_{n+1} - P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n\big\| \\
&= \|\mathcal{Z}_n - \mathcal{Z}_{n+1}\| + \delta_n\big\|\gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}P_C\big(I - \psi_n(I - \mathcal{T})\big)\mathcal{Y}_n\big\|,
\end{aligned}
$$

which implies by (3.6) and the condition (i) that

$$\lim_{n \to \infty} \left\| \mathcal{Z}_n - P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n \right\| = 0. \tag{3.12}$$

Since

$$\left\| \mathcal{Z}_n - P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Z}_n \right\|$$
$$\leq \left\| \mathcal{Z}_n - P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n \right\| + \left\| P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n - P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Z}_n \right\|$$
$$\leq \left\| \mathcal{Z}_n - P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n \right\| + \left\| \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n - \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Z}_n \right\|$$
$$\leq \left\| \mathcal{Z}_n - P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n \right\| + \left\| \mathcal{Y}_n - \mathcal{Z}_n \right\| + \psi_n \left\| (I - \mathcal{T}) \mathcal{Y}_n - (I - \mathcal{T}) \mathcal{Z}_n \right\|,$$

by (3.11), (3.12), and the condition (iii), we obtain

$$\lim_{n \to \infty} \left\| \mathcal{Z}_n - P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Z}_n \right\| = 0. \tag{3.13}$$

Step 4. Show that $\limsup_{n \to \infty} \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_n - q \rangle \leq 0$, where $q = P_\Omega (I - \mathcal{A} + \gamma \mathcal{F}) q$. First, take a subsequence $\{\mathcal{Z}_{n_k}\}$ of $\{\mathcal{Z}_n\}$ such that

$$\limsup_{n \to \infty} \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_n - q \rangle = \lim_{k \to \infty} \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_{n_k} - q \rangle.$$

Since $\{\mathcal{Z}_n\}$ is bounded, we can assume that $\mathcal{Z}_{n_k} \rightharpoonup \omega$ as $k \to \infty$. By (3.8), it follows that $\mathcal{U}_{n_k} \rightharpoonup \omega$ as $k \to \infty$.

Assume $\omega \notin \mathrm{Fix}(\mathcal{T})$. Since $\mathrm{Fix}(\mathcal{T}) = \mathrm{Fix}(P_C(I - \psi_{n_k}(I - \mathcal{T})))$, we have $\omega \neq P_C(I - \psi_{n_k}(I - \mathcal{T}))\omega$. By the nonexpansiveness of P_C, the condition (iii), (3.13), and Opial's condition, we get

$$\liminf_{k \to \infty} \| \mathcal{Z}_{n_k} - \omega \| < \liminf_{k \to \infty} \left\| \mathcal{Z}_{n_k} - P_C \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \omega \right\|$$
$$\leq \liminf_{k \to \infty} \Big(\left\| \mathcal{Z}_{n_k} - P_C \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \mathcal{Z}_{n_k} \right\|$$
$$+ \left\| P_C \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \mathcal{Z}_{n_k} - P_C \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \omega \right\| \Big)$$
$$\leq \liminf_{k \to \infty} \Big(\left\| \mathcal{Z}_{n_k} - P_C \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \mathcal{Z}_{n_k} \right\|$$
$$+ \left\| \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \mathcal{Z}_{n_k} - \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \omega \right\| \Big)$$
$$\leq \liminf_{k \to \infty} \Big(\left\| \mathcal{Z}_{n_k} - P_C \big(I - \psi_{n_k} (I - \mathcal{T}) \big) \mathcal{Z}_{n_k} \right\|$$
$$+ \| \mathcal{Z}_{n_k} - \omega \| + \psi_{n_k} \left\| (I - \mathcal{T}) \mathcal{Z}_{n_k} - (I - \mathcal{T}) \omega \right\| \Big)$$
$$\leq \liminf_{k \to \infty} \| \mathcal{Z}_{n_k} - \omega \|.$$

This is a contradiction. Then we have

$$\omega \in \mathrm{Fix}(\mathcal{T}). \tag{3.14}$$

By continuing the same argument as in [12] (Step 4 of Theorem 3.1), we obtain

$$\omega \in \bigcap_{i=1}^{N} \mathrm{EP}(\Phi_i). \tag{3.15}$$

From (3.14) and (3.15), we get $\omega \in \Omega$. Since $\mathcal{Z}_{n_k} \rightharpoonup \omega$ as $k \to \infty$, by Lemma 2.1 we can conclude that

$$
\begin{aligned}
\limsup_{n \to \infty} \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_n - q \rangle &= \lim_{k \to \infty} \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_{n_k} - q \rangle \\
&= \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \omega - q \rangle \\
&\leq 0.
\end{aligned}
\tag{3.16}
$$

Step 5. Finally, claim that the sequence $\{\mathcal{Z}_n\}$ converges strongly to $q = P_{\Omega}(I - \mathcal{A} + \gamma \mathcal{F})q$. By Lemma 2.4, Lemma 2.6, and Lemma 2.9(ii), we obtain

$$
\begin{aligned}
&\|\mathcal{Z}_{n+1} - q\|^2 \\
&\quad = \left\| \delta_n \big(\gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}q \big) + (I - \delta_n \mathcal{A}) \big(P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n - q \big) \right\|^2 \\
&\quad \leq \left\| (I - \delta_n \mathcal{A}) \big(P_C \big(I - \psi_n (I - \mathcal{T}) \big) \mathcal{Y}_n - q \big) \right\|^2 \\
&\qquad + 2 \delta_n \langle \gamma \mathcal{F}(\mathcal{Z}_n) - \mathcal{A}q, \mathcal{Z}_{n+1} - q \rangle \\
&\quad \leq (1 - \delta_n \bar{\gamma})^2 \|\mathcal{Y}_n - q\|^2 + 2 \delta_n \gamma \| \mathcal{F}(\mathcal{Z}_n) - \mathcal{F}(q) \| \| \mathcal{Z}_{n+1} - q \| \\
&\qquad + 2 \delta_n \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_{n+1} - q \rangle \\
&\quad \leq (1 - \delta_n \bar{\gamma})^2 \big(\theta_n \| P_C \mathcal{Z}_n - q \|^2 + (1 - \theta_n) \| \mathcal{V}_n - q \|^2 \big) \\
&\qquad + 2 \delta_n \gamma \alpha \| \mathcal{Z}_n - q \| \| \mathcal{Z}_{n+1} - q \| + 2 \delta_n \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_{n+1} - q \rangle \\
&\quad \leq (1 - \delta_n \bar{\gamma})^2 \| \mathcal{Z}_n - q \|^2 + \delta_n \gamma \alpha \big(\| \mathcal{Z}_n - q \|^2 + \| \mathcal{Z}_{n+1} - q \|^2 \big) \\
&\qquad + 2 \delta_n \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_{n+1} - q \rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|\mathcal{Z}_{n+1} - q\|^2 \\
&\quad \leq \frac{(1 - \delta_n \bar{\gamma})^2 + \delta_n \gamma \alpha}{1 - \delta_n \gamma \alpha} \| \mathcal{Z}_n - q \|^2 + \frac{2 \delta_n}{1 - \delta_n \gamma \alpha} \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_{n+1} - q \rangle \\
&\quad = \left(1 - \frac{2 \delta_n (\bar{\gamma} - \gamma \alpha)}{1 - \delta_n \gamma \alpha} \right) \| \mathcal{Z}_n - q \|^2 + \frac{2 \delta_n (\bar{\gamma} - \gamma \alpha)}{1 - \delta_n \gamma \alpha} \left(\frac{\delta_n \bar{\gamma}^2}{2(\bar{\gamma} - \gamma \alpha)} \| \mathcal{Z}_n - q \|^2 \right. \\
&\qquad \left. + \frac{1}{\bar{\gamma} - \gamma \alpha} \langle \gamma \mathcal{F}(q) - \mathcal{A}q, \mathcal{Z}_{n+1} - q \rangle \right).
\end{aligned}
$$

