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The existence of optimal approximate solution theorems for generalized α-proximal contraction non-self mappings and applications

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Abstract

In this paper, we investigate the sufficient conditions to find a best proximity point for a certain class of non-self mappings. It is well known that optimization problems can be transformed to the problems of the existence of a best proximity point. Hence, improvement in the best proximity point theory implicitly develops the theory of optimization. Our presented results generalize, extent and improve various well-known results on the topic in the literature. In particular, we consider some applications of our results to the best proximity point theorems on a class of metric spaces endowed with an arbitrary binary relation which involves the partially ordered metric spaces.

MSC: 47H10; 54H25

Keywords: approximately compact; best proximity point; binary relation; generalized α-proximal contraction of the first kind; generalized α-proximal contraction of the second kind

1 Introduction and preliminaries

Various nonlinear problems arising in several branches of mathematics, besides some other quantitative sciences such as economics, biology, physics and engineering, can be transformed to a fixed point problem of the form $Tx = x$ for a self-mapping T defined on a subset of the metric space (X, d). Among them, optimization problems, differential equation problems, integral equation problems, variational problems, equilibrium problems have attracted attention of researchers. The renowned Banach contraction principle of Banach [1] is a very crucial and popular tool for solving fixed point problems in the setting of a self-mapping T. On the other hand, if T is not a self-mapping ($T : A \to B$ where A and B are non-empty subsets of a metric space X), then T does not necessarily have a fixed point. Therefore, the equation $Tx = x$ could have no solution. In this case, it is natural to search for a point $x \in A$ in a way that the distance between x and its image Tx is as small as possible. In this discussion, the best approximation theorem plays an important role in studying the existence of an approximate solution that is optimal for the equation $Tx = x$.

Let A, B be subsets of the metric space (X, d). Under the setting of $d(x, Tx) = d(A, B) :=$ $\inf\{d(x, y) : x \in A$ and $y \in B\}$, x is called a best proximity point. Notice that the point $x \in A$ is the global minimum of the error involved for an approximate solution of the equation

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$Tx = x$. Therefore, we conclude that the purpose of the existence theorems for a best proximity point is to provide sufficient conditions to solve a minimization problem. One of the initial and most interesting best approximation theorems in this direction was given by Fan [2]. Following this result, several authors have reported a number of results in this field; see, *e.g.*, [3–14] and the references therein.

The notion of approximate compactness, introduced by Efimov and Stechkin [15], plays an important role in the theory of approximation. The properties of approximately compact sets have been studied intensively. The following definition is a generalization of this concept.

Definition 1.1 Let A and B be two non-empty subsets of the metric space (X, d). Then B is said to be approximately compact with respect to A if every sequence $\{y_n\}$ of B, satisfying the condition $d(x, y_n) \to d(x, B)$ as $n \to \infty$ for some $x \in A$, has a convergent subsequence.

Remark 1.1 Any non-empty subset of the metric space (X, d) is approximately compact with respect to itself. Moreover, if B is compact, then B is approximately compact with respect to A.

On the other hand, Samet *et al.* [16] introduced the concept of α-admissible mapping as follows.

Definition 1.2 [16] Let X be a non-empty set, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is an α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \quad \implies \quad \alpha(Tx, Ty) \geq 1.$$

Using this concept, they proved some fixed point theorems. The authors also showed that these results can be utilized to derive fixed point theorems in partially ordered spaces. Moreover, they applied their main results to solve certain types of ordinary differential equations. Afterward, a number of papers have reported on fixed point theory with applications to ordinary differential equations via the concept of α-admissible mapping in various directions (see [17–22] and references therein).

Recently, Jleli and Samet [23] introduced the notion of α-proximal admissible mapping which is a non-self version of an α-admissible mapping.

Definition 1.3 [23] Let A and B be two non-empty subsets of the metric space (X, d) and $\alpha : A \times A \to [0, \infty)$. A mapping $T : A \to B$ is said to be α-proximal admissible if for all $u, v, x, y \in A$,

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \quad \implies \quad \alpha(u, v) \geq 1. \\ d(v, Ty) = d(A, B), \end{cases}$$

Remark 1.2 If $A = B$, then the notion of α-proximal admissibility T coincides with the concept of α-admissibility.

Inspired by [23], we introduce the class of non-self mappings, generalized α-proximal contraction mappings. Also, we establish the existence theorems for a best proximity point for mappings in this class under the assumptions of approximate compactness and non-approximate compactness of subspaces. Our presented results generalize, extent and improve various well-known results such as the results of Banach [1], Kannan [24], Chatterjea [25], Berinde [26]. As an application, we apply our results to the existence theorems for a best proximity point on a metric space endowed with an arbitrary binary relation. Furthermore, we give the special case of these results in partially ordered metric spaces.

2 Best proximity point theorems for an α-proximal contraction non-self mapping

In the sequel, unless otherwise specified, $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of positive integers, respectively.

Let A and B be two non-empty subsets of a metric space (X, d). The following notations will be used in the sequel:

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

For a non-self mapping $T : A \to B$, we define the set of all best proximity points of T by $\mathcal{B}_{est}(T)$, that is,

$$\mathcal{B}_{est}(T) := \{x \in A : d(x, Tx) = d(A, B)\}.$$

Remark 2.1 Kirk *et al.* [27] gave sufficient conditions to ensure that A_0 and B_0 are non-empty. Also, we obtain that if A and B are closed subsets of a normed linear space such that $d(A, B) > 0$, then A_0 and B_0 are contained in the boundaries of A and B, respectively (see [28]).

Now, we introduce new classes of generalized proximal contraction non-self mappings.

