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System of variational inequalities and an accretive operator in Banach spaces

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Abstract

In this paper, we introduce composite Mann iteration methods for a general system of variational inequalities with solutions being also common fixed points of a countable family of nonexpansive mappings and zeros of an accretive operator in real smooth Banach spaces. Here, the composite Mann iteration methods are based on Korpelevich's extragradient method, viscosity approximation method and the Mann iteration method. We first consider and analyze a composite Mann iterative algorithm in the setting of uniformly convex and 2-uniformly smooth Banach space, and then another composite Mann iterative algorithm in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Under suitable assumptions, we derive some strong convergence theorems. The results presented in this paper improve, extend, supplement and develop the corresponding results announced in the earlier and very recent literature.

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Keywords: composite Mann iteration methods; general system of variational inequalities; accretive operator; nonexpansive mapping; sunny nonexpansive retraction; fixed point; uniformly Gâteaux differentiable norm; uniform smoothness; uniform convexity

1 Introduction

Let X be a real Banach space whose dual space is denoted by X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \Rightarrow \quad \|x + y\|/2 \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strict convex. A Banach space X is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$; in this case, X is also said to have a Gâteaux differentiable norm. X is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for $x \in U$. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of X is said to be the Frechet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. Let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. A mapping $A : C \to X$ is said to be accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that $\langle Ax - Ay, j(x - y)\rangle \geq 0$.

Recently, Yao *et al.* [1] combined the viscosity approximation method and the Mann iteration method, and gave the following hybrid viscosity approximation method:

Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X, $T : C \to C$ a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$ and $f \in \Xi_C$ with a contractive coefficient $\rho \in (0,1)$, where Ξ_C is the set of all contractive self-mappings on C. For an arbitrary $x_0 \in C$, define $\{x_n\}$ in the following way:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{YCY}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. They proved under certain control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of T. Subsequently, Ceng and Yao [2] under the convergence of no parameter sequences to zero proved that the sequence $\{x_n\}$ generated by (YCY) converges strongly to a fixed point of T. Such a result includes [1, Theorem 1] as a special case.

Theorem 1.1 (See [2, Theorem 3.1]) *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *and $f \in \Xi_C$ with contractive coefficient $\rho \in (0,1)$. Given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in* $[0,1]$, *the following control conditions are satisfied*:

(i) $1 \leq \beta_n \leq 1 - \rho$, $\forall n \geq n_0$ *for some integer* $n_0 \geq 1$;
(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iv) $\lim_{n\to\infty}(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}) = 0$.

For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by (YCY). *Then*

$$x_n \to q \quad \Longleftrightarrow \quad \beta_n\bigl(f(x_n) - x_n\bigr) \to 0,$$

where $q \in$ $\mathrm{Fix}(T)$ *solves the variational inequality problem* (*VIP*):

$$\bigl\langle q - f(q), J(q - p)\bigr\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T).$$

On the other hand, Cai and Bu [3] considered the following general system of variational inequalities (GSVI) in a real smooth Banach space X, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.1}$$

where C is a nonempty, closed and convex subset of X, $B_1, B_2 : C \to X$ are two nonlinear mappings and μ_1 and μ_2 are two positive constants. Here, the set of solutions of GSVI (1.1) is denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, if $X = H$ in a real Hilbert space, then GSVI (1.1) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

which μ_1 and μ_2 are two positive constants. The set of solutions of problem (1.2) is still denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, if $B_1 = B_2 = A$, then problem (1.2) reduces to the new system of variational inequalities (NSVI), introduced and studied by Verma [4]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

The solution set of VIP (1.3) is denoted by $\mathrm{VI}(C, A)$. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. It is now well known that the variational inequalities are equivalent to the fixed point problems, the origin of which can be traced back to Lions and Stampacchia [5]. This alternative formulation has been used to suggest and analyze projection iterative method for solving variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous.

Recently, Ceng *et al.* [6] transformed problem (1.2) into a fixed point problem in the following way.

Lemma 1.1 (See [6]) *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of problem* (1.2) *if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\big[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)\big], \quad \forall x \in C, \tag{1.4}$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$ and P_C is the projection of H onto C.

In particular, if the mappings $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$.

In 1976, Korpelevich [7] proposed an iterative algorithm for solving the VIP (1.3) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n), \quad n \geq 0, \end{cases}$$

with $\tau > 0$ a given number, which is known as the extragradient method. The literature on the VIP is vast, and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see, *e.g.*, [3, 8–14] and the references therein, to name but a few.

In particular, whenever X is still a real smooth Banach space, $B_1 = B_2 = A$ and $x^* = y^*$, then GSVI (1.1) reduces to the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C, \tag{1.5}$$

which was considered by Aoyama *et al.* [15]. Note that VIP (1.5) is connected with the fixed point problem for nonlinear mapping (see, *e.g.*, [16, 17]), the problem of finding a zero point of a nonlinear operator (see, *e.g.*, [18]) and so on. It is clear that VIP (1.5) extends VIP (1.3) from Hilbert spaces to Banach spaces.

In order to find a solution of VIP (1.5), Aoyama *et al.* [15] introduced the following Mann iterative scheme for an accretive operator A:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\Pi_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{1.6}$$

where Π_C is a sunny nonexpansive retraction from X onto C. Then they proved a weak convergence theorem.

Obviously, it is an interesting and valuable problem of constructing some algorithms with strong convergence for solving GSVI (1.1), which contains VIP (1.5) as a special case. Very recently, Cai and Bu [3] constructed an iterative algorithm for solving GSVI (1.1) and a common fixed point problem of a countable family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space.

Theorem 1.2 (See [3, Theorem 3.1]) *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be β_i-inverse-strongly accretive with $0 < \mu_i < \frac{\beta_i}{\kappa^2}$ for $i = 1, 2$. Let f be a contraction of C into itself with coefficient $\delta \in (0, 1)$. Let $\{T_n\}_{n=1}^\infty$ be a countable family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^\infty \operatorname{Fix}(T_i) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ on C. For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_{n+1} = \beta_n x_n + (1 - \beta_n) T_n y_n, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) z_n, \\ z_n = \Pi_C(u_n - \mu_1 B_1 u_n), \\ u_n = \Pi_C(x_n - \mu_2 B_2 x_n), \quad \forall n \geq 1. \end{cases}$$

Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\sum_{n=1}^\infty \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C, and let T be a mapping of C into X defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $\operatorname{Fix}(T) = \bigcap_{n=1}^\infty \operatorname{Fix}(T_n)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \leq 0, \quad \forall p \in F.$$

Furthermore, recall that a (possibly multivalued) operator $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ in a real Banach space X is accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists a $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \ge 0$. (Here J is the duality mapping.) An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$. An accretive operator A is m-accretive if $R(I + rA) = X$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I + rA) \to D(A)$ defined by $J_r = (I + rA)^{-1}$, which is called the resolvent of A. We know that J_r is nonexpansive and $\mathrm{Fix}(J_r) = A^{-1}0$ for all $r > 0$. Hence,

$$\mathrm{Fix}(J_r) = A^{-1}0 = \{z \in D(A) : 0 \in Az\}.$$

If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Az$ is solvable. The following resolvent identity is well known to us; see [19], where more details on accretive operators can be found.

Proposition 1.1 (Resolvent identity) *For $\lambda > 0$, $\mu > 0$ and $x \in X$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right).$$

Recently, Aoyama *et al.* [20] studied the following iterative scheme in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm: for resolvents J_{r_n} of an accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and $\{\alpha_n\} \subset (0, 1)$

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{r_n}x_n. \end{cases} \tag{1.7}$$

They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a zero of A under appropriate assumptions on $\{\alpha_n\}$ and $\{r_n\}$. Subsequently, Ceng *et al.* [21] introduced and analyzed the following composite iterative scheme in either a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping

$$\begin{cases} x_0 = x \in X, \\ y_n = \alpha_n u + (1 - \alpha_n)J_{r_n}x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n J_{r_n}y_n, \end{cases} \tag{1.8}$$

where $u \in \overline{D(A)}$ is an arbitrary (but fixed) element, under the following control conditions:

(H1) $\lim_{n\to\infty} \alpha_n = 0$;

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, or, equivalently, $\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$;

(H3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(H4) $r_n \ge \varepsilon$, $\forall n \ge 0$, for some $\varepsilon > 0$ and $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;

(H5) $\beta_n \in [0, a)$ for some $a \in (0, 1)$ and $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Further, as the viscosity approximation method, Jung [22] purposed and analyzed the following composite iterative scheme for finding a zero of an accretive operator A: for resolvent J_{r_n} of an accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$, $f \in \Xi_C$ (Ξ_C denotes the set of all contractions on C) and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$,

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n J_{r_n} y_n. \end{cases} \tag{JS}$$

Theorem 1.3 (See [22, Theorem 3.1]) *Let X be a strictly convex and reflexive Banach space having a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of X and $A \subset X \times X$ an accretive operator in X such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *which satisfy the conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \in [0, a)$ *for some* $0 < a < 1$ *for all* $n \geq 0$.