From (3.16), the condition (i), and Lemma 2.3, we can conclude that $\{\mathcal{Z}_n\}$ converges strongly to $q = P_{\Omega}(I - \mathcal{A} + \gamma \mathcal{F})q$. By (3.8) and (3.11), we see that $\{\mathcal{V}_n\}$ and $\{\mathcal{Y}_n\}$ converge strongly to $q = P_{\Omega}(I - \mathcal{A} + \gamma \mathcal{F})q$. This completes the proof. $\qquad \square$

The following corollaries are direct results from Theorem 3.1.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{F}$ be an α-contractive mapping on H and let $\mathcal{A}$ be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\mathcal{T} : C \to C$ be a nonspreading mapping. Let*

$\Phi : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4) *with* $\Omega := \mathrm{Fix}(\mathcal{T}) \cap \mathrm{EP}(\Phi) \neq \emptyset$. *Let* $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, *and* $\{\mathcal{V}_n\}$ *be sequences generated by* $\mathcal{Z}_1 \in H$ *and*

$$\begin{cases} \Phi(\mathcal{V}_n, y) + \frac{1}{\varphi_n}\langle y - \mathcal{V}_n, \mathcal{V}_n - \mathcal{Z}_n\rangle \geq 0, \quad \forall y \in C, \\ \mathcal{Y}_n = \theta_n P_C \mathcal{Z}_n + (1 - \theta_n)\mathcal{V}_n, \\ \mathcal{Z}_{n+1} = \delta_n \gamma \mathcal{F}(\mathcal{Z}_n) + (I - \delta_n \mathcal{A}) P_C (I - \psi_n(I - \mathcal{T}))\mathcal{Y}_n, \quad \forall n \in \mathbb{N}, \end{cases} \quad (3.17)$$

where $\{\delta_n\}, \{\theta_n\}, \{\varphi_n\}, \{\psi_n\} \subseteq (0,1)$. *Suppose the conditions* (i)-(vi) *hold.*

(i) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;

(ii) $0 < \tau \leq \theta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;

(iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, *for some* $\epsilon, \eta > 0$;

(v) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$, $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$,
$\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, *and* $\{\mathcal{V}_n\}$ *converge strongly to* $q = P_\Omega(I - \mathcal{A} + \gamma \mathcal{F})q$.

Proof Put $\Phi = \Phi_i$, for all $i = 1, 2, \ldots, N$. Using Theorem 3.1, the desired result is obtained.
□

In 2007, Plubtieng and Punpaeng [19] introduced the general iterative method for an equilibrium problem and a nonexpansive mapping in Hilbert spaces. Let S be a nonexpansive mapping on H with $\mathrm{Fix}(S) \cap \mathrm{EP}(F) \neq \emptyset$. With an initial value $z_1 \in H$, the sequences $\{z_n\}$ and $\{v_n\}$ are generated by

$$\begin{cases} F(v_n, y) + \frac{1}{\varphi_n}\langle y - v_n, v_n - z_n\rangle \geq 0, \quad \forall y \in H, \\ z_{n+1} = \alpha_n \gamma f(z_n) + (I - \alpha_n A)S v_n, \quad \forall n \in \mathbb{N}, \end{cases} \quad (3.18)$$

where $\{r_n\} \subset (0, \infty)$ and $\alpha_n \subset [0,1]$ satisfy some appropriate conditions. Then $\{z_n\}$ and $\{v_n\}$ converge strongly to a point z, where $z = P_{\mathrm{Fix}(S) \cap \mathrm{EP}(F)}(I - A + \gamma f)(z)$.

Later, in 2010, Ceng *et al.* [11] studied the iterative scheme for equilibrium problem and an infinite family of nonexpansive mappings. Let $0 < \gamma \alpha < \tilde{\gamma}$. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$. Starting with $z_1 \in H$, the sequences $\{z_n\}$ and $\{u_n\}$ are generated by the following iterative scheme:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \quad \forall y \in H, \\ v_n = (1 - \gamma_n)z_n + \gamma_n W_n u_n, \\ z_{n+1} = \alpha_n \gamma f(v_n) + (I - \alpha_n A)W_n v_n, \end{cases} \quad (3.19)$$

where W_n is a W-mapping generated by an infinite family of nonexpansive mappings and infinite real numbers. Then, under some suitable conditions, the sequences $\{z_n\}$ and $\{u_n\}$ converge strongly to $z^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(\phi)}\tilde{f}(z^*)$, where $\tilde{f} = I - \theta(A - \gamma f)$.

Remark 3.3 For Corollary 3.2, we prove the strong convergence theorem for equilibrium problem and a nonspreading mapping. Motivated by the results of Ceng *et al.* [11] and Plubtieng and Punpaeng [19], we consider the following statements, different from this work.

(i) We investigate the iterative algorithm for a nonspreading mapping instead of using a nonexpansive mapping.

(ii) We study the general iterative method by using the sequence
$$\mathcal{Y}_n = \theta_n P_C \mathcal{Z}_n + (1 - \theta_n)\mathcal{V}_n.$$

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{F}$ be an α-contractive mapping on H and let $\mathcal{A} : H \to H$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\mathcal{T} : C \to C$ be a nonspreading mapping with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Let $\{\mathcal{Z}_n\}$ be the sequence generated by $\mathcal{Z}_1 \in H$ and*

$$\mathcal{Z}_{n+1} = \delta_n \gamma \mathcal{F}(\mathcal{Z}_n) + (I - \delta_n \mathcal{A})P_C\big(I - \psi_n(I - \mathcal{T})\big)P_C \mathcal{Z}_n, \quad \forall n \in \mathbb{N}, \tag{3.20}$$

where $\{\delta_n\}, \{\psi_n\} \subseteq (0,1)$. Suppose the conditions (i)-(vi) *hold.*

(i) $\lim_{n \to \infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \psi_n < \infty$;

(iii) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$.

Then the sequence $\{\mathcal{Z}_n\}$ converges strongly to $q = P_{\mathrm{Fix}(\mathcal{T})}(I - \mathcal{A} + \gamma \mathcal{F})q$.

Proof Take $\Phi_i = 0$, for every $i = 1, 2, \ldots, N$. Then we have $\mathcal{V}_n = P_C \mathcal{Z}_n$, for every $n \in \mathbb{N}$. The result of Corollary 3.4 can be obtained by Theorem 3.1. □

4 Applications

By means of our main result, we obtain the strong convergence theorem for a finite family of nonspreading mappings and a finite family of equilibrium problems in the setting of Hilbert space. To prove this, the following definitions, remarks, and lemmas are needed.