Definition 2.1 Let A and B be two non-empty subsets of a metric space (X, d) and $\alpha : A \times A \to [0, \infty)$. A mapping $T : A \to B$ is said to be a *generalized α-proximal contraction of the first kind* if there exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \implies \\ d(v, Ty) = d(A, B), \end{cases} \quad \begin{aligned} d(u, v) &\leq \theta_1 d(x, y) + \theta_2\big[d(x, u) + d(y, v)\big] \\ &\quad + \theta_3\big[d(x, v) + d(y, u)\big] \\ &\quad + L \min\{d(y, u), d(x, v), d(x, u), d(y, v)\}. \end{aligned}$$

Definition 2.2 Let A and B be two non-empty subsets of the metric space (X, d) and $\alpha : A \times A \to [0, \infty)$. A mapping $T : A \to B$ is said to be a *generalized α-proximal contraction of the second kind* if there exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 <$

1 such that for all $u, v, x, y \in A$,

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B), \end{cases} \implies \begin{aligned} d(Tu, Tv) &\leq \theta_1 d(Tx, Ty) + \theta_2 \big[d(Tx, Tu) + d(Ty, Tv) \big] \\ &\quad + \theta_3 \big[d(Tx, Tv) + d(Ty, Tu) \big] \\ &\quad + L \min \big\{ d(Ty, Tu), d(Tx, Tv), \\ &\qquad d(Tx, Tu), d(Ty, Tv) \big\}. \end{aligned}$$

Next, we give our first main result which is the best proximity point theorem for a generalized α-proximal contraction.

Theorem 2.1 *Let (X, d) be a complete metric space, and let A and B be two non-empty, closed subsets of X such that B is approximately compact with respect to A. Assume that $\alpha : A \times A \to [0, \infty)$, A_0 and B_0 are non-empty sets and $T : A \to B$ is a generalized α-proximal contraction of the first kind such that the following conditions hold:*

(a) *T is α-proximal admissible;*
(b) *$T(A_0) \subseteq B_0$;*
(c) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;*
(d) *If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has at least one best proximity point. Moreover, if $\alpha(z_1, z_2) \geq 1$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$, then T has a unique best proximity point.

Proof From (c), there exist $x_0, x_1 \in A$ such that

$$d(x_1, Tx_0) = d(A, B) \tag{2.1}$$

and

$$\alpha(x_0, x_1) \geq 1. \tag{2.2}$$

Since $T(A_0) \subseteq B_0$, then by the definition of B_0, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{2.3}$$

Since T is α-proximal admissible, by (2.1), (2.2) and (2.3), we have

$$\alpha(x_1, x_2) \geq 1. \tag{2.4}$$

Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B). \tag{2.5}$$

Using (2.3), (2.4), (2.5) and the assumption that T is α-proximal admissible, we get

$$\alpha(x_2, x_3) \geq 1. \tag{2.6}$$

Continuing this process, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_n, Tx_{n-1}) = d(A, B) \tag{2.7}$$

and

$$\alpha(x_{n-1}, x_n) \geq 1 \tag{2.8}$$

for all $n \in \mathbb{N}$. Since T is a generalized α-proximal contraction of the first kind, for each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \theta_1 d(x_{n-1}, x_n) + \theta_2\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big] + \theta_3\big[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\big] \\
&\quad + L \min\big\{d(x_n, x_n), d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\} \\
&\leq \theta_1 d(x_{n-1}, x_n) + \theta_2\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big] + \theta_3\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big].
\end{aligned}
$$

It follows that $d(x_n, x_{n+1}) \leq K d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, where $K := \frac{\theta_1 + \theta_2 + \theta_3}{1 - \theta_2 - \theta_3} \in [0, 1)$. By induction we get

$$d(x_n, x_{n+1}) \leq K^n d(x_0, x_1)$$

for all $n \in \mathbb{N}$. It easily follows that for $m, n \in \mathbb{N}$ such that $m < n$,

$$
\begin{aligned}
d(x_m, x_n) &\leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \\
&\leq \big(K^m + K^{m+1} + \cdots + K^{n-1}\big) d(x_0, x_1) \\
&\leq \frac{K^m}{1 - h} d(x_0, x_1).
\end{aligned}
$$

This implies that $\{x_n\}$ is a Cauchy sequence. Now, since X is complete and A is closed, the sequence $\{x_n\}$ converges to some $z \in A$. Further, we have

$$
\begin{aligned}
d(z, B) &\leq d(z, Tx_n) \\
&\leq d(z, x_{n+1}) + d(x_{n+1}, Tx_n) \\
&= d(z, x_{n+1}) + d(A, B) \\
&\leq d(z, x_{n+1}) + d(z, B)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Therefore, $d(z, Tx_n) \to d(z, B)$ as $n \to \infty$. Since B is approximately compact with respect to A, then the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ converging to some element $w \in B$. Therefore, $d(z, w) = \lim_{k \to \infty} d(x_{n_k+1}, Tx_{n_k}) = d(A, B)$, and hence z must be a member of A_0. Because of the fact that $T(A_0)$ is contained in B_0, $d(u, Tz) = d(A, B)$ for some element u in A.

From the hypothesis (d) and (2.8), we get

$$\alpha(x_n, z) \geq 1 \tag{2.9}$$

for all $n \in \mathbb{N}$. Now, we have

$$
\begin{cases}
\alpha(x_n, z) \geq 1, \\
d(x_{n+1}, Tx_n) = d(A, B), \\
d(u, Tz) = d(A, B),
\end{cases}
$$

for all $n \in \mathbb{N}$. Since T is a generalized α-proximal contraction of the first kind, we get

$$
d(x_{n+1}, u) \leq \theta_1 d(x_n, z) + \theta_2 \big[d(u, z) + d(x_{n+1}, x_n) \big] + \theta_3 \big[d(x_{n+1}, z) + d(u, x_n) \big]
$$
$$
+ L \min \big\{ d(u, x_n), d(x_{n+1}, z), d(u, z), d(x_{n+1}, x_n) \big\}
$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we get $z = u$. Thus, it follows that $d(z, Tz) = d(u, Tz) = d(A, B)$. Therefore, z is at least one of the best proximity points of T.