Let $f \in \Xi_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by (JS) *for $r_n > 0$. If $\{x_n\}$ is asymptotically regular, i.e.,* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, *then $\{x_n\}$ converges strongly to $q \in A^{-1}0$, which is the unique solution of the variational inequality problem* (*VIP*)

$$\langle (I - f)q, J(q - p) \rangle \leq 0, \quad \forall f \in \Xi_C, p \in A^{-1}0.$$

Let C be a nonempty closed convex subset of a real smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Motivated and inspired by the research going on in this area, we introduce the composite Mann iteration methods for finding solutions of GSVI (1.1), which are also common fixed points of a countable family of nonexpansive mappings and zeros of an accretive operator $A \subset X \times X$ such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Here, the composite Mann iteration methods are based on Korpelevich's extragradient method, viscosity approximation method and the Mann iteration method. We first consider and analyze a composite Mann iterative algorithm in the setting of uniformly convex and 2-uniformly smooth Banach space, and then another composite Mann iterative algorithm in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Under suitable assumptions, we derive some strong convergence theorems. The results presented in this paper improve, extend, supplement and develop the corresponding results announced in the earlier and very recent literature; see, *e.g.*, [2, 3, 6, 8, 22].

2 Preliminaries

Let X be a real Banach space. We define a function $\rho : [0,\infty) \to [0,\infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup\left\{ \frac{1}{2}\big(\|x + y\| + \|x - y\|\big) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau\to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [23], no Banach space is q-uniformly smooth for $q > 2$. In addition, it is also known that J is single-valued if and only if X is smooth, whereas if X is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded subsets of X. If X has a uniformly Gâteaux differentiable norm, then the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X. We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence.

Let C be a nonempty closed convex subset of X. Recall that a mapping $A : C \to X$ is said to be

(i) α-strongly accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|x-y\|^2$$

for some $\alpha \in (0,1)$;

(ii) β-inverse-strongly-accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq \beta \|Ax - Ay\|^2$$

for some $\beta > 0$;

(iii) λ-strictly pseudocontractive [24] if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \leq \|x-y\|^2 - \lambda \bigl\|x - y - (Ax - Ay)\bigr\|^2$$

for some $\lambda \in (0,1)$.

It is worth emphasizing that the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping, which was studied by so many authors; see, *e.g.*, [9, 25, 26].

Proposition 2.1 (See [27]) *Let X be a 2-uniformly smooth Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + 2\|\kappa y\|^2, \quad \forall x, y \in X,$$

where κ is the 2-uniformly smooth constant of X, and J is the normalized duality mapping from X into X^*.

Proposition 2.2 (See [28]) *Let X be a real smooth and uniform convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g\bigl(\|x-y\|\bigr) \leq \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \quad \forall x, y \in B_r,$$

where $B_r = \{x \in X : \|x\| \leq r\}$.

Next, we list some lemmas that will be used in the sequel. Lemma 2.1 can be found in [29]. Lemma 2.2 is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2.1 *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \le 0$;

(iii) $\gamma_n \ge 0$, $\forall n \ge 0$, *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\limsup_{n\to\infty} s_n = 0$.

Lemma 2.2 *In a real smooth Banach space X, the following inequality holds*:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad \forall x,y \in X.$$

Let D be a subset of C, and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi\bigl[\Pi(x) + t\bigl(x - \Pi(x)\bigr)\bigr] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \ge 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.3 (See [30]) *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let D be a nonempty subset of C. Let Π be a retraction of C onto D. Then the following are equivalent*:

(i) *Π is sunny and nonexpansive*;

(ii) $\|\Pi(x) - \Pi(y)\|^2 \le \langle x - y, J(\Pi(x) - \Pi(y))\rangle$, $\forall x,y \in C$;

(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \le 0$, $\forall x \in C$, $y \in D$.

It is well known that if $X = H$ in a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X, and if $T : C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

Lemma 2.4 *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let $B_1, B_2 : C \to X$ be nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI* (1.1) *if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof We can rewrite GSVI (1.1) as

$$\begin{cases} \langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*)\rangle \ge 0, & \forall x \in C, \\ \langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*)\rangle \ge 0, & \forall x \in C, \end{cases}$$

which is obviously equivalent to

$$\begin{cases} x^* = \Pi_C(y^* - \mu_1 B_1 y^*), \\ y^* = \Pi_C(x^* - \mu_2 B_2 x^*), \end{cases}$$

because of Lemma 2.3. This completes the proof. □

In terms of Lemma 2.4, we observe that

$$x^* = \Pi_C\bigl[\Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr) - \mu_1 B_1 \Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr],$$

which implies that x^* is a fixed point of the mapping G. Throughout this paper, the set of fixed points of the mapping G is denoted by Ω.

Lemma 2.5 (See [27]) *Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\bigl\|\lambda x + (1-\lambda)y\bigr\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g\bigl(\|x-y\|\bigr)$$

for all $\lambda \in [0,1]$ and $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$.

Lemma 2.6 (See [31]) *Let C be a nonempty closed convex subset of a Banach space X. Let $S_0, S_1, \ldots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1}x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.*

Let C be a nonempty closed convex subset of a Banach space X, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. As previously, let Ξ_C be the set of all contractions on C. For $t \in (0,1)$ and $f \in \Xi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$ on C; that is,

$$x_t = tf(x_t) + (1-t)Tx_t.$$

Lemma 2.7 (See [17, 32]) *Let X be a uniformly smooth Banach space, or a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of X, let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Xi_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to a point in $\mathrm{Fix}(T)$. If we define a mapping $Q : \Xi_C \to \mathrm{Fix}(T)$ by $Q(f) := s - \lim_{t\to 0} x_t$, $\forall f \in \Xi_C$, then $Q(f)$ solves the VIP:*

$$\bigl\langle (I-f)Q(f), J\bigl(Q(f)-p\bigr)\bigr\rangle \le 0, \quad \forall f \in \Xi_C, p \in \mathrm{Fix}(T).$$

Lemma 2.8 (See [33]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that*

$\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ *is nonempty. Let* $\{\lambda_n\}$ *be a sequence of positive numbers with* $\sum_{n=0}^{\infty} \lambda_n = 1$. *Then a mapping* S *on* C *defined by* $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ *for* $x \in C$ *is defined well, nonexpansive and* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ *holds.*

Lemma 2.9 (See [15]) *Let* C *be a nonempty closed convex subset of a smooth Banach space* X. *Let* Π_C *be a sunny nonexpansive retraction from* X *onto* C, *and let* $A : C \to X$ *be an accretive mapping. Then for all* $\lambda > 0$,

$$\mathrm{VI}(C, A) = \mathrm{Fix}\big(\Pi_C(I - \lambda A)\big).$$

Lemma 2.10 (See [34]) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X, *and let* $\{\beta_n\}$ *be a sequence of nonnegative numbers in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then* $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.11 (See [35]) *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$. *Then there exists a continuous, strictly increasing and convex function* $g : [0, \infty] \to [0, \infty]$, $g(0) = 0$ *such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \le \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta g\big(\|x - y\|\big)$$

for all $x, y, z \in B_r$ *and all* $\alpha, \beta, \gamma \in [0, 1]$ *with* $\alpha + \beta + \gamma = 1$.

3 Composite Mann iterative algorithms in uniformly convex and 2-uniformly smooth Banach spaces

In this section, we introduce our composite Mann iterative algorithms in uniformly convex and 2-uniformly smooth Banach spaces and show the strong convergence theorems. We will use some useful lemmas in the sequel.

Lemma 3.1 (See [3, Lemma 2.8]) *Let* C *be a nonempty closed convex subset of a real* 2-*uniformly smooth Banach space* X. *Let the mapping* $B_i : C \to X$ *be* α_i-*inverse-strongly accretive. Then we have*

$$\big\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\big\|^2 \le \|x - y\|^2 + 2\mu_i\big(\mu_i \kappa^2 - \alpha_i\big)\|B_i x - B_i y\|^2, \quad \forall x, y \in C,$$

for $i = 1, 2$, *where* $\mu_i > 0$. *In particular, if* $0 < \mu_i \le \frac{\alpha_i}{\kappa^2}$, *then* $I - \mu_i B_i$ *is nonexpansive for* $i = 1, 2$.

Lemma 3.2 (See [3, Lemma 2.9]) *Let* C *be a nonempty closed convex subset of a real* 2-uniformly *smooth Banach space* X. *Let* Π_C *be a sunny nonexpansive retraction from* X *onto* C. *Let the mapping* $B_i : C \to X$ *be* α_i-*inverse-strongly accretive for* $i = 1, 2$. *Let* $G : C \to C$ *be the mapping defined by*

$$Gx = \Pi_C\big[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)\big], \quad \forall x \in C.$$

If $0 < \mu_i \le \frac{\alpha_i}{\kappa^2}$ *for* $i = 1, 2$, *then* $G : C \to C$ *is nonexpansive.*

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $A \subset X \times X$ be an accretive operator in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $B_i : C \to X$ be α_i-inverse strongly accretive for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \operatorname{Fix}(S_i) \cap \Omega \cap A^{-1}0 \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $0 \leq \alpha_n \leq 1 - \rho$, $\forall n \geq n_0$ *for some integer* $n_0 \geq 0$;
(ii) $\liminf_{n\to\infty} \gamma_n > 0$ *and* $\liminf_{n\to\infty} \delta_n > 0$;
(iii) $\lim_{n\to\infty}(|\frac{\alpha_{n+1}}{1-(1-\alpha_{n+1})\beta_{n+1}} - \frac{\alpha_n}{1-(1-\alpha_n)\beta_n}| + |\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}|) = 0$;
(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n \geq \varepsilon > 0$ *for all* $n \geq 0$;
(v) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that $\operatorname{Fix}(S) = \bigcap_{i=0}^{\infty} \operatorname{Fix}(S_i)$. *Then,*

$$x_n \to q \quad \Longleftrightarrow \quad \alpha_n\bigl(f(x_n) - x_n\bigr) \to 0,$$

where $q \in F$ solves the following VIP:

$$\bigl\langle q - f(q), J(q - p)\bigr\rangle \leq 0, \quad \forall p \in F.$$

Proof First of all, let us show that the sequence $\{x_n\}$ is bounded. Indeed, take a fixed $p \in F$ arbitrarily. Then we get $p = Gp$, $p = S_n p$ and $p = J_{r_n} p$ for all $n \geq 0$. By Lemma 3.2, we know that G is nonexpansive. Then from (3.1), we have