Definition 4.1 A mapping $\mathcal{T}$ is *quasi-nonexpansive* if

$$\|\mathcal{T}x - p\| \leq \|x - p\|, \quad \text{for every } x \in C \text{ and } p \in \mathrm{Fix}(\mathcal{T}).$$

Remark 4.1 If $\mathcal{T} : C \to C$ is nonspreading with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$, then $\mathcal{T}$ is quasi-nonexpansive.

Example 4.2 Let an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ be given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let $I = [1, 100]$ and let $\mathcal{T} : I^2 \to I^2$ be defined by

$$\mathcal{T}u = \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right), \quad \text{for all } u = (u_1, u_2) \in I^2.$$

First, we show that $\mathcal{T}$ is a nonspreading mapping.

For every $u, v \in I^2$, we obtain

$$\|\mathcal{T}u - \mathcal{T}v\|^2 = \left\| \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right) - \left(\frac{v_1 + 1}{2}, \frac{5v_2 + 1}{6} \right) \right\|^2$$

$$= \left\| \left(\frac{1}{2}(u_1 - v_1), \frac{5}{6}(u_2 - v_2) \right) \right\|^2$$

$$= \frac{1}{4}(u_1 - v_1)^2 + \frac{25}{36}(u_2 - v_2)^2$$

and

$$2\langle u - \mathcal{T}u, v - \mathcal{T}v \rangle$$

$$= 2\left\langle (u_1, u_2) - \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right), (v_1, v_2) - \left(\frac{v_1 + 1}{2}, \frac{5v_2 + 1}{6} \right) \right\rangle$$

$$= 2\left\langle \left(\frac{u_1 - 1}{2}, \frac{u_2 - 1}{6} \right), \left(\frac{v_1 - 1}{2}, \frac{v_2 - 1}{6} \right) \right\rangle$$

$$= 2\left(\frac{u_1 - 1}{2}, \frac{u_2 - 1}{6} \right) \cdot \left(\frac{v_1 - 1}{2}, \frac{v_2 - 1}{6} \right)$$

$$= 2\left[\left(\frac{u_1 - 1}{2} \right)\left(\frac{v_1 - 1}{2} \right) + \left(\frac{u_2 - 1}{6} \right)\left(\frac{v_2 - 1}{6} \right) \right]$$

$$= \frac{(u_1 - 1)(v_1 - 1)}{2} + \frac{(u_2 - 1)(v_2 - 1)}{18}$$

$$\geq 0.$$

This yields

$$\|u - v\|^2 + 2\langle u - \mathcal{T}u, v - \mathcal{T}v \rangle \geq \|u - v\|^2$$

$$= \left\| (u_1 - v_1, u_2 - v_2) \right\|^2$$

$$= (u_1 - v_1)^2 + (u_2 - v_2)^2$$

$$> \frac{1}{4}(u_1 - v_1)^2 + \frac{25}{36}(u_2 - v_2)^2$$

$$= \|\mathcal{T}u - Tv\|^2. \tag{4.1}$$

Then $\mathcal{T}$ is a nonspreading mapping and we observe that $\mathrm{Fix}(\mathcal{T}) = \{\mathbf{1}\}$, where $\mathbf{1} = (1, 1)$. For every $u \in I \times I$ and $\mathbf{1} \in \mathrm{Fix}(\mathcal{T})$, from (4.1), we have

$$\|\mathcal{T}u - \mathcal{T}\mathbf{1}\|^2 \leq \|u - \mathbf{1}\|^2 + 2\langle u - \mathcal{T}u, \mathbf{1} - \mathcal{T}\mathbf{1} \rangle$$

$$= \|u - \mathbf{1}\|^2.$$

Therefore $\mathcal{T}$ is a quasi-nonexpansive mapping.

The following example shows that the converse of Remark 4.1 does not hold.

Example 4.3 Let $I = [0, 2]$ and let $\mathcal{T} : I^2 \to I^2$ be defined by

$$\mathcal{T}u = \begin{cases} \left(\frac{u_1 + 2}{2}, \frac{u_2 + 2}{2} \right) & \text{if } u \in (1, 2] \times (1, 2], \\ \left(\frac{u_1}{2}, \frac{u_2}{2} \right) & \text{if } u \in [0, 1] \times [0, 1]. \end{cases}$$

First, show that $\mathcal{T}$ is quasi-nonexpansive for all $u \in I^2$.

Observe that $\mathrm{Fix}(\mathcal{T}) = \{\mathbf{2}\}$ if $x \in (1, 2] \times (1, 2]$ and $\mathrm{Fix}(\mathcal{T}) = \{\mathbf{0}\}$ if $u \in [0, 1] \times [0, 1]$, where $\mathbf{2} = (2, 2)$ and $\mathbf{0} = (0, 0)$.

For any $u \in (1,2] \times (1,2]$, we have

$$
\left\| \left(\frac{u_1 + 2}{2}, \frac{u_2 + 2}{2} \right) - (2,2) \right\| = \left\| \left(\frac{u_1 - 2}{2}, \frac{u_2 - 2}{2} \right) \right\|
$$

$$
= \frac{1}{2} \| (u_1 - 2, u_2 - 2) \|
$$

$$
= \frac{1}{2} \| (u_1, u_2) - (2,2) \|
$$

$$
< \| u - \mathbf{2} \|.
$$

For every $u \in [0,1] \times [0,1]$, we obtain

$$
\left\| \left(\frac{u_1}{2}, \frac{u_2}{2} \right) - (0,0) \right\| = \left\| \left(\frac{u_1}{2}, \frac{u_2}{2} \right) \right\|
$$

$$
= \frac{1}{2} \| (u_1, u_2) \|
$$

$$
< \| u \|.
$$

Therefore $\mathcal{T}$ is a quasi-nonexpansive for all $u \in I^2$.

Choose $u = (\frac{3}{2}, \frac{3}{2})$ and $v = (\frac{1}{2}, \frac{1}{2})$, we have

$$
\left\| \mathcal{T} \left(\frac{3}{2}, \frac{3}{2} \right) - \mathcal{T} \left(\frac{1}{2}, \frac{1}{2} \right) \right\|^2 = \left\| \left(\frac{7}{4}, \frac{7}{4} \right) - \left(\frac{1}{4}, \frac{1}{4} \right) \right\|^2
$$

$$
= \left\| \left(\frac{6}{4}, \frac{6}{4} \right) \right\|^2 = \left(\frac{6}{4} \right)^2 + \left(\frac{6}{4} \right)^2 = \frac{9}{2}.
$$

Thus we get

$$
\| u - v \|^2 + 2 \langle u - \mathcal{T}u, v - \mathcal{T}v \rangle
$$

$$
= \left\| \left(\frac{3}{2}, \frac{3}{2} \right) - \left(\frac{1}{2}, \frac{1}{2} \right) \right\|^2 + 2 \left\langle \left(\frac{3}{2}, \frac{3}{2} \right) - \mathcal{T} \left(\left(\frac{3}{2}, \frac{3}{2} \right) \right), \left(\frac{1}{2}, \frac{1}{2} \right) - \mathcal{T} \left(\frac{1}{2}, \frac{1}{2} \right) \right\rangle
$$

$$
= \| (1,1) \|^2 + 2 \left\langle \left(\frac{3}{2}, \frac{3}{2} \right) - \left(\frac{7}{4}, \frac{7}{4} \right), \left(\frac{1}{2}, \frac{1}{2} \right) - \left(\frac{1}{4}, \frac{1}{4} \right) \right\rangle
$$

$$
= 2 + 2 \left(\frac{3}{2} - \frac{7}{4}, \frac{3}{2} - \frac{7}{4} \right) \cdot \left(\frac{1}{2} - \frac{1}{4}, \frac{1}{2} - \frac{1}{4} \right)
$$

$$
= 2 + 2 \left(-\frac{1}{4}, -\frac{1}{4} \right) \cdot \left(\frac{1}{4}, \frac{1}{4} \right)
$$

$$
= 2 - \frac{2}{8} = \frac{14}{8}.
$$

Hence we have

$$
\| \mathcal{T}u - \mathcal{T}v \|^2 > \| u - v \|^2 + 2 \langle u - \mathcal{T}u, v - \mathcal{T}v \rangle.
$$

By changing $\mathcal{T}$ from a nonspreading mapping to a quasi-nonexpansive mapping with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$, we obtain the same result as shown in Lemma 2.9.