Finally, we prove the uniqueness of the best proximity point. Assume that $\tilde{z}$ is another best proximity point of T such that $\alpha(z, \tilde{z}) \geq 1$, then we have

$$
\begin{cases}
\alpha(z, \tilde{z}) \geq 1, \\
d(z, Tz) = d(A, B), \\
d(\tilde{z}, T\tilde{z}) = d(A, B).
\end{cases}
$$

Again, since T is a generalized α-proximal contraction of the first kind, we obtain that

$$
d(z, \tilde{z}) \leq \theta_1 d(z, \tilde{z}) + \theta_2 \big[d(z, z) + d(\tilde{z}, \tilde{z}) \big] + \theta_3 \big[d(z, \tilde{z}) + d(\tilde{z}, z) \big]
$$
$$
+ L \min \big\{ d(\tilde{z}, z), d(z, \tilde{z}), d(z, z), d(\tilde{z}, \tilde{z}) \big\},
$$

which implies that $d(z, \tilde{z}) = 0$, and then $z = \tilde{z}$. Hence T has a unique best proximity point. This completes the proof. □

Now we give some examples to support our result.

Example 2.2 Let $X = \mathbb{R}$ with the usual metric $d : X \times X \to [0, \infty)$ defined by $d(x, y) := |x - y|$. Clearly, (X, d) is a complete metric space. Also, let $A = (-\infty, -2]$ and $B = [\frac{1}{3}, 200]$ be two closed subsets of X. It is obtained that B is compact and so B is approximately compact with respect to A. Define a non-self mapping $T : A \to B$ by

$$
Tx = \begin{cases}
\frac{1}{2}, & x \in (-\infty, -5), \\
\frac{1}{3} - x^3, & x \in [-5, -3), \\
-\frac{1}{5}x - \frac{1}{15}, & x \in [-3, -2].
\end{cases}
$$

Clearly, $d(A, B) = \frac{7}{3}$ and

$$
A_0 = \left\{ x \in A : d(x, y) = d(A, B) = \frac{7}{3} \text{ for some } y \in B \right\} = \{-2\},
$$

$$
B_0 = \left\{ y \in B : d(x, y) = d(A, B) = \frac{7}{3} \text{ for some } x \in A \right\} = \left\{ \frac{1}{3} \right\}.
$$

It is easy to see that $T(A_0) \subseteq B_0$. Define $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} e^{|x-y|}, & x, y \in [-3, -2], \\ 0, & \text{otherwise.} \end{cases}$$

Now we show that T is a generalized α-proximal contraction of the first kind. Assume that $u, v, x, y \in A$ such that

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Then

$$\begin{cases} x, y \in [-3, -2], \\ d(u, Tx) = \frac{7}{3}, \\ d(v, Ty) = \frac{7}{3}. \end{cases}$$

Since $Tz \in [\frac{1}{3}, \frac{8}{15}]$ for all $z \in [-3, -2]$, we get $u = v = -2$. Therefore,

$$0 = d(u, v) \leq \theta_1 d(x, y) + \theta_2 [d(x, u) + d(y, v)] + \theta_3 [d(x, v) + d(y, u)]$$
$$+ L \min \{d(y, u), d(x, v), d(x, u), d(y, v)\},$$

where $\theta_1, \theta_2, \theta_3$ and L are non-negative real numbers with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$. This shows that T is a generalized α-proximal contraction of the first kind.

Next, we prove that T is α-proximal admissible. Assume that $u, v, x, y \in A$ such that

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Then we have $x, y \in [-3, -2]$ and $u = v = -2$. So $\alpha(u, v) = \alpha(-2, -2) = 1$ and thus T is α-proximal admissible.

It is easy to see that there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.

Assume that $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Therefore, $x_n \in [-3, -2]$ for all $n \in \mathbb{N}$. By the closeness of $[-3, -2]$, we get $x \in [-3, -2]$ and hence $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Consequently, all the hypotheses of Theorem 2.1 are satisfied and so T has at least one best proximity point, that is, a point -2 such that

$$d(-2, T(-2)) = d\left(-2, \frac{1}{3}\right) = \frac{7}{3} = d(A, B).$$

Example 2.3 Let $X = \mathbb{R}$ with the usual metric $d : X \times X \to [0, \infty)$ defined by $d(x, y) := |x - y|$. Clearly, (X, d) is a complete metric space. Also, let $A = [-1, 1] \cup \{5, 10\}$ and $B =$

$[-\frac{1}{3}, \frac{1}{3}] \cup \{5, 10\}$ be two closed subsets of X. It is obtained that B is compact and so B is approximately compact with respect to A. Define a non-self mapping $T : A \to B$ by

$$Tx = \begin{cases} \frac{1}{3}x, & x \in [-1, 1], \\ 0, & x = 5, \\ 5, & x = 10. \end{cases}$$

Clearly, $d(A, B) = 0$ and

$$A_0 = \{x \in A : d(x, y) = d(A, B) = 0 \text{ for some } y \in B\} = B,$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) = 0 \text{ for some } x \in A\} = B.$$

It is easy to see that $T(A_0) \subseteq B_0$. Define $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} x^2 + y^2 + \pi, & x, y \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Now we show that T is a generalized α-proximal contraction of the first kind with $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{10}$, $\theta_3 = \frac{1}{10}$ and $L = 2$. Assume that $u, v, x, y \in A$ such that

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Then

$$\begin{cases} x, y \in [-1, 1], \\ d(u, Tx) = 0, \\ d(v, Ty) = 0. \end{cases}$$

Hence, $u = Tx = \frac{1}{3}x$ and $v = Ty = \frac{1}{3}y$. Therefore,

$$\begin{aligned} d(u, v) &= d(Tx, Ty) \\ &= \frac{1}{3}|x - y| \\ &\leq \frac{1}{2}d(x, y) + \frac{1}{10}\big[d(x, u) + d(y, v)\big] + \frac{1}{10}\big[d(x, v) + d(y, u)\big] \\ &\quad + 2 \min\{d(y, u), d(x, v), d(x, u), d(y, v)\}. \end{aligned}$$

This shows that T is a generalized α-proximal contraction of the first kind.

Next, we prove that T is α-proximal admissible. Assume that $u, v, x, y \in A$ such that

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Then we have $x, y \in [-1, 1]$ and so

$$u = Tx \in \left[-\frac{1}{3}, \frac{1}{3} \right] \subseteq [-1, 1]$$

and

$$v = Ty \in \left[-\frac{1}{3}, \frac{1}{3} \right] \subseteq [-1, 1].$$

Thus $\alpha(u, v) = \alpha(Tx, Ty) \geq 1$. This implies that T is α-proximal admissible.