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|x_n - p\| + \gamma_n \|S_n x_n - p\| + \delta_n \|J_{r_n} G x_n - p\| \\ &\leq \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|G x_n - p\| \\ &\leq \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|x_n - p\| \\ &= \|x_n - p\|, \end{aligned} \tag{3.2}$$

and hence

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \bigl\|f(x_n) - p\bigr\| + (1 - \alpha_n)\|y_n - p\| \\ &\leq \alpha_n \bigl(\bigl\|f(x_n) - f(p)\bigr\| + \bigl\|f(p) - p\bigr\|\bigr) + (1 - \alpha_n)\|y_n - p\| \\ &\leq \alpha_n \bigl(\rho\|x_n - p\| + \bigl\|f(p) - p\bigr\|\bigr) + (1 - \alpha_n)\|x_n - p\| \end{aligned}$$

$$
\begin{aligned}
&= \big(1-\alpha_n(1-\rho)\big)\|x_n-p\| + \alpha_n(1-\rho)\frac{\|f(p)-p\|}{1-\rho} \\
&\le \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-\rho}\right\}.
\end{aligned}
$$

By induction, we obtain

$$
\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\rho}\right\}, \quad \forall n \ge 0. \tag{3.3}
$$

Thus, $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{Gx_n\}$ and $\{f(x_n)\}$.

Let us show that

$$
\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0. \tag{3.4}
$$

As a matter of fact, put $\sigma_n = (1-\alpha_n)\beta_n$, $\forall n \ge 0$. Then it follows from (i) and (v) that

$$
\beta_n \ge \sigma_n = (1-\alpha_n)\beta_n \ge \big(1-(1-\rho)\big)\beta_n = \rho\beta_n, \quad \forall n \ge n_0,
$$

and hence

$$
0 < \liminf_{n\to\infty} \sigma_n \le \liminf_{n\to\infty} \sigma_n < 1. \tag{3.5}
$$

Define

$$
x_{n+1} = \sigma_n x_n + (1-\sigma_n) z_n. \tag{3.6}
$$

Observe that

$$
\begin{aligned}
&z_{n+1} - z_n \\
&\quad = \frac{x_{n+2}-\sigma_{n+1}x_{n+1}}{1-\sigma_{n+1}} - \frac{x_{n+1}-\sigma_n x_n}{1-\sigma_n} \\
&\quad = \frac{\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1})y_{n+1} - \sigma_{n+1}x_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n) + (1-\alpha_n)y_n - \sigma_n x_n}{1-\sigma_n} \\
&\quad = \left(\frac{\alpha_{n+1}f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}\right) - \frac{(1-\alpha_n)[\beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n] - \sigma_n x_n}{1-\sigma_n} \\
&\qquad + \frac{(1-\alpha_{n+1})[\beta_{n+1}x_{n+1} + \gamma_{n+1}S_{n+1}x_{n+1} + \delta_{n+1}J_{r_{n+1}}Gx_{n+1}] - \sigma_{n+1}x_{n+1}}{1-\sigma_{n+1}} \\
&\quad = \left(\frac{\alpha_{n+1}f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}\right) + \frac{1-\alpha_{n+1}}{1-\sigma_{n+1}}(\gamma_{n+1}S_{n+1}x_{n+1} + \delta_{n+1}J_{r_{n+1}}Gx_{n+1}) \\
&\qquad - \frac{1-\alpha_n}{1-\sigma_n}(\gamma_n S_n x_n + \delta_n J_{r_n} G x_n) \\
&\quad = \left(\frac{\alpha_{n+1}f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}\right) \\
&\qquad + \frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{1-\sigma_{n+1}}\left[\frac{\gamma_{n+1}S_{n+1}x_{n+1} + \delta_{n+1}J_{r_{n+1}}Gx_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{1-\beta_n}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{1-\sigma_{n+1}}-\frac{(1-\alpha_n)(1-\beta_n)}{1-\sigma_n}\right]\frac{\gamma_n S_n x_n+\delta_n J_{r_n}Gx_n}{1-\beta_n}\\
=&\frac{\alpha_{n+1}}{1-\sigma_{n+1}}\bigl(f(x_{n+1})-f(x_n)\bigr)+\left(\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right)f(x_n)\\
&+\frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{1-\sigma_{n+1}}\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(S_{n+1}x_{n+1}-S_nx_n)\right.\\
&+\left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right)S_nx_n\\
&\left.+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(J_{r_{n+1}}Gx_{n+1}-J_{r_n}Gx_n)+\left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right)J_{r_n}Gx_n\right]\\
&-\left(\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right)\frac{\gamma_n S_n x_n+\delta_n J_{r_n}Gx_n}{\gamma_n+\delta_n}\\
=&\frac{\alpha_{n+1}}{1-\sigma_{n+1}}\bigl(f(x_{n+1})-f(x_n)\bigr)+\left(\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right)\left(f(x_n)-\frac{\gamma_n S_n x_n+\delta_n J_{r_n}Gx_n}{\gamma_n+\delta_n}\right)\\
&+\frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(S_{n+1}x_{n+1}-S_nx_n)+\left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right)S_nx_n\right.\\
&\left.+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(J_{r_{n+1}}Gx_{n+1}-J_{r_n}Gx_n)+\left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right)J_{r_n}Gx_n\right],
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|z_{n+1}-z_n\|\\
&\quad\le\frac{\alpha_{n+1}}{1-\sigma_{n+1}}\bigl\|f(x_{n+1})-f(x_n)\bigr\|+\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right|\left\|f(x_n)-\frac{\gamma_n S_n x_n+\delta_n J_{r_n}Gx_n}{\gamma_n+\delta_n}\right\|\\
&\qquad+\frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}\left\|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(S_{n+1}x_{n+1}-S_nx_n)+\left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right)S_nx_n\right.\\
&\qquad\left.+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(J_{r_{n+1}}Gx_{n+1}-J_{r_n}Gx_n)+\left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right)J_{r_n}Gx_n\right\|\\
&\quad\le\frac{\rho\alpha_{n+1}}{1-\sigma_{n+1}}\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right|\bigl(\bigl\|f(x_n)\bigr\|+\|S_nx_n\|+\|J_{r_n}Gx_n\|\bigr)\\
&\qquad+\frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\|S_{n+1}x_{n+1}-S_nx_n\|+\left|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right|\|S_nx_n\|\right.\\
&\qquad\left.+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\|J_{r_{n+1}}Gx_{n+1}-J_{r_n}Gx_n\|+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|\|J_{r_n}Gx_n\|\right].
\end{aligned}
\tag{3.7}
$$

On the other hand, if $r_n\le r_{n+1}$, using the resolvent identity in Proposition 1.1,

$$
J_{r_{n+1}}Gx_{n+1}=J_{r_n}\left(\frac{r_n}{r_{n+1}}Gx_{n+1}+\left(1-\frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}Gx_{n+1}\right),
$$

we get

$$
\begin{aligned}
\|J_{r_{n+1}}Gx_{n+1}-J_{r_n}Gx_n\|&=\left\|J_{r_n}\left(\frac{r_n}{r_{n+1}}Gx_{n+1}+\left(1-\frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}Gx_{n+1}\right)-J_{r_n}Gx_n\right\|\\
&\le\frac{r_n}{r_{n+1}}\|Gx_{n+1}-Gx_n\|+\left(1-\frac{r_n}{r_{n+1}}\right)\|J_{r_{n+1}}Gx_{n+1}-Gx_n\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \|x_{n+1}-x_n\| + \frac{r_{n+1}-r_n}{r_{n+1}}\|J_{r_{n+1}}Gx_{n+1}-Gx_n\| \\
&\le \|x_{n+1}-x_n\| + \frac{1}{\varepsilon}|r_{n+1}-r_n|\|J_{r_{n+1}}Gx_{n+1}-Gx_n\|.
\end{aligned}
$$

If $r_{n+1} \le r_n$, we derive in the similar way

$$
\|J_{r_{n+1}}Gx_{n+1}-J_{r_n}Gx_n\| \le \|x_n-x_{n+1}\| + \frac{1}{\varepsilon}|r_n-r_{n+1}|\|J_{r_n}Gx_n-Gx_{n+1}\|.
$$