Remark 4.4 Let C be a nonempty closed convex subset of a real Hilbert space H and let $\mathcal{T} : C \to C$ be a quasi-nonexpansive mapping with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Then we have the following statement:

  (i) $\mathrm{Fix}(\mathcal{T}) = \mathrm{VI}(C, I - \mathcal{T})$;

  (ii) for every $u \in C$ and $v \in \mathrm{Fix}(\mathcal{T})$,

$$\left\| P_C \left(I - \lambda (I - \mathcal{T}) \right) u - v \right\| \leq \| u - v \|, \quad \text{where } \lambda \in (0, 1).$$

Definition 4.2 ([24]) Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, \ldots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$U_0 = I,$$

$$U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,$$

$$U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,$$

$$U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,$$

$$\vdots$$

$$U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,$$

$$S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.$$

This mapping is called the *S-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 4.5 ([25]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $\mathrm{Fix}(S) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ and S is a quasi-nonexpansive mapping.*

Theorem 4.6 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{F} : C \to C$ be an α-contractive mapping, let $\mathcal{A} : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. For $i = 1, 2, \ldots, \bar{N}$, let $\Phi_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $\mathcal{T}_i : C \to C$, for $i = 1, 2, \ldots, N$ be a finite family of nonspreading mappings with $\Omega := \bigcap_{i=1}^N \mathrm{Fix}(\mathcal{T}_i) \cap \bigcap_{i=1}^{\bar{N}} \mathrm{EP}(\Phi_i) \neq \emptyset$. Let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \ldots, N$, and let S be the S-mapping generated by $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_N$ and $\rho_1, \rho_2, \ldots, \rho_N$. Let $\{\mathcal{Z}_n\}, \{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ be sequences generated by $\mathcal{Z}_1 \in H$ and*

$$\begin{cases} \sum_{i=1}^{\bar{N}} a_i \Phi_i(\mathcal{V}_n, y) + \frac{1}{\varphi_n} \langle y - \mathcal{V}_n, \mathcal{V}_n - \mathcal{Z}_n \rangle \geq 0, & \forall y \in C, \\ \mathcal{Y}_n = \theta_n P_C \mathcal{Z}_n + (1 - \theta_n) \mathcal{V}_n, \\ \mathcal{Z}_{n+1} = \delta_n \gamma \mathcal{F}(\mathcal{Z}_n) + (I - \delta_n \mathcal{A}) P_C (I - \psi_n (I - S)) \mathcal{Y}_n, & \forall n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\{\delta_n\}, \{\theta_n\}, \{\varphi_n\}, \{\psi_n\} \subseteq (0,1)$, $0 < a_i < 1$, *for all* $i = 1, 2, \ldots, \bar{N}$. *Suppose the conditions* (i)-(vi) *hold.*

   (i) $\lim_{n \to \infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;

   (ii) $0 < \tau \le \theta_n \le \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

   (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;

   (iv) $0 < \epsilon \le \varphi_n \le \eta < 1$, *for some* $\epsilon, \eta > 0$;

   (v) $\sum_{n=1}^{\bar{N}} a_i = 1$;

   (vi) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$, $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$,
      $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, *and* $\{\mathcal{V}_n\}$ *converge strongly to* $q = P_\Omega(I - \mathcal{A} + \gamma \mathcal{F})q$.

Proof Using Remark 4.4, Lemma 4.5, and the same method as in Theorem 3.1, we have the desired conclusion. □

Remark 4.7 Theorem 4.6 can be considered as an improvement of Theorem 3.1 in the result of Tian and Jin [26] in the sense that some conditions are not assumed.

   (i) $T_\omega := (1 - \omega)I + \omega T$, $\omega \in (0, \frac{1}{2})$,

   (ii) T is demi-closed on H,

where T is a quasi-nonexpansive mapping on H.

5 Examples for equilibrium problems and numerical results

In this section, the numerical examples are given for supporting Theorem 3.1. Using these examples, we see that our iteration for the combination of equilibrium problem converges faster than our iteration for the classical equilibrium problem.

Example 5.1 Let the mappings $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, $\mathcal{F} : \mathbb{R} \to \mathbb{R}$, be defined by

$$\mathcal{A}x = \frac{x}{2},$$

$$\mathcal{F}x = \frac{x}{4}, \quad \text{for all } x \in \mathbb{R}.$$

For every $i = 1, 2, \ldots, N$, let $\Phi_i : [1, 100] \times [1, 100] \to \mathbb{R}$ and $\mathcal{T} : [1, 100] \to [1, 100]$ be defined by

$$\mathcal{T}x = \frac{2x + 5}{7},$$

$$\Phi_i(x, y) = i(y - x)(y + 2x - 3), \quad \text{for all } x, y \in [1, 100].$$

Put $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, for every $i = 1, 2, \ldots, N$. Let $\gamma = \frac{1}{3}$, $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$, and $\psi_n = \frac{1}{n^2}$ for every $n \in \mathbb{N}$. Let the initial values be defined as in the following cases:

   (i) $\mathcal{Z}_1 = 50$, $N = 1$, and $n = 10$,

   (ii) $\mathcal{Z}_1 = 50$ and $n = N = 10$.

Then, for both cases, the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ converge strongly to 1.

Solution. It is obvious that $\mathcal{T}$ is a nonspreading mapping and $\text{Fix}(\mathcal{T}) = \{1\}$.

Since $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, we obtain

$$\sum_{i=1}^{N} a_i \Phi_i(x, y) = \sum_{i=1}^{N} \left(\frac{4}{5^i} + \frac{1}{N5^N} \right) i(y - x)(y + 2x - 3) = \mu(y - x)(y + 2x - 3),$$

where $\mu = \sum_{i=1}^{N}(\frac{4}{5^i} + \frac{1}{N5^N})i$. It is clear that $\sum_{i=1}^{N} a_i \Phi_i$ satisfies all conditions in Theorem 3.1 and $\mathrm{EP}(\sum_{i=1}^{N} a_i \Phi_i) = \bigcap_{i=1}^{N} \mathrm{EP}(\Phi_i) = \{1\}$. Then we have

$$\mathrm{Fix}(\mathcal{T}) \cap \bigcap_{i=1}^{N} \mathrm{EP}(\Phi_i) = \{1\}.$$