It is easy to see that there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.

Suppose that $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Therefore, $x_n \in [-1, 1]$ for all $n \in \mathbb{N}$. Since $[-1, 1]$ is closed, we get $x \in [-1, 1]$ and hence $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Consequently, all the hypotheses of Theorem 2.1 are satisfied and so T has at least one best proximity point, that is, a point 0 such that

$$d\big(0, T(0)\big) = d(A, B).$$

It is easy to see that Theorem 2.1 yields the following corollary.

Corollary 2.4 *Let (X, d) be a complete metric space, and let A and B be two non-empty, closed subsets of X such that B is approximately compact with respect to A. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ satisfies the following conditions:*

(a′) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{cases} \implies \begin{aligned} d(u, v) &\leq \theta_1 d(x, y) + \theta_2 \big[d(x, u) + d(y, v) \big] \\ &\quad + \theta_3 \big[d(x, v) + d(y, u) \big] \\ &\quad + L \min \big\{ d(y, u), d(x, v), d(x, u), d(y, v) \big\}; \end{aligned}$$

(b) *$T(A_0) \subseteq B_0$;*

(c) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$.*

Then there exists unique $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has a unique best proximity point.

Proof By taking $\alpha(x, y) = 1$ for all $x, y \in A$ in Theorem 2.1, we get this result. □

In Corollary 2.4, if T is a self-mapping, then we get the following fixed point theorem.

Corollary 2.5 *Let (X, d) be a complete metric space, A be a non-empty, closed subset of X and $T : A \to A$. If there exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that*

$$d(Tx, Ty) \leq \theta_1 d(x, y) + \theta_2 \big[d(x, Tx) + d(y, Ty) \big] + \theta_3 \big[d(x, Ty) + d(y, Tx) \big]$$

$$+ L \min \big\{ d(y, Tx), d(x, Tv), d(x, Tx), d(y, Ty) \big\}$$

for all $x, y \in A$, then T has a unique fixed point.

Remark 2.2

1. If $\theta_2 = \theta_3 = L = 0$ in Corollary 2.5, we get the generalized and improved result of Banach [1].
2. If $\theta_1 = \theta_3 = L = 0$ in Corollary 2.5, we get the generalized and improved result of Kannan [24].
3. If $\theta_1 = \theta_2 = L = 0$ in Corollary 2.5, we get the generalized and improved result of Chatterjea [25].
4. If $\theta_2 = \theta_3 = 0$ in Corollary 2.5, we get the generalized and improved result of Berinde [26].

Next, we give the existence theorem of a best proximity point for a generalized α-proximal contraction of the second kind.

Theorem 2.6 *Let (X, d) be a complete metric space, and let A and B be two non-empty, closed subsets of X such that A is approximately compact with respect to B. Assume that $\alpha : A \times A \to [0, \infty)$, A_0 and B_0 are non-empty sets and $T : A \to B$ is a continuous generalized α-proximal contraction of the second kind such that the following conditions hold:*

(a) *T is α-proximal admissible;*
(b) *$T(A_0) \subseteq B_0$;*
(c) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.*

Then there exists an element $z \in \mathcal{B}_{est}(T)$. Further, $Tz_1 = Tz_2$ whenever $z_1, z_2 \in \mathcal{B}_{est}(T)$ and $\alpha(z_1, z_2) \geq 1$.

Proof Following the arguments in Theorem 2.1, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_n, Tx_{n-1}) = d(A, B) \tag{2.10}$$

and

$$\alpha(x_{n-1}, x_n) \geq 1 \tag{2.11}$$

for all $n \in \mathbb{N}$. Since T is a generalized α-proximal contraction of the second kind, we have

$$
\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq \theta_1 d(Tx_{n-1}, Tx_n) + \theta_2 \big[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) \big] \\
&\quad + \theta_3 \big[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n) \big] \\
&\quad + L \min \big\{ d(Tx_n, Tx_n), d(Tx_{n-1}, Tx_{n+1}), \\
&\quad\quad d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}) \big\} \\
&\leq \theta_1 d(Tx_{n-1}, Tx_n) + \theta_2 \big[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) \big] \\
&\quad + \theta_3 \big[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) \big]
\end{aligned}
$$

for all $n \in \mathbb{N}$. It follows that $d(Tx_n, Tx_{n+1}) \leq K d(Tx_{n-1}, Tx_n)$ for all $n \in \mathbb{N}$, where $K := \frac{\theta_1 + \theta_2 + \theta_3}{1 - \theta_2 - \theta_3} < 1$. Similar to the proof in Theorem 2.1, we obtain that the sequence $\{Tx_n\}$ is a

Cauchy sequence in B. Since B is a closed subspace of the complete metric space X, B is complete. Then the sequence $\{Tx_n\}$ converges to some $\hat{y} \in B$. Further, we have

$$
\begin{aligned}
d(\hat{y}, A) &\leq d(\hat{y}, x_{n+1}) \\
&\leq d(\hat{y}, Tx_n) + d(Tx_n, x_{n+1}) \\
&= d(\hat{y}, Tx_n) + d(A, B) \\
&\leq d(\hat{y}, Tx_n) + d(\hat{y}, A)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Therefore, $d(\hat{y}, x_n) \to d(\hat{y}, A)$ as $n \to \infty$. Since A is approximately compact with respect to B, then the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some element $z \in A$. Now, using the continuity of T, we obtain that $d(z, Tz) = \lim_{k \to \infty} d(x_{n_k+1}, Tx_{n_k}) = d(A, B)$. Hence $z \in \mathcal{B}_{est}(T)$.