Thus, combining the above cases, we obtain

$$
\|J_{r_{n+1}}Gx_{n+1}-J_{r_n}Gx_n\| \le \|x_n-x_{n+1}\| + M_0|r_n-r_{n+1}|, \quad \forall n \ge 0, \tag{3.8}
$$

where $\sup_{n\ge 0}\{\frac{1}{\varepsilon}(\|J_{r_{n+1}}Gx_{n+1}-Gx_n\| + \|J_{r_n}Gx_n-Gx_{n+1}\|)\} \le M_0$ for some $M_0 > 0$. Substituting (3.8) for (3.7), we have

$$
\begin{aligned}
&\|z_{n+1}-z_n\| \\
&\quad\le \frac{\rho\alpha_{n+1}}{1-\sigma_{n+1}}\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right|\bigl(\|f(x_n)\| + \|S_nx_n\| + \|J_{r_n}Gx_n\|\bigr) \\
&\qquad + \frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}\biggl[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\bigl(\|S_{n+1}x_{n+1}-S_{n+1}x_n\| + \|S_{n+1}x_n-S_nx_n\|\bigr) \\
&\qquad + \left|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\gamma_n}{\gamma_n+\delta_n}\right|\|S_nx_n\| + \frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\bigl(\|x_n-x_{n+1}\| + M_0|r_n-r_{n+1}|\bigr) \\
&\qquad + \left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right|\|J_{r_n}Gx_n\|\biggr] \\
&\quad\le \frac{\rho\alpha_{n+1}}{1-\sigma_{n+1}}\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right|\bigl(\|f(x_n)\| + \|S_nx_n\| + \|J_{r_n}Gx_n\|\bigr) \\
&\qquad + \frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}\biggl[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\bigl(\|x_{n+1}-x_n\| + \|S_{n+1}x_n-S_nx_n\|\bigr) \\
&\qquad + \frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\bigl(\|x_n-x_{n+1}\| + M_0|r_n-r_{n+1}|\bigr) \\
&\qquad + \left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right|\bigl(\|S_nx_n\| + \|J_{r_n}Gx_n\|\bigr)\biggr] \\
&\quad= \frac{1-\sigma_{n+1}-\alpha_{n+1}(1-\rho)}{1-\sigma_{n+1}}\|x_{n+1}-x_n\| \\
&\qquad + \left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right|\bigl(\|f(x_n)\| + \|S_nx_n\| + \|J_{r_n}Gx_n\|\bigr) \\
&\qquad + \frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}\biggl[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\|S_{n+1}x_n-S_nx_n\| + \frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}M_0|r_n-r_{n+1}| \\
&\qquad + \left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right|\bigl(\|S_nx_n\| + \|J_{r_n}Gx_n\|\bigr)\biggr] \\
&\quad\le \|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right|M + \|S_{n+1}x_n-S_nx_n\| + M|r_n-r_{n+1}| \\
&\qquad + \left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right|M
\end{aligned}
$$

$$= \|x_{n+1} - x_n\| + M\left(\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right| + \left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right| + |r_{n+1} - r_n|\right) + \|S_{n+1}x_n - S_nx_n\|,$$

which hence yields

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\quad\le M\left(\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right| + \left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right| + |r_{n+1} - r_n|\right) \\ &\qquad + \|S_{n+1}x_n - S_nx_n\|, \end{aligned} \tag{3.9}$$

where $\sup_{n\ge0}\{\|f(x_n)\| + \|S_nx_n\| + \|J_{r_n}Gx_n\| + M_0\} \le M$ for some $M > 0$. So, from (3.9), conditions (iii), (iv) and the assumption on $\{S_n\}$, it follows that

$$\limsup_{n\to\infty}\bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\bigr) \le 0.$$

Consequently, by Lemma 2.10, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.10}$$

It follows from (3.5) and (3.6) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\sigma_n)\|z_n - x_n\| = 0. \tag{3.11}$$

From (3.1), we have

$$x_{n+1} - x_n = \alpha_n\bigl(f(x_n) - x_n\bigr) + (1-\alpha_n)(y_n - x_n),$$

which hence implies that

$$\begin{aligned} \rho\|y_n - x_n\| &= \bigl(1-(1-\rho)\bigr)\|y_n - x_n\| \le (1-\alpha_n)\|y_n - x_n\| \\ &= \bigl\|x_{n+1} - x_n - \alpha_n\bigl(f(x_n) - x_n\bigr)\bigr\| \\ &\le \|x_{n+1} - x_n\| + \bigl\|\alpha_n\bigl(f(x_n) - x_n\bigr)\bigr\|. \end{aligned}$$

Since $x_{n+1} - x_n \to 0$ and $\alpha_n(f(x_n) - x_n) \to 0$, we get

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.12}$$

Next, we show that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$.

Indeed, for simplicity, put $q = \Pi_C(p - \mu_2B_2p)$, $u_n = \Pi_C(x_n - \mu_2B_2x_n)$ and $v_n = \Pi_C(u_n - \mu_1B_1u_n)$. Then $v_n = Gx_n$ for all $n \ge 0$. From Lemma 3.1, we have

$$\begin{aligned} \|u_n - q\|^2 &= \bigl\|\Pi_C(x_n - \mu_2B_2x_n) - \Pi_C(p - \mu_2B_2p)\bigr\|^2 \le \bigl\|x_n - p - \mu_2(B_2x_n - B_2p)\bigr\|^2 \\ &\le \|x_n - p\|^2 - 2\mu_2\bigl(\alpha_2 - \kappa^2\mu_2\bigr)\|B_2x_n - B_2p\|^2, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\|v_n - p\|^2 &= \big\|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q)\big\|^2 \\ &\le \big\|u_n - q - \mu_1(B_1 u_n - B_1 q)\big\|^2 \\ &\le \|u_n - q\|^2 - 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2. \end{aligned}\tag{3.14}$$

Substituting (3.13) for (3.14), we obtain

$$\begin{aligned}\|v_n - p\|^2 \le{}& \|x_n - p\|^2 - 2\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 \\ &- 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2. \end{aligned}\tag{3.15}$$

From (3.1) and (3.15), we have

$$\begin{aligned}&\|y_n - p\|^2 \\ &\quad\le \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + \delta_n\|J_{r_n} G x_n - p\|^2 \\ &\quad\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|v_n - p\|^2 \\ &\quad\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\big[\|x_n - p\|^2 \\ &\qquad - 2\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 - 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big] \\ &\quad= \|x_n - p\|^2 - 2\delta_n\big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 \\ &\qquad + 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big], \end{aligned}\tag{3.16}$$

which hence implies that

$$\begin{aligned}&2\delta_n\big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 + \mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big] \\ &\quad\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad\le \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|. \end{aligned}\tag{3.17}$$

Since $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$, and $\{x_n\}$ is bounded, we obtain from (3.12), (3.17) and condition (ii) that

$$\lim_{n\to\infty}\|B_2 x_n - B_2 p\| = 0 \quad\text{and}\quad \lim_{n\to\infty}\|B_1 u_n - B_1 q\| = 0.\tag{3.18}$$

Utilizing Proposition 2.2 and Lemma 2.3, we have

$$\begin{aligned}\|u_n - q\|^2 &= \big\|\Pi_C(x_n - \mu_2 B_2 x_n) - \Pi_C(p - \mu_2 B_2 p)\big\|^2 \\ &\le \big\langle x_n - \mu_2 B_2 x_n - (p - \mu_2 B_2 p), J(u_n - q)\big\rangle \\ &= \big\langle x_n - p, J(u_n - q)\big\rangle + \mu_2\big\langle B_2 p - B_2 x_n, J(u_n - q)\big\rangle \\ &\le \frac{1}{2}\big[\|x_n - p\|^2 + \|u_n - q\|^2 - g_1\big(\big\|x_n - u_n - (p - q)\big\|\big)\big] \\ &\qquad + \mu_2\|B_2 p - B_2 x_n\|\|u_n - q\|, \end{aligned}$$

which implies that

$$\|u_n - q\|^2 \le \|x_n - p\|^2 - g_1\bigl(\|x_n - u_n - (p-q)\|\bigr) + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\|. \tag{3.19}$$

In the same way, we derive

$$\begin{aligned}
\|v_n - p\|^2 &= \|\Pi_C(u_n - \mu_1B_1u_n) - \Pi_C(q - \mu_1B_1q)\|^2 \\
&\le \langle u_n - \mu_1B_1u_n - (q - \mu_1B_1q), J(v_n - p)\rangle \\
&= \langle u_n - q, J(v_n - p)\rangle + \mu_1\langle B_1q - B_1u_n, J(v_n - p)\rangle \\
&\le \frac{1}{2}\bigl[\|u_n - q\|^2 + \|v_n - p\|^2 - g_2\bigl(\|u_n - v_n + (p-q)\|\bigr)\bigr] \\
&\quad + \mu_1\|B_1q - B_1u_n\|\|v_n - p\|,
\end{aligned}$$

which implies that

$$\|v_n - p\|^2 \le \|u_n - q\|^2 - g_2\bigl(\|u_n - v_n + (p-q)\|\bigr) + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|. \tag{3.20}$$

Substituting (3.19) for (3.20), we get

$$\begin{aligned}
\|v_n - p\|^2 &\le \|x_n - p\|^2 - g_1\bigl(\|x_n - u_n - (p-q)\|\bigr) - g_2\bigl(\|u_n - v_n + (p-q)\|\bigr) \\
&\quad + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|.
\end{aligned} \tag{3.21}$$

By Lemma 2.2, we have from (3.16) and (3.21)

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|v_n - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\bigl[\|x_n - p\|^2 - g_1\bigl(\|x_n - u_n - (p-q)\|\bigr) \\
&\qquad - g_2\bigl(\|u_n - v_n + (p-q)\|\bigr) + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| \\
&\qquad + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|\bigr] \\
&\quad \le \|x_n - p\|^2 - \delta_n\bigl[g_1\bigl(\|x_n - u_n - (p-q)\|\bigr) + g_2\bigl(\|u_n - v_n + (p-q)\|\bigr)\bigr] \\
&\qquad + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|,
\end{aligned}$$

which hence leads to

$$\begin{aligned}
&\delta_n\bigl[g_1\bigl(\|x_n - u_n - (p-q)\|\bigr) + g_2\bigl(\|u_n - v_n + (p-q)\|\bigr)\bigr] \\
&\quad \le \|x_n - p\|^2 - \|y_n - p\|^2 + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\| \\
&\quad \le \bigl(\|x_n - p\| + \|y_n - p\|\bigr)\|x_n - y_n\| + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| \\
&\qquad + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|.
\end{aligned} \tag{3.22}$$