Observe that

$$0 \le \sum_{i=1}^{N} a_i \Phi_i(\mathcal{V}_n, y) + \frac{1}{\varphi_n} \langle y - \mathcal{V}_n, \mathcal{V}_n - \mathcal{Z}_n \rangle$$

$$= \mu(y - \mathcal{V}_n)(y + 2\mathcal{V}_n - 3) + \frac{1}{\varphi_n}(y - \mathcal{V}_n)(\mathcal{V}_n - \mathcal{Z}_n)$$

$$\Leftrightarrow \quad 0 \le \mu\varphi_n(y - \mathcal{V}_n)(y + 2\mathcal{V}_n - 3) + (y - \mathcal{V}_n)(\mathcal{V}_n - \mathcal{Z}_n)$$

$$= \mu\varphi_n y^2 + (\mu\mathcal{V}_n\varphi_n + \mathcal{V}_n - \mathcal{Z}_n - 3\mu\varphi_n)y$$

$$+ 3\mu\varphi_n\mathcal{V}_n - \mathcal{V}_n^2 - 2\mu\varphi_n\mathcal{V}_n^2 + \mathcal{V}_n\mathcal{Z}_n. \tag{5.1}$$

Let $G(y) = \mu\varphi_n y^2 + (\mu\mathcal{V}_n\varphi_n + \mathcal{V}_n - \mathcal{Z}_n - 3\mu\varphi_n)y + 3\mu\varphi_n\mathcal{V}_n - \mathcal{V}_n^2 - 2\mu\varphi_n\mathcal{V}_n^2 + \mathcal{V}_n\mathcal{Z}_n$. $G(y)$ is a quadratic function of y with coefficients $a = \mu\varphi_n$, $b = \mu\mathcal{V}_n\varphi_n + \mathcal{V}_n - \mathcal{Z}_n - 3\mu\varphi_n$, and $c = 3\mu\varphi_n\mathcal{V}_n - \mathcal{V}_n^2 - 2\mu\varphi_n\mathcal{V}_n^2 + \mathcal{V}_n\mathcal{Z}_n$. Determine the discriminant Δ of G as follows:

$$\Delta = b^2 - 4ac$$

$$= (\mu\mathcal{V}_n\varphi_n + \mathcal{V}_n - \mathcal{Z}_n - 3\mu\varphi_n)^2 - 4(\mu\varphi_n)(3\mu\varphi_n\mathcal{V}_n - \mathcal{V}_n^2 - 2\mu\varphi_n\mathcal{V}_n^2 + \mathcal{V}_n\mathcal{Z}_n)$$

$$= 9\mu^2\varphi_n^2 - 6\mu\varphi_n\mathcal{V}_n - 18\mu^2\varphi_n^2\mathcal{V}_n + \mathcal{V}_n^2 + 6\mu\varphi_n\mathcal{V}_n^2 + 9\mu^2\varphi_n^2\mathcal{V}_n^2 + 6\mu\varphi_n\mathcal{Z}_n - 2\mathcal{V}_n\mathcal{Z}_n$$

$$- 6\mu\varphi_n\mathcal{V}_n\mathcal{Z}_n + \mathcal{Z}_n^2$$

$$= (\mathcal{V}_n - 3\mu\varphi_n + 3\mu\varphi_n\mathcal{V}_n - \mathcal{Z}_n)^2.$$

From (5.1), we have $G(y) \ge 0$, for every $y \in \mathbb{R}$. If $G(y)$ has at most one solution in $\mathbb{R}$, thus we have $\Delta \le 0$. This implies that

$$\mathcal{V}_n = \frac{\mathcal{Z}_n + 3\mu\varphi_n}{1 + 3\mu\varphi_n}, \tag{5.2}$$

where $\mu = \sum_{i=1}^{N}(\frac{4}{5^i} + \frac{1}{N5^N})i$. Put $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$, $\psi_n = \frac{1}{n^2}$, $\forall n \in \mathbb{N}$. It is clear to see that the sequences $\{\delta_n\}$, $\{\theta_n\}$, $\{\varphi_n\}$, and $\{\psi_n\}$ satisfy all conditions in Theorem 3.1. For every $n \in \mathbb{N}$, from (5.2), we rewrite (3.1) as follows:

$$\begin{cases} \mathcal{Y}_n = \frac{n}{2n+3} P_{[1,100]} \mathcal{Z}_n + (1 - \frac{n}{2n+3}) \frac{1}{1 + 3\mu\frac{2n}{3n+2}}(\mathcal{Z}_n + 3\mu\frac{2n}{3n+2}), \\ \mathcal{Z}_{n+1} = \frac{1}{36n}\mathcal{Z}_n + (I - \frac{1}{3n}\mathcal{A})P_{[1,100]}(I - \frac{1}{n^2}(I - \mathcal{T}))\mathcal{Y}_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{5.3}$$

From Theorem 3.1, we can conclude that the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ generated by (5.3) converge strongly to 1.

For case (i), with $N = 1$, we have $\mu = 1$. Then (5.2) becomes

$$\mathcal{V}_n = \frac{\mathcal{Z}_n + 3\varphi_n}{1 + 3\varphi_n}. \tag{5.4}$$

Table 1 The values of $\{\mathcal{V}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{Z}_n\}$ with $\mathcal{Z}_1 = 50$

n	N = 1			N = 10		
	$\mathcal{V}_n$	$\mathcal{Y}_n$	$\mathcal{Z}_n$	$\mathcal{V}_n$	$\mathcal{Y}_n$	$\mathcal{Z}_n$
1	23.272727	28.618182	50.000000	20.600006	26.480004	50.000000
2	4.119192	5.455988	8.797980	3.535267	4.893445	8.288890
3	2.287423	2.989653	4.394114	1.973077	2.636539	3.963461
4	1.631883	2.025784	2.715111	1.446758	1.794880	2.404095
5	1.331444	1.556406	1.916346	1.219216	1.405203	1.702781
6	1.178502	1.307023	1.499805	1.109733	1.208492	1.356631
7	1.096153	1.168453	1.271738	1.053976	1.104708	1.177182
8	1.050342	1.089475	1.143282	1.024666	1.048632	1.081587
9	1.024377	1.043831	1.069769	1.009019	1.018015	1.030010
10	1.009539	1.017314	1.027423	1.000654	1.001320	1.002186

(a) N = 1   (b) N = 10
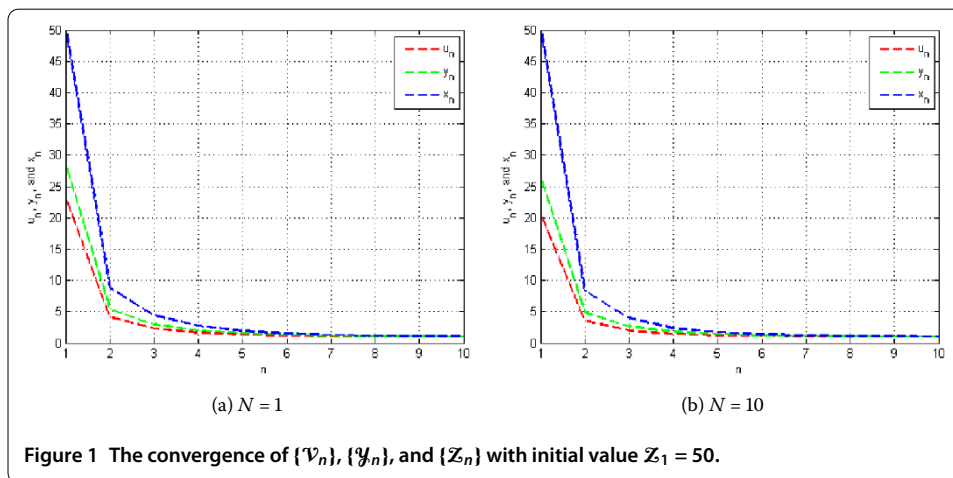
Figure 1 The convergence of $\{\mathcal{V}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{Z}_n\}$ with initial value $\mathcal{Z}_1 = 50$.