Finally, we may assume that $z_1, z_2 \in \mathcal{B}_{est}(T)$ and $\alpha(z_1, z_2) \geq 1$. Therefore, we get

$$
\begin{cases}
\alpha(z_1, z_2) \geq 1, \\
d(z_1, Tz_1) = d(A, B), \\
d(z_2, Tz_2) = d(A, B).
\end{cases}
$$

Since T is a generalized α-proximal contraction of the second kind, we have

$$
\begin{aligned}
d(Tz_1, Tz_2) &\leq \theta_1 d(Tz_1, Tz_2) + \theta_2 \big[d(Tz_1, Tz_1) + d(Tz_2, Tz_2) \big] \\
&\quad + \theta_3 \big[d(Tz_1, Tz_2) + d(Tz_2, Tz_1) \big] \\
&\quad + L \min \big\{ d(Tz_2, Tz_1), d(Tz_1, Tz_2), d(Tz_1, Tz_1), d(Tz_2, Tz_2) \big\},
\end{aligned}
$$

which implies that

$$
d(Tz_1, Tz_2) \leq \theta_1 + 2\theta_3 d(Tz_1, Tz_2).
$$

It follows from $\theta_1 + 2\theta_3 \in [0, 1)$ that $d(Tz_1, Tz_2) = 0$ and hence $Tz_1 = Tz_2$. This completes the proof. □

As a consequence of Theorem 2.6, we state the following corollary.

Corollary 2.7 *Let (X, d) be a complete metric space, and let A and B be two non-empty, closed subsets of X such that A is approximately compact with respect to B. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ is continuous such that the following conditions hold:*

(a″) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(u, v) &\leq \theta_1 d(Tx, Ty) + \theta_2 \big[d(Tx, Tu) + d(Ty, Tv) \big] \\
&\quad + \theta_3 \big[d(Tx, Tv) + d(Ty, Tu) \big] \\
&\quad + L \min \big\{ d(Ty, Tu), d(Tx, Tv), \\
&\qquad\qquad d(Tx, Tu), d(Ty, Tv) \big\};
\end{aligned}
$$

(b) $T(A_0) \subseteq B_0$;

(c)   *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$.*

Then there exists an element $z \in \mathcal{B}_{est}(T)$. Further, $Tz_1 = Tz_2$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$.

Proof By taking $\alpha(x, y) = 1$ for all $x, y \in A$ in Theorem 2.6, we get this result.   □

In Corollary 2.7, if T is a self-mapping, then we get the following fixed point theorem.

Corollary 2.8 *Let (X, d) be a complete metric space, A be a non-empty, closed subset of X, and let $T : A \to A$ be a continuous mapping. If there exist non-negative real numbers θ_1, θ_2, θ_3 and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that*

$$d(TTx, TTy) \leq \theta_1 d(Tx, Ty) + \theta_2 \big[d(Tx, TTx) + d(Ty, TTy) \big]$$
$$+ \theta_3 \big[d(Tx, TTy) + d(Ty, TTx) \big]$$
$$+ L \min \big\{ d(Ty, TTx), d(Tx, TTv), d(Tx, TTx), d(Ty, TTy) \big\}$$

for all $x, y \in A$, then T has a unique fixed point.

In the next theorem, we give conditions for the existence of a best proximity point for a non-self mapping that is a generalized α-proximal contraction of the first and second kinds. In this theorem, we consider only a completeness hypothesis without assuming the continuity of the non-self mapping and the approximate compactness of the subspace.

Theorem 2.9 *Let (X, d) be a complete metric space, and let A and B be two non-empty, closed subsets of X. Assume that $\alpha : A \times A \to [0, \infty)$, A_0 and B_0 are non-empty sets and $T : A \to B$ is a generalized α-proximal contraction of the first and second kinds such that the following conditions hold*:

(a)   *T is α-proximal admissible*;

(b)   *$T(A_0) \subseteq B_0$*;

(c)   *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$*;

(d)   *If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has at least one best proximity point. Moreover, if $\alpha(z_1, z_2) \geq 1$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$, then T has a unique best proximity point.

Proof Following the arguments in Theorem 2.1, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_n, Tx_{n-1}) = d(A, B) \tag{2.12}$$

and

$$\alpha(x_{n-1}, x_n) \geq 1 \tag{2.13}$$

for all $n \in \mathbb{N}$.

Also, using similar arguments as in the proof of Theorem 2.1, we conclude that the sequence $\{x_n\}$ is a Cauchy sequence, and hence converges to some $z \in A$. Moreover, on the lines of Theorem 2.6, we obtain that the sequence $\{Tx_n\}$ is a Cauchy sequence and hence converges to some $\hat{y} \in B$. Therefore, we have $d(z, \hat{y}) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = d(A, B)$, hence z must be in A_0. Since $T(A_0) \subseteq B_0$, then $d(\hat{u}, Tz) = d(A, B)$ for some $\hat{u} \in A$.

From the assumption (d) and (2.12), we have

$$\alpha(x_n, z) \geq 1 \tag{2.14}$$

for all $n \in \mathbb{N}$.

Now we have

$$\begin{cases} \alpha(x_n, z) \geq 1, \\ d(x_{n+1}, Tx_n) = d(A, B), \\ d(\hat{u}, Tz) = d(A, B), \end{cases}$$

for all $n \in \mathbb{N}$. Using the fact that T is a generalized α-proximal contraction of the first kind, we have

$$d(x_{n+1}, \hat{u}) \leq \theta_1 d(x_n, z) + \theta_2 \big[d(x_n, x_{n+1}) + d(z, \hat{u}) \big] + \theta_3 \big[d(x_n, \hat{u}) + d(z, x_{n+1}) \big]$$
$$+ L \min \big\{ d(z, x_{n+1}), d(x_n, \hat{u}), d(x_n, x_{n+1}), d(z, \hat{u}) \big\}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the previous inequality, we have

$$d(z, \hat{u}) \leq (\theta_2 + \theta_3) d(z, \hat{u}).$$

Since $\theta_2 + \theta_3 \in [0, 1)$, we get $d(z, \hat{u}) = 0$ and then $z = \hat{u}$. Thus, it follows that $d(z, Tz) = d(\hat{u}, Tz) = d(A, B)$. Therefore, $z \in \mathcal{B}_{est}(T)$.