From (3.18), (3.22), condition (ii) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$, we deduce that

$$\lim_{n\to\infty} g_1\bigl(\bigl\|x_n - u_n - (p-q)\bigr\|\bigr) = 0 \quad \text{and} \quad \lim_{n\to\infty} g_2\bigl(\bigl\|u_n - v_n + (p-q)\bigr\|\bigr) = 0.$$

Utilizing the properties of g_1 and g_2, we deduce that

$$\lim_{n\to\infty} \bigl\|x_n - u_n - (p-q)\bigr\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \bigl\|u_n - v_n + (p-q)\bigr\| = 0. \tag{3.23}$$

From (3.23), we get

$$\|x_n - v_n\| \le \bigl\|x_n - u_n - (p-q)\bigr\| + \bigl\|u_n - v_n + (p-q)\bigr\| \to 0 \quad \text{as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{3.24}$$

Next, let us show that

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|J_{r_n} x_n - x_n\| = 0.$$

Indeed, utilizing Lemma 2.5 and (3.1), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \left\| \delta_n (J_{r_n} G x_n - p) + (\beta_n + \gamma_n)\left(\frac{\beta_n x_n + \gamma_n S_n x_n}{\beta_n + \gamma_n} - p\right)\right\|^2 \\
&\le \delta_n \|J_{r_n} G x_n - p\|^2 + (\beta_n + \gamma_n)\left\|\frac{\beta_n x_n + \gamma_n S_n x_n}{\beta_n + \gamma_n} - p\right\|^2 \\
&= \delta_n \|J_{r_n} G x_n - p\|^2 + (\beta_n + \gamma_n)\left\|\frac{\beta_n}{\beta_n + \gamma_n}(x_n - p) + \frac{\gamma_n}{\beta_n + \gamma_n}(S_n x_n - p)\right\|^2 \\
&\le \delta_n \|G x_n - p\|^2 + (\beta_n + \gamma_n)\left[\frac{\beta_n}{\beta_n + \gamma_n}\|x_n - p\|^2 + \frac{\gamma_n}{\beta_n + \gamma_n}\|S_n x_n - p\|^2 \right. \\
&\quad \left. - \frac{\beta_n \gamma_n}{(\beta_n + \gamma_n)^2} g_3\bigl(\|x_n - S_n x_n\|\bigr)\right] \\
&\le \delta_n \|x_n - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 - \frac{\beta_n \gamma_n}{\beta_n + \gamma_n} g_3\bigl(\|x_n - S_n x_n\|\bigr) \\
&= \|x_n - p\|^2 - \frac{\beta_n \gamma_n}{\beta_n + \gamma_n} g_3\bigl(\|x_n - S_n x_n\|\bigr),
\end{aligned}$$

which immediately implies that

$$\begin{aligned}
\beta_n \gamma_n g_3\bigl(\|x_n - S_n x_n\|\bigr) &\le \frac{\beta_n \gamma_n}{\beta_n + \gamma_n} g_3\bigl(\|x_n - S_n x_n\|\bigr) \\
&\le \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\le \bigl(\|x_n - p\| + \|y_n - p\|\bigr)\|x_n - y_n\|.
\end{aligned}$$

So, from (3.12), the boundedness of $\{x_n\}$, $\{y_n\}$ and conditions (ii), (v), it follows that

$$\lim_{n\to\infty} g_3\bigl(\|x_n - S_n x_n\|\bigr) = 0.$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{3.25}$$

Taking into account that

$$y_n - x_n = \gamma_n(S_n x_n - x_n) + \delta_n(J_{r_n} G x_n - x_n),$$

we have

$$\begin{aligned}
\delta_n \|J_{r_n} G x_n - x_n\| &= \bigl\|y_n - x_n - \gamma_n(S_n x_n - x_n)\bigr\| \\
&\le \|y_n - x_n\| + \gamma_n \|S_n x_n - x_n\| \\
&\le \|y_n - x_n\| + \|S_n x_n - x_n\|.
\end{aligned}$$

From (3.12), (3.25) and condition (ii), it follows that

$$\lim_{n\to\infty} \|J_{r_n} G x_n - x_n\| = 0. \tag{3.26}$$

Note that

$$\|x_n - S x_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - S x_n\|.$$

So, in terms of (3.25) and Lemma 2.6, we have

$$\lim_{n\to\infty} \|x_n - S x_n\| = 0. \tag{3.27}$$

Also, note that

$$\begin{aligned}
\|x_n - J_{r_n} x_n\| &\le \|x_n - J_{r_n} G x_n\| + \|J_{r_n} G x_n - J_{r_n} x_n\| \\
&\le \|x_n - J_{r_n} G x_n\| + \|G x_n - x_n\|.
\end{aligned}$$

From (3.24) and (3.26), we have

$$\lim_{n\to\infty} \|x_n - J_{r_n} x_n\| = 0. \tag{3.28}$$

Furthermore, we claim that $\lim_{n\to\infty} \|x_n - J_r x_n\| = 0$ for a fixed number r such that $\varepsilon > r > 0$. In fact, taking into account the resolvent identity in Proposition 1.1, we have

$$\begin{aligned}
\|J_{r_n} x_n - J_r x_n\| &= \left\| J_r\left(\frac{r}{r_n} x_n + \left(1 - \frac{r}{r_n}\right) J_{r_n} x_n\right) - J_r x_n \right\| \\
&\le \left(1 - \frac{r}{r_n}\right) \|x_n - J_{r_n} x_n\| \\
&\le \|x_n - J_{r_n} x_n\|.
\end{aligned} \tag{3.29}$$

Thus, we get from (3.28) and (3.29)

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - J_{r_n} x_n\| + \|J_{r_n} x_n - J_r x_n\| \\ &\le \|x_n - J_{r_n} x_n\| + \|x_n - J_{r_n} x_n\| \\ &= 2\|x_n - J_{r_n} x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

That is,

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0. \tag{3.30}$$

Define a mapping $Wx = (1-\theta_1-\theta_2)J_r x + \theta_1 Sx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then by Lemma 2.8, we have that $\mathrm{Fix}(W) = \mathrm{Fix}(J_r) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned}\|x_n - Wx_n\| &= \big\|(1-\theta_1-\theta_2)(x_n - J_r x_n) + \theta_1(x_n - Sx_n) + \theta_2(x_n - Gx_n)\big\| \\ &\le (1-\theta_1-\theta_2)\|x_n - J_r x_n\| + \theta_1\|x_n - Sx_n\| + \theta_2\|x_n - Gx_n\|.\end{aligned}$$

From (3.24), (3.27) and (3.30), we obtain

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{3.31}$$

Now, we claim that

$$\limsup_{n\to\infty} \big\langle f(q) - q, J(x_n - q)\big\rangle \le 0, \tag{3.32}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus, we have

$$x_t - x_n = (1-t)(Wx_t - x_n) + t\big(f(x_t) - x_n\big).$$

By Lemma 2.2, we conclude that

$$\begin{aligned}&\|x_t - x_n\|^2 \\ &\quad = \big\|(1-t)(Wx_t - x_n) + t\big(f(x_t) - x_n\big)\big\|^2 \\ &\quad \le (1-t)^2\|Wx_t - x_n\|^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\ &\quad \le (1-t)^2\big(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|\big)^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\ &\quad \le (1-t)^2\big(\|x_t - x_n\| + \|Wx_n - x_n\|\big)^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\ &\quad = (1-t)^2\big[\|x_t - x_n\|^2 + 2\|x_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2\big] \\ &\qquad + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\big\langle x_t - x_n, J(x_t - x_n)\big\rangle\end{aligned}$$

$$\begin{aligned}&= \left(1-2t+t^2\right)\|x_t-x_n\|^2 + f_n(t)\\&\quad + 2t\left\langle f(x_t)-x_t, J(x_t-x_n)\right\rangle + 2t\|x_t-x_n\|^2,\end{aligned} \tag{3.33}$$

where

$$f_n(t) = (1-t)^2\left(2\|x_t-x_n\| + \|x_n-Wx_n\|\right)\|x_n-Wx_n\| \to 0, \quad \text{as } n\to\infty. \tag{3.34}$$

It follows from (3.33) that

$$\left\langle x_t - f(x_t), J(x_t-x_n)\right\rangle \le \frac{t}{2}\|x_t-x_n\|^2 + \frac{1}{2t}f_n(t). \tag{3.35}$$

Letting $n\to\infty$ in (3.35) and noticing (3.34), we derive

$$\limsup_{n\to\infty}\left\langle x_t - f(x_t), J(x_t-x_n)\right\rangle \le \frac{t}{2}M_2, \tag{3.36}$$

where $M_2 > 0$ is a constant such that $\|x_t-x_n\|^2 \le M_2$ for all $t\in(0,1)$ and $n\ge 0$. Taking $t\to 0$ in (3.36), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\left\langle x_t - f(x_t), J(x_t-x_n)\right\rangle \le 0.$$