Then we have

$$\begin{cases} \mathcal{Y}_n = \frac{n}{2n+3} P_{[1,100]} \mathcal{Z}_n + (1 - \frac{n}{2n+3}) \frac{1}{1+3\frac{2n}{3n+2}} (\mathcal{Z}_n + 3\frac{2n}{3n+2}), \\ \mathcal{Z}_{n+1} = \frac{1}{36n} \mathcal{Z}_n + (I - \frac{1}{3n}\mathcal{A}) P_{[1,100]} (I - \frac{1}{n^2}(I - \mathcal{T})) \mathcal{Y}_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{5.5}$$

From Corollary 3.2, we can conclude that the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ generated by (5.5) converge strongly to 1.

Table 1 and Figure 1 show the values of sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ in two cases.

Remark 5.2

(i) From Table 1 and Figure 1, the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ converge to 1, where $\{1\} = \mathrm{Fix}(\mathcal{T}) \cap \bigcap_{i=1}^N \mathrm{EP}(\Phi_i)$.

(ii) For case (i), Corollary 3.2 guarantees the convergence of $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$.

(iii) For case (ii), the convergence of $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ can be guaranteed by Theorem 3.1.

(iv) The iteration (5.3) for the combination of equilibrium problem converges faster than the iteration (5.5) for the classical equilibrium problem.

Finally, we give the numerical example for our main theorem in two-dimensional space of real numbers.

Example 5.3 Let $\mathbb{R}^2$ be the two-dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let the mappings $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathcal{A}u = \left(\frac{u_1}{2}, \frac{u_2}{2} \right),$$

$$\mathcal{F}u = \left(\frac{u_1}{4}, \frac{u_2}{4} \right), \quad \text{for all } u = (u_1, u_2) \in \mathbb{R}^2.$$

For every $i = 1, 2, \ldots, N$ and $I = [0, 1]$, let $\Phi_i : I^2 \times I^2 \to \mathbb{R}$ and $\mathcal{T} : I^2 \to I^2$ be defined by

$$\mathcal{T}u = \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right),$$

$$\Phi_i(u, v) = i(v - u) \cdot (v + 6u - 7), \quad \text{for all } u = (u_1, u_2), v = (v_1, v_2) \in I^2,$$

where $7 = (7, 7)$. Let $\gamma = \frac{1}{3}$, $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$, and $\psi_n = \frac{1}{n^2}$ for every $n \in \mathbb{N}$.

It is clear that $\mathcal{T}$ is a nonspreading mapping and $\text{Fix}(\mathcal{T}) = \{1\}$, where $1 = (1, 1)$.

Put $a_i = \frac{8}{9^i} + \frac{1}{N9^N}$, for every $i = 1, 2, \ldots, N$. It is obvious that $\sum_{i=1}^N a_i \Phi_i$ satisfies all conditions in Theorem 3.1 and $\text{EP}(\sum_{i=1}^N a_i \Phi_i) = \bigcap_{i=1}^N \text{EP}(\Phi_i) = \{1\}$, where $1 = (1, 1)$. Then we have

$$\text{Fix}(\mathcal{T}) \cap \bigcap_{i=1}^N \text{EP}(\Phi_i) = \{1\}.$$

Then, by Theorem 3.1, the sequences $\mathbb{Z}_n = (\mathbb{Z}_n^1, \mathbb{Z}_n^2)$, $\mathcal{Y}_n = (\mathcal{Y}_n^1, \mathcal{Y}_n^2)$, and $\mathcal{V}_n = (\mathcal{V}_n^1, \mathcal{V}_n^2)$ converge strongly to $\{1\}$.

Remark 5.4 From Example 5.3, putting $\rho = \sum_{i=1}^N (\frac{8}{9^i} + \frac{1}{N9^N})i$, we obtain

$$0 \le \sum_{i=1}^N a_i \Phi_i(\mathcal{V}_n, y) + \frac{1}{\varphi_n} \langle y - \mathcal{V}_n, \mathcal{V}_n - \mathbb{Z}_n \rangle$$

$$= \rho(y - \mathcal{V}_n) \cdot (y + 6\mathcal{V}_n - (7, 7)) + \frac{1}{\varphi_n}(y - \mathcal{V}_n) \cdot (\mathcal{V}_n - \mathbb{Z}_n)$$

$$= \rho(y_1 - \mathcal{V}_n^1, y_2 - \mathcal{V}_n^2) \cdot (y_1 + 6\mathcal{V}_n^1 - 7, y_2 + 6\mathcal{V}_n^2 - 7)$$

$$\quad + \frac{1}{\varphi_n}(y_1 - \mathcal{V}_n^1, y_2 - \mathcal{V}_n^2) \cdot (\mathcal{V}_n^1 - \mathbb{Z}_n^1, \mathcal{V}_n^2 - \mathbb{Z}_n^2)$$

$$= \rho\left((y_1 - \mathcal{V}_n^1)(y_1 + 6\mathcal{V}_n^1 - 7) + (y_2 - \mathcal{V}_n^2)(y_2 + 6\mathcal{V}_n^2 - 7) \right)$$

$$\quad + \frac{1}{\varphi_n}\left((y_1 - \mathcal{V}_n^1)(\mathcal{V}_n^1 - \mathbb{Z}_n^1) + (y_2 - \mathcal{V}_n^2)(\mathcal{V}_n^2 - \mathbb{Z}_n^2) \right)$$

$$= \left(\rho(y_1 - \mathcal{V}_n^1)(y_1 + 6\mathcal{V}_n^1 - 7) + \frac{1}{\varphi_n}(y_1 - \mathcal{V}_n^1)(\mathcal{V}_n^1 - \mathbb{Z}_n^1) \right)$$

$$\quad + \left(\rho(y_2 - \mathcal{V}_n^2)(y_2 + 6\mathcal{V}_n^2 - 7) + \frac{1}{\varphi_n}(y_2 - \mathcal{V}_n^2)(\mathcal{V}_n^2 - \mathbb{Z}_n^2) \right)$$

$$\Leftrightarrow \quad 0 \le \left(\rho\varphi_n(y_1 - \mathcal{V}_n^1)(y_1 + 6\mathcal{V}_n^1 - 7) + (y_1 - \mathcal{V}_n^1)(\mathcal{V}_n^1 - \mathbb{Z}_n^1) \right)$$

$$+ \left(\rho \varphi_n \left(y_2 - \mathcal{V}_n^2 \right) \left(y_2 + 6 \mathcal{V}_n^2 - 7 \right) + \left(y_2 - \mathcal{V}_n^2 \right) \left(\mathcal{V}_n^2 - \mathcal{Z}_n^2 \right) \right)$$

$$= \left(\rho \varphi_n (y_1)^2 + \left(5 \rho \mathcal{V}_n^1 \varphi_n + \mathcal{V}_n^1 - \mathcal{Z}_n^1 - 7 \rho \varphi_n \right) y_1 \right.$$