For the uniqueness of a best proximity point of T, we proceed similarly to the proof of Theorem 2.1. Then, in order to avoid repetition, the details are omitted. $\square$

Corollary 2.10 *Let (X, d) be a complete metric space, and let A and B be two non-empty, closed subsets of X. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ satisfies the following conditions:*

(a′) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{cases} \implies \begin{aligned} d(u, v) &\leq \theta_1 d(x, y) + \theta_2 \big[d(x, u) + d(y, v) \big] \\ &\quad + \theta_3 \big[d(x, v) + d(y, u) \big] \\ &\quad + L \min \big\{ d(y, u), d(x, v), \\ &\qquad d(x, u), d(y, v) \big\}; \end{aligned}$$

(a″) *There exist non-negative real numbers $\overline{\theta}_1$, $\overline{\theta}_2$, $\overline{\theta}_3$ and $\overline{L}$ with $\overline{\theta}_1 + 2\overline{\theta}_2 + 2\overline{\theta}_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(u, v) &\le \overline{\theta}_1 d(Tx, Ty) + \overline{\theta}_2\big[d(Tx, Tu) + d(Ty, Tv)\big] \\
&\quad + \overline{\theta}_3\big[d(Tx, Tv) + d(Ty, Tu)\big] \\
&\quad + \overline{L}\min\big\{d(Ty, Tu), d(Tx, Tv), \\
&\qquad\quad d(Tx, Tu), d(Ty, Tv)\big\};
\end{aligned}
$$

(b) $T(A_0) \subseteq B_0$;

(c) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$.*

Then there exists a unique $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has a unique best proximity point.

Proof By taking $\alpha(x, y) = 1$ for all $x, y \in A$ in Theorem 2.9, we get this result. □

3 Best proximity point theorems on a metric space endowed with an arbitrary binary relation

In this section, we apply our results in the previous section (Theorems 2.1, 2.6 and 2.9) to the best proximity point theorems on a metric space endowed with an arbitrary binary relation. Moreover, we obtain a special case of these results as corollaries in partially ordered metric spaces.

Before presenting our results, we need a few preliminaries. Let (X, d) be a metric space and $\mathcal{R}$ be an arbitrary binary relation over X. Denote

$$
\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1};
$$

this is the symmetric relation attached to R. Clearly,

$$
x, y \in X, \quad x\mathcal{S}y \iff x\mathcal{R}y \quad \text{or} \quad y\mathcal{R}x.
$$

Definition 3.1 Let (X, d) be a metric space, let $\mathcal{R}$ be a binary relation over X, and let A and B be two non-empty subsets of X. We say that $T : A \to B$ is a proximal comparative mapping if

$$
\left.
\begin{array}{l}
x\mathcal{S}y, \\
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{array}
\right\}
\implies u\mathcal{S}v
$$

for all $u, v, x, y \in A$.

We have the following best proximity point result.

Theorem 3.1 *Let (X, d) be a complete metric space, let $\mathcal{R}$ be a binary relation over X, and let A and B be two non-empty, closed subsets of X such that B is approximately compact with respect to A. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ such that the following conditions hold:*

(A) *T is proximal comparative mapping*;

(A$'$) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
x \mathcal{S} y, \\
d(u, Tx) = d(A, B), \quad \Longrightarrow \\
d(v, Ty) = d(A, B),
\end{cases}
\begin{aligned}
d(u, v) &\leq \theta_1 d(x, y) + \theta_2 \big[d(x, u) + d(y, v) \big] \\
&\quad + \theta_3 \big[d(x, v) + d(y, u) \big] \\
&\quad + L \min \big\{ d(y, u), d(x, v), d(x, u), d(y, v) \big\};
\end{aligned}
$$

(B) $T(A_0) \subseteq B_0$;

(C) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \mathcal{S} x_1$*;

(D) *If $\{x_n\}$ is a sequence in A such that $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then $x_n \mathcal{S} x$ for all $n \in \mathbb{N}$.*

Then there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has at least one best proximity point. Moreover, if $z_1 \mathcal{S} z_2$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$, then T has a unique best proximity point.

Proof Define the mapping $\alpha : A \times A \to [0, \infty)$ by

$$
\alpha(x, y) = \begin{cases} 1, & x \mathcal{S} y, \\ 0, & \text{otherwise.} \end{cases}
\tag{3.1}
$$

Since T is a proximal comparative mapping, we have

$$
\left.
\begin{aligned}
&x \mathcal{S} y, \\
&d(u, Tx) = d(A, B), \\
&d(v, Ty) = d(A, B),
\end{aligned}
\right\} \quad \Longrightarrow \quad u \mathcal{S} v
$$

for all $u, v, x, y \in A$. By the definition of α, we obtain that

$$
\left.
\begin{aligned}
&\alpha(x, y) \geq 1, \\
&d(u, Tx) = d(A, B), \\
&d(v, Ty) = d(A, B),
\end{aligned}
\right\} \quad \Longrightarrow \quad \alpha(u, v) \geq 1
$$

for all $u, v, x, y \in A$. This shows that T is α-proximal admissible.

From (A$'$) and the construction of mapping α, we get

$$
\begin{cases}
\alpha(x, y) \geq 1, \\
d(u, Tx) = d(A, B), \quad \Longrightarrow \\
d(v, Ty) = d(A, B),
\end{cases}
\begin{aligned}
d(u, v) &\leq \theta_1 d(x, y) + \theta_2 \big[d(x, u) + d(y, v) \big] \\
&\quad + \theta_3 \big[d(x, v) + d(y, u) \big] \\
&\quad + L \min \big\{ d(y, u), d(x, v), d(x, u), d(y, v) \big\}
\end{aligned}
$$

for all $u, v, x, y \in A$, that is, T is a generalized α-proximal contraction of the first kind.

Since condition (C) implies that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$, condition (c) in Theorem 2.1 holds. It is easy to see that condition (D) implies condition (d) in Theorem 2.1.