On the other hand, we have

$$\begin{aligned}&\left\langle f(q)-q, J(x_n-q)\right\rangle\\&\quad = \left\langle f(q)-q, J(x_n-q)\right\rangle - \left\langle f(q)-q, J(x_n-x_t)\right\rangle + \left\langle f(q)-q, J(x_n-x_t)\right\rangle\\&\qquad - \left\langle f(q)-x_t, J(x_n-x_t)\right\rangle + \left\langle f(q)-x_t, J(x_n-x_t)\right\rangle - \left\langle f(x_t)-x_t, J(x_n-x_t)\right\rangle\\&\qquad + \left\langle f(x_t)-x_t, J(x_n-x_t)\right\rangle\\&\quad = \left\langle f(q)-q, J(x_n-q)-J(x_n-x_t)\right\rangle + \left\langle x_t-q, J(x_n-x_t)\right\rangle\\&\qquad + \left\langle f(q)-f(x_t), J(x_n-x_t)\right\rangle + \left\langle f(x_t)-x_t, J(x_n-x_t)\right\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\left\langle f(q)-q, J(x_n-q)\right\rangle &\le \limsup_{n\to\infty}\left\langle f(q)-q, J(x_n-q)-J(x_n-x_t)\right\rangle\\&\quad + \|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\| + \rho\|q-x_t\|\limsup_{n\to\infty}\|x_n-x_t\|\\&\quad + \limsup_{n\to\infty}\left\langle f(x_t)-x_t, J(x_n-x_t)\right\rangle.\end{aligned}$$

Taking into account that $x_t\to q$ as $t\to 0$, we have

$$\begin{aligned}\limsup_{n\to\infty}\left\langle f(q)-q, J(x_n-q)\right\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\left\langle f(q)-q, J(x_n-q)\right\rangle\\&\le \limsup_{t\to 0}\limsup_{n\to\infty}\left\langle f(q)-q, J(x_n-q)-J(x_n-x_t)\right\rangle.\end{aligned} \tag{3.37}$$

Since X has a uniformly Frechet differentiable norm, the duality mapping J is norm-to-norm uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable, and hence (3.32) holds. From (3.4), we get $(x_{n+1}-q)-(x_n-q)\to 0$. Noticing the norm-to-norm uniform continuity of J on bounded subsets of X, we deduce from (3.32) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(q)-q,J(x_{n+1}-q)\rangle\\&\quad=\limsup_{n\to\infty}\bigl(\langle f(q)-q,J(x_n-q)\rangle+\langle f(q)-q,J(x_{n+1}-q)-J(x_n-q)\rangle\bigr)\\&\quad=\limsup_{n\to\infty}\langle f(q)-q,J(x_n-q)\rangle\le 0.\end{aligned}\tag{3.38}$$

Finally, let us show that $x_n\to q$ as $n\to\infty$. Utilizing Lemma 2.2, from (3.1) and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}\|y_n-q\|^2&\le\beta_n\|x_n-q\|^2+\gamma_n\|S_nx_n-q\|^2+\delta_n\|J_{r_n}Gx_n-q\|^2\\&\le\beta_n\|x_n-q\|^2+\gamma_n\|x_n-q\|^2+\delta_n\|x_n-q\|^2\\&=\|x_n-q\|^2,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-q\|^2&=\bigl\|\alpha_n\bigl(f(x_n)-f(q)\bigr)+(1-\alpha_n)(y_n-q)+\alpha_n\bigl(f(q)-q\bigr)\bigr\|^2\\&\le\bigl\|\alpha_n\bigl(f(x_n)-f(q)\bigr)+(1-\alpha_n)(y_n-q)\bigr\|^2+2\alpha_n\langle f(q)-q,J(x_{n+1}-q)\rangle\\&\le\alpha_n\bigl\|f(x_n)-f(q)\bigr\|^2+(1-\alpha_n)\|y_n-q\|^2+2\alpha_n\langle f(q)-q,J(x_{n+1}-q)\rangle\\&\le\alpha_n\rho\|x_n-q\|^2+(1-\alpha_n)\|x_n-q\|^2+2\alpha_n\langle f(q)-q,J(x_{n+1}-q)\rangle\\&=\bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-q\|^2+2\alpha_n\langle f(q)-q,J(x_{n+1}-q)\rangle\\&=\bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-q\|^2+\alpha_n(1-\rho)\frac{2\langle f(q)-q,J(x_{n+1}-q)\rangle}{1-\rho}.\end{aligned}\tag{3.39}$$

Applying Lemma 2.1 to (3.39), we obtain that $x_n\to q$ as $n\to\infty$. This completes the proof. □

Corollary 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $A\subset X\times X$ be an accretive operator in X such that $\overline{D(A)}\subset C\subset\bigcap_{r>0}R(I+rA)$. Let $V:C\to C$ be an α-strictly pseudocontractive mapping. Let $f:C\to C$ be a contraction with coefficient $\rho\in(0,1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F=\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(V)\cap A^{-1}0\ne\emptyset$. For arbitrarily given $x_0\in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases}y_n=\beta_nx_n+\gamma_nS_nx_n+\delta_nJ_{r_n}((1-l)I+lV)x_n,\\x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)y_n,\quad\forall n\ge 0,\end{cases}\tag{3.40}$$

where $0<l<\frac{\alpha}{\kappa^2}$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0,1]$ such that $\beta_n+\gamma_n+\delta_n=1$ for all $n\ge 0$. Suppose that the following conditions hold:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $0 \le \alpha_n \le 1 - \rho$, $\forall n \ge n_0$ *for some integer* $n_0 \ge 0$;
(ii) $\liminf_{n\to\infty} \gamma_n > 0$ *and* $\liminf_{n\to\infty} \delta_n > 0$;
(iii) $\lim_{n\to\infty}(|\frac{\alpha_{n+1}}{1-(1-\alpha_{n+1})\beta_{n+1}} - \frac{\alpha_n}{1-(1-\alpha_n)\beta_n}| + |\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}|) = 0$;
(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n \ge \varepsilon > 0$ *for all* $n \ge 0$;
(v) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset* D *of* C, *and let* S *be a mapping of* C *into itself defined by* $Sx = \lim_{n\to\infty} S_n x$ *for all* $x \in C$, *and suppose that* $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. *Then*

$$x_n \to q \quad \Longleftrightarrow \quad \alpha_n\big(f(x_n) - x_n\big) \to 0,$$

where $q \in F$ *solves the following VIP*:

$$\big\langle q - f(q), J(q - p)\big\rangle \le 0, \quad \forall p \in F.$$

Proof In Theorem 3.1, we put $B_1 = I - V$, $B_2 = 0$ and $\mu_1 = l$, where $0 < l < \frac{\alpha}{\kappa^2}$. Then GSVI (1.1) is equivalent to the VIP of finding $x^* \in C$ such that

$$\big\langle B_1 x^*, J\big(x - x^*\big)\big\rangle \ge 0, \quad \forall x \in C.$$

In this case, $B_1 : C \to X$ is α-inverse strongly accretive. It is not hard to see that $\mathrm{Fix}(V) = \mathrm{VI}(C, B_1)$. As a matter of fact, we have, for $l > 0$,

$$\begin{aligned}
u \in \mathrm{VI}(C, B_1) \quad &\Leftrightarrow \quad \big\langle B_1 u, J(y - u)\big\rangle \ge 0, \quad \forall y \in C \\
&\Leftrightarrow \quad \big\langle u - lB_1 u - u, J(u - y)\big\rangle \ge 0, \quad \forall y \in C \\
&\Leftrightarrow \quad u = \Pi_C(u - lB_1 u) \\
&\Leftrightarrow \quad u = \Pi_C(u - lu + lVu) \\
&\Leftrightarrow \quad \big\langle u - lu + lVu - u, J(u - y)\big\rangle \ge 0, \quad \forall y \in C \\
&\Leftrightarrow \quad \big\langle u - Vu, J(u - y)\big\rangle \le 0, \quad \forall y \in C \\
&\Leftrightarrow \quad u = Vu \\
&\Leftrightarrow \quad u \in \mathrm{Fix}(V).
\end{aligned}$$

Accordingly, we know that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \cap A^{-1}0 = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(V) \cap A^{-1}0$, and

$$\begin{aligned}
&\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_n \\
&\quad = \Pi_C(I - \mu_1 B_1)x_n \\
&\quad = \Pi_C\big((1 - l)x_n + lVx_n\big) = \big((1 - l)I + lV\big)x_n.
\end{aligned}$$

So, the scheme (3.1) reduces to (3.40). Therefore, the desired result follows from Theorem 3.1. □

Remark 3.1 Theorem 3.1 improves, extends, supplements and develops Jung [22, Theorem 3.1], Ceng and Yao [2, Theorem 3.1] and Cai and Bu [3, Theorem 3.1] in the following aspects.

(i) The problem of finding a point $q \in \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \cap A^{-1}0$ in our Theorem 3.1 is more general and more subtle than any of the problems of finding a point $q \in A^{-1}0$ in [22, Theorem 3.1], the problem of finding a point $q \in \mathrm{Fix}(T)$ in [2, Theorem 3.1], and the problem of finding a point $q \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \Omega$ in [3, Theorem 3.1].

(ii) The iterative scheme in [2, Theorem 3.1] is extended to develop the iterative scheme (3.1) of Theorem 3.1 by virtue of the iterative schemes of [22, Theorem 3.1] and [3, Theorem 3.1]. The iterative scheme (3.1) of Theorem 3.1 is more advantageous and more flexible than the iterative scheme of [2, Theorem 3.1], because it can be applied to solving three problems (*i.e.*, GSVI (1.1), fixed point problem and zero point problem) and involves several parameter sequences $\{r_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$.

(iii) Our Theorem 3.1 extends and generalizes Ceng and Yao [2, Theorem 3.1] from a nonexpansive mapping to a countable family of nonexpansive mappings, and Jung [22, Theorems 3.1] to the setting of a countable family of nonexpansive mappings and GSVI (1.1) for two inverse-strongly accretive mappings. In the meantime, our Theorem 3.1 extends and generalizes Cai and Bu [3, Theorem 3.1] to the setting of an accretive operator.