$$+ 7 \rho \varphi_n \mathcal{V}_n^1 - \left(\mathcal{V}_n^1 \right)^2 - 6 \rho \varphi_n \left(\mathcal{V}_n^1 \right)^2 + \mathcal{V}_n^1 \mathcal{Z}_n^1)$$

$$+ \left(\rho \varphi_n (y_2)^2 + \left(5 \rho \mathcal{V}_n^2 \varphi_n + \mathcal{V}_n^2 - \mathcal{Z}_n^2 - 7 \rho \varphi_n \right) y_2 \right.$$

$$+ 7 \rho \varphi_n \mathcal{V}_n^2 - \left(\mathcal{V}_n^2 \right)^2 - 6 \rho \varphi_n \left(\mathcal{V}_n^2 \right)^2 + \mathcal{V}_n^2 \mathcal{Z}_n^2)$$

$$= G_1(y_1) + G_2(y_2), \tag{5.6}$$

where $G_1(y_1) = \rho \varphi_n (y_1)^2 + (5 \rho \mathcal{V}_n^1 \varphi_n + \mathcal{V}_n^1 - \mathcal{Z}_n^1 - 7 \rho \varphi_n) y_1 + 7 \rho \varphi_n \mathcal{V}_n^1 - (\mathcal{V}_n^1)^2 - 6 \rho \varphi_n (\mathcal{V}_n^1)^2 + \mathcal{V}_n^1 \mathcal{Z}_n^1$ and $G_2(y_2) = \rho \varphi_n (y_2)^2 + (5 \rho \mathcal{V}_n^2 \varphi_n + \mathcal{V}_n^2 - \mathcal{Z}_n^2 - 7 \rho \varphi_n) y_2 + 7 \rho \varphi_n \mathcal{V}_n^2 - (\mathcal{V}_n^2)^2 - 6 \rho \varphi_n (\mathcal{V}_n^2)^2 + \mathcal{V}_n^2 \mathcal{Z}_n^2$. Then $G_1(y_1)$ and $G_2(y_2)$ are quadratic functions with coefficients $a_1 = \rho \varphi_n$, $b_1 = 5 \rho \mathcal{V}_n^1 \varphi_n + \mathcal{V}_n^1 - \mathcal{Z}_n^1 - 7 \rho \varphi_n$, and $c_1 = 7 \rho \varphi_n \mathcal{V}_n^1 - (\mathcal{V}_n^1)^2 - 6 \rho \varphi_n (\mathcal{V}_n^1)^2 + \mathcal{V}_n^1 \mathcal{Z}_n^1$, and $a_2 = \rho \varphi_n$, $b_2 = 5 \rho \mathcal{V}_n^2 \varphi_n + \mathcal{V}_n^2 - \mathcal{Z}_n^2 - 7 \rho \varphi_n$, and $c_2 = 7 \rho \varphi_n \mathcal{V}_n^2 - (\mathcal{V}_n^2)^2 - 6 \rho \varphi_n (\mathcal{V}_n^2)^2 + \mathcal{V}_n^2 \mathcal{Z}_n^2$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$\Delta_1 = (b_1)^2 - 4 a_1 c_1$$

$$= \left(5 \rho \mathcal{V}_n^1 \varphi_n + \mathcal{V}_n^1 - \mathcal{Z}_n^1 - 7 \rho \varphi_n \right)^2 - 4 \rho \varphi_n \left(7 \rho \varphi_n \mathcal{V}_n^1 - \left(\mathcal{V}_n^1 \right)^2 - 6 \rho \varphi_n \left(\mathcal{V}_n^1 \right)^2 + \mathcal{V}_n^1 \mathcal{Z}_n^1 \right)$$

$$= \left(\mathcal{V}_n^1 - 7 \rho \varphi_n + 7 \rho \varphi_n \mathcal{V}_n^1 - \mathcal{Z}_n^1 \right)^2.$$

From (5.6), if $G_1(y_1) \geq 0$, $\forall y_1 \in \mathbb{R}$, it has at most one solution in $\mathbb{R}$, thus $\Delta_1 \leq 0$. It follows that

$$\mathcal{V}_n^1 = \frac{\mathcal{Z}_n^1 + 7 \rho \varphi_n}{1 + 7 \rho \varphi_n}. \tag{5.7}$$

Next, we determine the discriminant Δ_2 of G_2 as follows:

$$\Delta_2 = (b_2)^2 - 4 a_2 c_2$$

$$= \left(5 \rho \mathcal{V}_n^2 \varphi_n + \mathcal{V}_n^2 - \mathcal{Z}_n^2 - 7 \rho \varphi_n \right)^2 - 4 \rho \varphi_n \left(7 \rho \varphi_n \mathcal{V}_n^2 - \left(\mathcal{V}_n^2 \right)^2 - 6 \rho \varphi_n \left(\mathcal{V}_n^2 \right)^2 + \mathcal{V}_n^2 \mathcal{Z}_n^2 \right)$$

$$= \left(\mathcal{V}_n^2 - 7 \rho \varphi_n + 7 \rho \varphi_n \mathcal{V}_n^2 - \mathcal{Z}_n^2 \right)^2.$$

From (5.6), if $G_2(y_2) \geq 0$, $\forall y_2 \in \mathbb{R}$ and it has at most one solution in $\mathbb{R}$, then $\Delta_2 \leq 0$. This yields

$$\mathcal{V}_n^2 = \frac{\mathcal{Z}_n^2 + 7 \rho \varphi_n}{1 + 7 \rho \varphi_n}. \tag{5.8}$$

Put $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$, $\psi_n = \frac{1}{n^2}$, for all $n \in \mathbb{N}$. It is obvious that the sequences $\{\delta_n\}$, $\{\theta_n\}$, $\{\varphi_n\}$, and $\{\psi_n\}$ satisfy all conditions in Theorem 3.1. For every $n \in \mathbb{N}$, from (5.7) and (5.8), the iterative scheme (3.1) becomes

$$\begin{cases} \mathcal{Y}_n = \frac{n}{2n+3} P_C \mathcal{Z}_n + \left(1 - \frac{n}{2n+3} \right) \mathcal{U}_n, \\ \mathcal{Z}_{n+1} = \frac{1}{36n} \mathcal{Z}_n + \left(I - \frac{1}{3n} A \right) P_C \left(I - \frac{1}{n^2} (I - \mathcal{T}) \right) \mathcal{Y}_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{5.9}$$

where $\mathcal{Z}_n = (\mathcal{Z}_n^1, \mathcal{Z}_n^2)$, $\mathcal{Y}_n = (\mathcal{Y}_n^1, \mathcal{Y}_n^2)$, and $\mathcal{V}_n = (\mathcal{V}_n^1, \mathcal{V}_n^2) = \left(\frac{\mathcal{Z}_n^1 + 7 \rho \varphi_n}{1 + 7 \rho \varphi_n}, \frac{\mathcal{Z}_n^2 + 7 \rho \varphi_n}{1 + 7 \rho \varphi_n} \right)$.

Let the initial values be defined as in the following cases.

(i) $\mathcal{Z}_1 = (\mathcal{Z}_1^1, \mathcal{Z}_1^2) = (-1, 0)$, $N = 1$, and $n = 20$,

(ii) $\mathcal{Z}_1 = (\mathcal{Z}_1^1, \mathcal{Z}_1^2) = (-1, 0)$ and $n = N = 20$.