Now, all the hypotheses of Theorem 2.1 are satisfied, and the desired result follows immediately from this theorem. $\qquad \square$

Theorem 3.2 *Let (X, d) be a complete metric space, let $\mathcal{R}$ be a binary relation over X, and let A and B be two non-empty, closed subsets of X such that A is approximately compact with respect to B. Assume that A_0 and B_0 are non-empty and $T : A \to B$ is continuous such that the following conditions hold:*

(A) *T is a proximal comparative mapping;*

(A″) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
x\mathcal{S}y, \\
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(Tu, Tv) &\le \theta_1 d(Tx, Ty) + \theta_2\big[d(Tx, Tu) + d(Ty, Tv)\big] \\
&\quad + \theta_3\big[d(Tx, Tv) + d(Ty, Tu)\big] \\
&\quad + L \min\big\{d(Ty, Tu), d(Tx, Tv), \\
&\qquad\quad d(Tx, Tu), d(Ty, Tv)\big\};
\end{aligned}
$$

(b) *$T(A_0) \subseteq B_0$;*

(c) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \mathcal{S} x_1$.*

Then there exists an element $z \in \mathcal{B}_{est}(T)$. Further, $Tz_1 = Tz_2$ whenever $z_1, z_2 \in \mathcal{B}_{est}(T)$ and $z_1 \mathcal{S} z_2$.

Proof The result follows from Theorem 2.6 by considering the mapping α as in the proof of Theorem 3.1. □

Theorem 3.3 *Let (X, d) be a complete metric space, let $\mathcal{R}$ be a binary relation over X, and let A and B be two non-empty, closed subsets of X. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ such that the following conditions hold:*

(A) *T is a proximal comparative mapping;*

(A′) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
x\mathcal{S}y, \\
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(u, v) &\le \theta_1 d(x, y) + \theta_2\big[d(x, u) + d(y, v)\big] \\
&\quad + \theta_3\big[d(x, v) + d(y, u)\big] \\
&\quad + L \min\big\{d(y, u), d(x, v), d(x, u), d(y, v)\big\};
\end{aligned}
$$

(A″) *There exist non-negative real numbers $\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3$ and L with $\overline{\theta}_1 + 2\overline{\theta}_2 + 2\overline{\theta}_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
x\mathcal{S}y, \\
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(Tu, Tv) &\le \overline{\theta}_1 d(Tx, Ty) + \overline{\theta}_2\big[d(Tx, Tu) + d(Ty, Tv)\big] \\
&\quad + \overline{\theta}_3\big[d(Tx, Tv) + d(Ty, Tu)\big] \\
&\quad + L \min\big\{d(Ty, Tu), d(Tx, Tv), \\
&\qquad\quad d(Tx, Tu), d(Ty, Tv)\big\};
\end{aligned}
$$

(B) *$T(A_0) \subseteq B_0$;*

(C) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \mathcal{S} x_1$;*

(D) *If $\{x_n\}$ is a sequence in A such that $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then $x_n \mathcal{S} x$ for all $n \in \mathbb{N}$.*

Then there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has at least one best proximity point. Moreover, if $z_1 \mathcal{S} z_2$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$, then T has a unique best proximity point.

Proof This result follows from Theorem 2.9 by considering the mapping α given in the proof of Theorem 3.1. □

Next, we deduce Theorems 3.1, 3.2 and 3.3 to the special case in the context of partially ordered metric spaces. Before studying the next results, we give the following definitions.

Definition 3.2 Let X be a non-empty set. Then $(X, d, \preceq)$ is called a partially ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.

For the partially ordered set $(X, \preceq)$, we define

$$\asymp := \big\{ (x, y) \in X \times X : x \preceq y \text{ or } y \preceq x \big\}.$$

Definition 3.3 Let $(X, d, \preceq)$ be a partially ordered metric space, and let A and B be two non-empty subsets of X. We say that $T : A \to B$ is a proximal comparative mapping with respect to $\preceq$ if

$$\left. \begin{array}{l} x \asymp y, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{array} \right\} \quad \Longrightarrow \quad u \asymp v$$

for all $u, v, x, y \in A$.

It is obtained that $\preceq$ is a binary operation on X and $\asymp = \preceq \cup \preceq^{-1}$. Therefore, we get the following best proximity point results in a partially ordered metric space.

Corollary 3.4 *Let $(X, d, \preceq)$ be a complete partially ordered metric space, and let A and B be two non-empty, closed subsets of X such that B is approximately compact with respect to A. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ such that the following conditions hold:*

(A) *T is proximal comparative with respect to $\preceq$;*

(A′) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$\begin{cases} x \asymp y, & d(u, v) \le \theta_1 d(x, y) + \theta_2 \big[d(x, u) + d(y, v) \big] \\ d(u, Tx) = d(A, B), \quad \Longrightarrow & \quad + \theta_3 \big[d(x, v) + d(y, u) \big] \\ d(v, Ty) = d(A, B), & \quad + L \min \big\{ d(y, u), d(x, v), d(x, u), d(y, v) \big\}; \end{cases}$$

(B) *$T(A_0) \subseteq B_0$;*

(C) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \asymp x_1$;*

(D) *If $\{x_n\}$ is a sequence in A such that $x_n \asymp x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then $x_n \asymp x$ for all $n \in \mathbb{N}$.*

Then there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has at least one best proximity point. Moreover, if $z_1 \asymp z_2$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$, then T has a unique best proximity point.

Corollary 3.5 *Let $(X, d, \preceq)$ be a complete partially ordered metric space, and let A and B be two non-empty, closed subsets of X such that A is approximately compact with respect to B. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ is continuous such that the following conditions hold*:

(A)   *T is a proximal comparative mapping with respect to $\preceq$*;
(A″) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
x \asymp y, \\
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(Tu, Tv) \leq{} & \theta_1 d(Tx, Ty) + \theta_2\big[d(Tx, Tu) + d(Ty, Tv)\big] \\
& + \theta_3\big[d(Tx, Tv) + d(Ty, Tu)\big] \\
& + L \min\big\{d(Ty, Tu), d(Tx, Tv), \\
& \quad d(Tx, Tu), d(Ty, Tv)\big\};
\end{aligned}
$$

(B)   *$T(A_0) \subseteq B_0$*;
(C)   *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \asymp x_1$.*

Then there exists an element $z \in \mathcal{B}_{est}(T)$. Further, $Tz_1 = Tz_2$ whenever $z_1, z_2 \in \mathcal{B}_{est}(T)$ and $z_1 \asymp z_2$.