(iv) The iterative scheme (3.1) in Theorem 3.1 is very different from any in [22, Theorem 3.1], [2, Theorem 3.1] and [3, Theorem 3.1], because the mapping G in [3, Theorem 3.1] and the mapping J_{r_n} in [22, Theorem 3.1] are replaced by the same composite mapping $J_{r_n}G$ in the iterative scheme (3.1) of our Theorem 3.1.

(v) Cai and Bu's proof in [3, Theorem 3.1] depends on the argument techniques in [6], the inequality in 2-uniformly smooth Banach spaces (see Proposition 2.1) and the inequality in smooth and uniform convex Banach spaces (see Proposition 2.2). Because the composite mapping $J_{r_n}G$ appears in the iterative scheme (3.1) of our Theorem 3.1, the proof of our Theorem 3.1 depends on the argument techniques in [6], the inequality in 2-uniformly smooth Banach spaces (see Proposition 2.1), the inequality in smooth and uniform convex Banach spaces (see Proposition 2.2), the inequalities in uniform convex Banach spaces (see Lemmas 2.5 and 2.9 in Section 2 of this paper), and the resolvent identity for accretive operators (see Proposition 1.1).

(vi) It is worth emphasizing that the assumption of asymptotic regularity on $\{x_n\}$ in [22, Theorem 3.1] is dropped by Theorem 3.1, and there is no assumption of the convergence of parameter sequences to zero in our Theorem 3.1.

4 Composite Mann iterative algorithms in uniformly convex Banach spaces having uniformly Gâteaux differentiable norms

In this section, we introduce our composite Mann iterative algorithms in uniformly convex Banach spaces having uniformly Gâteaux differentiable norms and show the strong convergence theorems. First, we give some useful lemmas whose proofs will be omitted because they can be obtained by standard argument.

Lemma 4.1 *Let C be a nonempty closed convex subset of a smooth Banach space X, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0, 1]$ we have*

$$\big\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\big\| \leq \left\{\sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\left(1 + \frac{1}{\lambda_i}\right)\right\}\|x - y\|, \quad \forall x, y \in C,$$

for $i = 1,2$. In particular, if $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \le \mu_i \le 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1,2$.

Lemma 4.2 *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \ge 1$ for $i = 1,2$. Let $G : C \to C$ be the mapping defined by*

$$G(x) = \Pi_C\bigl[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)\bigr], \quad \forall x \in C.$$

If $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \le \mu_i \le 1$, then $G : C \to C$ is nonexpansive.

We now state and prove the main result of this section.

Theorem 4.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $A \subset X \times X$ be an accretive operator in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\lambda_i + \alpha_i \ge 1$ for $i = 1,2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \cap A^{-1}0 \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \le \mu_i \le 1$ for $i = 1,2$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n, \\ x_{n+1} = \sigma_n G x_n + (1 - \sigma_n) y_n, \quad \forall n \ge 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\sigma_n\}$ are the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \ge 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c,d]$ *for some* $c, d \in (0,1)$;
(iii) $\sum_{n=1}^{\infty} (|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;
(iv) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ *and* $r_n \ge \varepsilon > 0$ *for all* $n \ge 0$;
(v) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1} x\| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\bigl\langle q - f(q), J(q - p)\bigr\rangle \le 0, \quad \forall p \in F.$$

Proof First of all, take a fixed $p \in F$ arbitrarily. Then we obtain $p = Gp$, $p = S_n p$ and $J_{r_n} p = p$ for all $n \ge 0$. By Lemma 4.2, we get from (4.1)

$$\begin{aligned} \|y_n - p\| &\le \alpha_n \bigl\|f(x_n) - p\bigr\| + \beta_n \|x_n - p\| + \gamma_n \|S_n x_n - p\| + \delta_n \|J_{r_n} G x_n - p\| \\ &\le \alpha_n \bigl(\rho \|x_n - p\| + \bigl\|f(p) - p\bigr\|\bigr) + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|x_n - p\| \\ &= \bigl(1 - \alpha_n(1 - \rho)\bigr)\|x_n - p\| + \alpha_n \bigl\|f(p) - p\bigr\|, \end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1}-p\| &\le \sigma_n\|Gx_n-p\|+(1-\sigma_n)\|y_n-p\|\\ &\le \sigma_n\|x_n-p\|+(1-\sigma_n)\big[\big(1-\alpha_n(1-\rho)\big)\|x_n-p\|+\alpha_n\big\|f(p)-p\big\|\big]\\ &=\big(1-(1-\sigma_n)\alpha_n(1-\rho)\big)\|x_n-p\|+(1-\sigma_n)\alpha_n\big\|f(p)-p\big\|\\ &=\big(1-(1-\sigma_n)\alpha_n(1-\rho)\big)\|x_n-p\|+(1-\sigma_n)\alpha_n(1-\rho)\frac{\|f(p)-p\|}{1-\rho}\\ &\le \max\left\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\}.\end{aligned}\tag{4.2}$$

By induction, we have

$$\|x_n-p\|\le\max\left\{\|x_0-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\},\quad \forall n\ge 0,$$

which implies that $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{Gx_n\}$, $\{f(x_n)\}$.

Let us show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{4.3}$$

As a matter of fact, observe that y_n can be rewritten as follows

$$y_n=\beta_nx_n+(1-\beta_n)z_n,$$

where $z_n=\frac{\alpha_nf(x_n)+\gamma_nS_nx_n+\delta_nJ_{r_n}Gx_n}{1-\beta_n}$. Observe that

$$\begin{aligned}&\|z_n-z_{n-1}\|\\ &\quad=\left\|\frac{\alpha_nf(x_n)+\gamma_nS_nx_n+\delta_nJ_{r_n}Gx_n}{1-\beta_n}-\frac{\alpha_{n-1}f(x_{n-1})+\gamma_{n-1}S_{n-1}x_{n-1}+\delta_{n-1}J_{r_{n-1}}Gx_{n-1}}{1-\beta_{n-1}}\right\|\\ &\quad=\left\|\frac{y_n-\beta_nx_n}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\|\\ &\quad=\left\|\frac{y_n-\beta_nx_n}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n}+\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\|\\ &\quad\le\left\|\frac{y_n-\beta_nx_n}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n}\right\|+\left\|\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\|\\ &\quad=\frac{1}{1-\beta_n}\big\|y_n-\beta_nx_n-(y_{n-1}-\beta_{n-1}x_{n-1})\big\|+\left|\frac{1}{1-\beta_n}-\frac{1}{1-\beta_{n-1}}\right|\|y_{n-1}-\beta_{n-1}x_{n-1}\|\\ &\quad=\frac{1}{1-\beta_n}\big\|y_n-\beta_nx_n-(y_{n-1}-\beta_{n-1}x_{n-1})\big\|+\frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1}-\beta_{n-1}x_{n-1}\|\\ &\quad=\frac{1}{1-\beta_n}\big\|\alpha_nf(x_n)+\gamma_nS_nx_n+\delta_nJ_{r_n}Gx_n-\alpha_{n-1}f(x_{n-1})-\gamma_{n-1}S_{n-1}x_{n-1}-\delta_{n-1}J_{r_{n-1}}Gx_{n-1}\big\|\\ &\qquad+\frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1}-\beta_{n-1}x_{n-1}\|\\ &\quad\le\frac{1}{1-\beta_n}\big[\alpha_n\big\|f(x_n)-f(x_{n-1})\big\|+\gamma_n\|S_nx_n-S_{n-1}x_{n-1}\|+\delta_n\|Jr_nGx_n-J_{r_{n-1}}Gx_{n-1}\|\end{aligned}$$

$$
\begin{aligned}
&+ |\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + |\gamma_n - \gamma_{n-1}| \|S_{n-1}x_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} Gx_{n-1}\| \big] \\
&+ \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1}x_{n-1}\|. \qquad (4.4)
\end{aligned}
$$

On the other hand, repeating the same arguments as those of (3.8) in the proof of Theorem 3.1, we can deduce that for all $n \geq 0$,

$$
\|J_{r_{n+1}} Gx_{n+1} - J_{r_n} Gx_n\| \leq \|x_n - x_{n+1}\| + M_0 |r_n - r_{n+1}|, \quad \forall n \geq 0, \qquad (4.5)
$$

where $\sup_{n\geq 0}\{\frac{1}{\varepsilon}(\|J_{r_{n+1}}Gx_{n+1} - Gx_n\| + \|J_{r_n}Gx_n - Gx_{n+1}\|)\} \leq M_0$ for some $M_0 > 0$. Taking into account $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [\hat{c}, \hat{d}]$. So, from (4.4) and (4.5), we have