For case (i), with $N = 1$, we have $\rho = 1$. Then, from (5.7) and (5.8), we obtain

$$\mathcal{V}_n^1 = \frac{\mathcal{Z}_n^1 + 3\varphi_n}{1 + 7\varphi_n}$$

and

$$\mathcal{V}_n^2 = \frac{\mathcal{Z}_n^2 + 3\varphi_n}{1 + 7\varphi_n}.$$

Then we have

$$\begin{cases} \mathcal{Y}_n = \frac{n}{2n+3} P_C \mathcal{Z}_n + (1 - \frac{n}{2n+3}) \mathcal{V}_n, \\ \mathcal{Z}_{n+1} = \frac{1}{36n} \mathcal{Z}_n + (I - \frac{1}{3n} A) P_C (I - \frac{1}{n^2}(I - \mathcal{T})) \mathcal{Y}_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{5.10}$$

where $\mathcal{Z}_n = (\mathcal{Z}_n^1, \mathcal{Z}_n^2)$, $\mathcal{Y}_n = (\mathcal{Y}_n^1, \mathcal{Y}_n^2)$, and $\mathcal{V}_n = (\mathcal{V}_n^1, \mathcal{V}_n^2) = (\frac{\mathcal{Z}_n^1 + 3\varphi_n}{1 + 7\varphi_n}, \frac{\mathcal{Z}_n^2 + 3\varphi_n}{1 + 7\varphi_n})$.
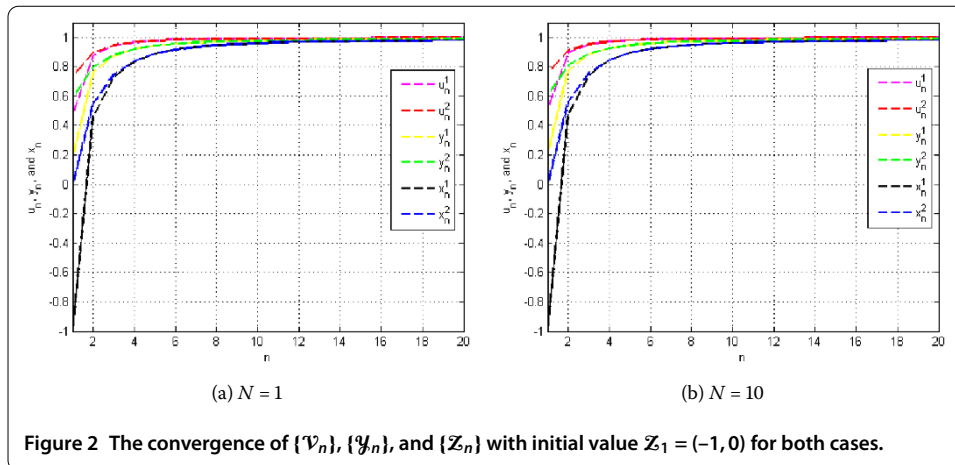
Tables 2 and 3 and Figure 2 show the values of the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ in the two cases.

Table 2 The values of $\{\mathcal{V}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{Z}_n\}$ with $\mathcal{Z}_1 = (-1, 0)$ and $N = 1$

n	$\mathcal{V}_n = (\mathcal{V}_n^1, \mathcal{V}_n^2)$	$\mathcal{Y}_n = (\mathcal{Y}_n^1, \mathcal{Y}_n^2)$	$\mathcal{Z}_n = (\mathcal{Z}_n^1, \mathcal{Z}_n^2)$
1	(0.473684, 0.736842)	(0.178947, 0.589474)	(−1.000000, 0.000000)
2	(0.880767, 0.899610)	(0.761533, 0.799220)	(0.463450, 0.548246)
3	(0.944343, 0.947678)	(0.873506, 0.881086)	(0.731833, 0.747902)
4	(0.967678, 0.968228)	(0.920665, 0.922015)	(0.838391, 0.841141)
5	(0.978604, 0.978548)	(0.944718, 0.944575)	(0.890502, 0.890217)
⋮	⋮	⋮	⋮
10	(0.992787, 0.992723)	(0.979067, 0.978881)	(0.961231, 0.960887)
⋮	⋮	⋮	⋮
16	(0.995890, 0.995878)	(0.987473, 0.987437)	(0.977477, 0.977412)
17	(0.996158, 0.996149)	(0.988232, 0.988203)	(0.978908, 0.978856)
18	(0.996393, 0.996386)	(0.988903, 0.988879)	(0.980164, 0.980121)
19	(0.996601, 0.996595)	(0.989499, 0.989479)	(0.981276, 0.981241)
20	(0.996785, 0.996780)	(0.990033, 0.990017)	(0.982268, 0.982239)

Table 3 The values of $\{\mathcal{V}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{Z}_n\}$ with $\mathcal{Z}_1 = (-1, 0)$, $N = 20$

n	$\mathcal{V}_n = (\mathcal{V}_n^1, \mathcal{V}_n^2)$	$\mathcal{Y}_n = (\mathcal{Y}_n^1, \mathcal{Y}_n^2)$	$\mathcal{Z}_n = (\mathcal{Z}_n^1, \mathcal{Z}_n^2)$
1	(0.518072, 0.759036)	(0.214458, 0.607229)	(−1.000000, 0.000000)
2	(0.894328, 0.911003)	(0.775448, 0.810881)	(0.478246, 0.560576)
3	(0.951505, 0.954360)	(0.882070, 0.889012)	(0.743199, 0.758316)
4	(0.972025, 0.972470)	(0.926247, 0.927421)	(0.846135, 0.848585)
5	(0.981489, 0.981428)	(0.948508, 0.948339)	(0.895738, 0.895396)
⋮	⋮	⋮	⋮
10	(0.993655, 0.993602)	(0.980077, 0.979909)	(0.962426, 0.962110)
⋮	⋮	⋮	⋮
16	(0.996360, 0.996351)	(0.987975, 0.987942)	(0.978017, 0.977957)
17	(0.996597, 0.996589)	(0.988698, 0.988671)	(0.979405, 0.979357)
18	(0.996804, 0.996798)	(0.989336, 0.989315)	(0.980624, 0.980585)
19	(0.996987, 0.996982)	(0.989905, 0.989888)	(0.981706, 0.981673)
20	(0.997150, 0.997146)	(0.990416, 0.990401)	(0.982671, 0.982644)

(a) $N = 1$ (b) $N = 10$

Figure 2 The convergence of $\{\mathcal{V}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{Z}_n\}$ with initial value $\mathcal{Z}_1 = (-1, 0)$ for both cases.

Remark 5.5

(i) Tables 2 and 3, and Figure 2 show that the sequences $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ converge to **1**, where $\{\mathbf{1}\} = \{(1,1)\} = \mathrm{Fix}(\mathcal{T}) \cap \bigcap_{i=1}^{N} \mathrm{EP}(\Phi_i)$.

(ii) For case (i), Corollary 3.2 guarantees the convergence of $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$.

(iii) For case (ii), the convergence of $\{\mathcal{Z}_n\}$, $\{\mathcal{Y}_n\}$, and $\{\mathcal{V}_n\}$ can be guaranteed by Theorem 3.1.

(iv) The iteration (5.9) for the combination of equilibrium problem converges faster than the iteration (5.10) for the classical equilibrium problem.

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