Corollary 3.6 *Let $(X, d, \preceq)$ be a complete partially ordered metric space, and let A and B be two non-empty, closed subsets of X. Assume that A_0 and B_0 are non-empty sets and $T : A \to B$ such that the following conditions hold*:

(A)   *T is a proximal comparative mapping with respect to $\preceq$*;
(A′) *There exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
x \asymp y, \\
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(u, v) \leq{} & \theta_1 d(x, y) + \theta_2\big[d(x, u) + d(y, v)\big] \\
& + \theta_3\big[d(x, v) + d(y, u)\big] \\
& + L \min\big\{d(y, u), d(x, v), d(x, u), d(y, v)\big\};
\end{aligned}
$$

(A″) *There exist non-negative real numbers $\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3$ and L with $\overline{\theta}_1 + 2\overline{\theta}_2 + 2\overline{\theta}_3 < 1$ such that for all $u, v, x, y \in A$,*

$$
\begin{cases}
x \asymp y, \\
d(u, Tx) = d(A, B), \\
d(v, Ty) = d(A, B),
\end{cases}
\implies
\begin{aligned}
d(Tu, Tv) \leq{} & \overline{\theta}_1 d(Tx, Ty) + \overline{\theta}_2\big[d(Tx, Tu) + d(Ty, Tv)\big] \\
& + \overline{\theta}_3\big[d(Tx, Tv) + d(Ty, Tu)\big] \\
& + L \min\big\{d(Ty, Tu), d(Tx, Tv), \\
& \quad d(Tx, Tu), d(Ty, Tv)\big\};
\end{aligned}
$$

(B)   *$T(A_0) \subseteq B_0$*;
(C)   *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \asymp x_1$;*

(D) *If $\{x_n\}$ is a sequence in A such that $x_n \asymp x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then $x_n \asymp x$ for all $n \in \mathbb{N}$.*

Then there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has at least one best proximity point. Moreover, if $z_1 \asymp z_2$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$, then T has a unique best proximity point.

4 Remarks and conclusions

Recently, Salimi *et al.* [21] presented some fixed point results by modifying $\alpha - \psi$ contraction types in [16] and [29].

Definition 4.1 [21] Let $G : X \to X$ be a mapping on a metric space (X, d), and let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. We say that G is an α-admissible mapping with respect to η if for $x, y \in X$, we have

$$\alpha(x, y) \geq \eta(x, y) \quad \Rightarrow \quad \alpha(Gx, Gy) \geq \eta(Gx, Gy).$$

Inspired by Salimi *et al.* [21] one can suggest the following definitions.

Definition 4.2 Let A and B be two non-empty subsets of a metric space (X, d) and $\alpha, \beta : A \times A \to [0, \infty)$. A mapping $T : A \to B$ is said to be an (α, β)-*proximal contraction of the first kind* if there exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,

$$\begin{cases} \alpha(x, y) \geq \beta(x, y), & d(u, v) \leq \theta_1 d(x, y) + \theta_2 [d(x, u) + d(y, v)] \\ d(u, Tx) = d(A, B), \quad \Longrightarrow & \quad + \theta_3 [d(x, v) + d(y, u)] \\ d(v, Ty) = d(A, B), & \quad + L \min \{d(y, u), d(x, v), d(x, u), d(y, v)\}. \end{cases}$$

Definition 4.3 Let A and B be two non-empty subsets of a metric space (X, d) and $\alpha, \beta : A \times A \to [0, \infty)$. A mapping $T : A \to B$ is said to be an (α, β)-*proximal contraction of the second kind* if there exist non-negative real numbers $\theta_1, \theta_2, \theta_3$ and L with $\theta_1 + 2\theta_2 + 2\theta_3 < 1$ such that for all $u, v, x, y \in A$,

$$\begin{cases} \alpha(x, y) \geq \beta(x, y), & d(Tu, Tv) \leq \theta_1 d(Tx, Ty) + \theta_2 [d(Tx, Tu) + d(Ty, Tv)] \\ d(u, Tx) = d(A, B), \quad \Longrightarrow & \quad + \theta_3 [d(Tx, Tv) + d(Ty, Tu)] \\ d(v, Ty) = d(A, B), & \quad + L \min \{d(Ty, Tu), d(Tx, Tv), \\ & \quad d(Tx, Tu), d(Ty, Tv)\}. \end{cases}$$

By replacing an (α, β)-*proximal contraction of the first kind* with an α-*proximal contraction of the first kind* (that is, by replacing Definition 4.2 with Definition 2.1) in the statements of Theorem 2.1, one can get the following theorem.

Theorem 4.1 *Let (X, d) be a complete metric space, and let A and B be two non-empty, closed subsets of X such that B is approximately compact with respect to A. Assume that $\alpha, \beta : A \times A \to [0, \infty)$, A_0 and B_0 are non-empty sets and $T : A \to B$ is an (α, β)-proximal contraction of the first kind such that the following conditions hold:*

(a) *T is α-proximal admissible with respect to β;*

(b) *$T(A_0) \subseteq B_0$;*

(c) *There exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \beta(x_0, x_1)$;*

(d) *If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then $\alpha(x_n, x) \geq \beta(x_n, x)$ for all $n \in \mathbb{N}$.*

Then there exists an element $z \in A_0$ such that $d(z, Tz) = d(A, B)$, that is, T has at least one best proximity point. Moreover, if $\alpha(z_1, z_2) \geq \beta(z_1, z_2)$ for all $z_1, z_2 \in \mathcal{B}_{est}(T)$, then T has a unique best proximity point.

In this case, the proof is the analog of the proof of Theorem 2.1. In the same setting, the analogues of Theorem 2.6, Theorem 2.9, Theorem 3.1, Theorem 3.2, Theorem 3.3 can be obtained easily.

Very recently, Berzig and Karapınar [30] proved that the first main result of Salimi *et al.* [21] follows from a result of Karapınar and Samet [29]. Thus, Definition 4.2 and Definition 4.3 can be considered as a consequence of Definition 2.1 and Definition 2.2.

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