$$
\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\leq \frac{1}{1-\beta_n} \big[\alpha_n \| f(x_n) - f(x_{n-1}) \| + \gamma_n \big(\|S_n x_n - S_n x_{n-1}\| + \|S_n x_{n-1} - S_{n-1}x_{n-1}\| \big) \\
&\quad + \delta_n \big(\|x_{n-1} - x_n\| + M_0 |r_{n-1} - r_n| \big) + |\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + |\gamma_n - \gamma_{n-1}| \|S_{n-1}x_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} Gx_{n-1}\| \big] + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1}x_{n-1}\| \\
&\leq \frac{1}{1-\beta_n} \big[\alpha_n \rho \|x_n - x_{n-1}\| + \gamma_n \big(\|x_n - x_{n-1}\| + \|S_n x_{n-1} - S_{n-1}x_{n-1}\| \big) \\
&\quad + \delta_n \big(\|x_{n-1} - x_n\| + M_0 |r_{n-1} - r_n| \big) + |\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + |\gamma_n - \gamma_{n-1}| \|S_{n-1}x_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} Gx_{n-1}\| \big] + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1}x_{n-1}\| \\
&= \frac{1}{1-\beta_n} \big\{ \big(1 - \beta_n - \alpha_n(1-\rho)\big) \|x_n - x_{n-1}\| + \gamma_n \|S_n x_{n-1} - S_{n-1}x_{n-1}\| + \delta_n M_0 |r_{n-1} - r_n| \\
&\quad + |\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + |\gamma_n - \gamma_{n-1}| \|S_{n-1}x_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} Gx_{n-1}\| \big\} \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1}x_{n-1}\| \\
&= \left(1 - \frac{\alpha_n(1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| + \frac{\gamma_n}{1-\beta_n} \|S_n x_{n-1} - S_{n-1}x_{n-1}\| + \frac{\delta_n M_0}{1-\beta_n} |r_{n-1} - r_n| \\
&\quad + \frac{1}{1-\beta_n} \big[|\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + |\gamma_n - \gamma_{n-1}| \|S_{n-1}x_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} Gx_{n-1}\| \big] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1}x_{n-1}\| \\
&\leq \left(1 - \frac{\alpha_n(1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| + \|S_n x_{n-1} - S_{n-1}x_{n-1}\| + M_0 |r_{n-1} - r_n| \\
&\quad + \frac{1}{1-\beta_n} \big[|\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + |\gamma_n - \gamma_{n-1}| \|S_{n-1}x_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} Gx_{n-1}\| \big] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \big\| \alpha_{n-1} f(x_{n-1}) + \gamma_{n-1} S_{n-1}x_{n-1} + \delta_{n-1} J_{r_{n-1}} Gx_{n-1} \big\| \\
&\leq \left(1 - \frac{\alpha_n(1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| + M_1 \big[|r_{n-1} - r_n| + |\alpha_n - \alpha_{n-1}|
\end{aligned}
$$

$$+ |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|$$
$$+ |\delta_n - \delta_{n-1}|\big] + \|S_n x_{n-1} - S_{n-1}x_{n-1}\|, \tag{4.6}$$

where $\sup_{n\ge0}\{\frac{1}{(1-\hat{d})^2}(\|f(x_n)\| + \|S_n x_n\| + \|J_{r_n}Gx_n\| + M_0)\} \le M_1$ for some $M_1 > 0$. In the meantime, observe that

$$x_{n+1} - x_n = \sigma_n(Gx_n - Gx_{n-1}) + (\sigma_n - \sigma_{n-1})(Gx_{n-1} - z_{n-1}) + (1-\sigma_n)(z_n - z_{n-1}).$$

This together with (4.6) implies that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad\le \sigma_n\|Gx_n - Gx_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - z_{n-1}\| + (1-\sigma_n)\|z_n - z_{n-1}\| \\
&\quad\le \sigma_n\|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - z_{n-1}\| + (1-\sigma_n)\left\{\left(1 - \frac{\alpha_n(1-\rho)}{1-\beta_n}\right)\|x_n - x_{n-1}\|\right. \\
&\qquad + M_1\big[|r_n - r_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|\big] \\
&\qquad \left. + \|S_n x_{n-1} - S_{n-1}x_{n-1}\|\right\} \\
&\quad\le \left(1 - \frac{(1-\sigma_n)\alpha_n(1-\rho)}{1-\beta_n}\right)\|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - z_{n-1}\| + M_1\big[|r_n - r_{n-1}| \\
&\qquad + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|\big] + \|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\
&\quad\le \left(1 - \frac{(1-\sigma_n)\alpha_n(1-\rho)}{1-\beta_n}\right)\|x_n - x_{n-1}\| + M_2\big[|r_n - r_{n-1}| + |\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| \\
&\qquad + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|\big] + \|S_n x_{n-1} - S_{n-1}x_{n-1}\|,
\end{aligned} \tag{4.7}$$

where $\sup_{n\ge0}\{M_1 + \|Gx_n - z_n\|\} \le M_2$ for some $M_2 > 0$. Since $\frac{(1-\sigma_n)\alpha_n(1-\rho)}{1-\beta_n} \ge (1-\sigma_n)\alpha_n(1-\rho)$, we obtain from conditions (i) and (v) that $\sum_{n=0}^{\infty}\frac{(1-\sigma_n)\alpha_n(1-\rho)}{1-\beta_n} = \infty$. Thus, applying Lemma 2.1 to (4.7), we deduce from conditions (iii), (iv) and the assumption on $\{S_n\}$ that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Next, we show that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$.

Indeed, according to Lemma 2.2, we have from (4.1)

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad= \big\|\alpha_n\big(f(x_n) - f(p)\big) + \beta_n(x_n - p) + \gamma_n(S_n x_n - p) + \delta_n(J_{r_n}Gx_n - p) + \alpha_n\big(f(p) - p\big)\big\|^2 \\
&\quad\le \big\|\alpha_n\big(f(x_n) - f(p)\big) + \beta_n(x_n - p) + \gamma_n(S_n x_n - p) + \delta_n(J_{r_n}Gx_n - p)\big\|^2 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, J(y_n - p)\big\rangle \\
&\quad\le \alpha_n\big\|f(x_n) - f(p)\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + \delta_n\|J_{r_n}Gx_n - p\|^2 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, J(y_n - p)\big\rangle \\
&\quad\le \alpha_n\rho\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|x_n - p\|^2
\end{aligned}$$

$$\begin{aligned}
&\quad + 2\alpha_n \| f(p) - p \| \|y_n - p\| \\
&= \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\|^2 + 2\alpha_n \| f(p) - p \| \|y_n - p\| \\
&\le \|x_n - p\|^2 + 2\alpha_n \| f(p) - p \| \|y_n - p\|.
\end{aligned} \tag{4.8}$$

Utilizing Lemma 2.5, we get from (4.1) and (4.8)

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|\sigma_n(Gx_n - p) + (1-\sigma_n)(y_n - p)\big\|^2 \\
&\le \sigma_n\|Gx_n - p\|^2 + (1-\sigma_n)\|y_n - p\|^2 - \sigma_n(1-\sigma_n)g\big(\|Gx_n - y_n\|\big) \\
&\le \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\big[\|x_n - p\|^2 + 2\alpha_n \| f(p) - p \| \|y_n - p\|\big] \\
&\quad - \sigma_n(1-\sigma_n)g\big(\|Gx_n - y_n\|\big) \\
&\le \|x_n - p\|^2 + 2\alpha_n \| f(p) - p \| \|y_n - p\| - \sigma_n(1-\sigma_n)g\big(\|Gx_n - y_n\|\big),
\end{aligned}$$

which hence yields

$$\begin{aligned}
&\sigma_n(1-\sigma_n)g\big(\|Gx_n - y_n\|\big) \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \| f(p) - p \| \|y_n - p\| \\
&\quad \le \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| + 2\alpha_n \| f(p) - p \| \|y_n - p\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, from condition (v) and the boundedness of $\{x_n\}$ and $\{y_n\}$, it follows that

$$\lim_{n\to\infty} g\big(\|Gx_n - y_n\|\big) = 0.$$

Utilizing the properties of g, we have

$$\lim_{n\to\infty} \|Gx_n - y_n\| = 0, \tag{4.9}$$

which together with (4.1) and (4.3) implies that

$$\begin{aligned}
\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\
&= \|x_n - x_{n+1}\| + \sigma_n\|Gx_n - y_n\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

That is,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.10}$$

Since

$$\|x_n - Gx_n\| \le \|x_n - y_n\| + \|y_n - Gx_n\|,$$

it immediately follows from (4.9) and (4.10) that

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{4.11}$$

On the other hand, observe that y_n can be rewritten as follows:

$$\begin{aligned} y_n &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n \\ &= \alpha_n f(x_n) + \beta_n x_n + (\gamma_n + \delta_n) \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n} \\ &= \alpha_n f(x_n) + \beta_n x_n + e_n \hat{z}_n, \end{aligned}$$

where $e_n = \gamma_n + \delta_n$ and $\hat{z}_n = \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n}$. Utilizing Lemma 2.11, we have

$$\begin{aligned} \|y_n - p\|^2 &= \big\|\alpha_n\big(f(x_n) - p\big) + \beta_n(x_n - p) + e_n(\hat{z}_n - p)\big\|^2 \\ &\le \alpha_n \big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + e_n\|\hat{z}_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big) \\ &= \alpha_n \big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left\| \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n} - p \right\|^2 \\ &= \alpha_n \big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left\| \frac{\gamma_n}{\gamma_n + \delta_n}(S_n x_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G x_n - p) \right\|^2 \\ &\le \alpha_n \big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n}\|S_n x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G x_n - p\|^2 \right] \\ &\le \alpha_n \big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n}\|x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|x_n - p\|^2 \right] \\ &= \alpha_n \big\|f(x_n) - p\big\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big) \\ &\le \alpha_n \big\|f(x_n) - p\big\|^2 + \|x_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big), \end{aligned}$$

which hence implies that

$$\begin{aligned} \beta_n e_n g_1\big(\|\hat{z}_n - x_n\|\big) &\le \alpha_n \big\|f(x_n) - p\big\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le \alpha_n \big\|f(x_n) - p\big\|^2 + \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|. \end{aligned}$$

Utilizing (4.10), conditions (i), (ii), (v) and the boundedness of $\{x_n\}$, $\{y_n\}$ and $\{f(x_n)\}$, we get

$$\lim_{n\to\infty} g_1\big(\|\hat{z}_n - x_n\|\big) = 0.$$

From the properties of g_1, we have

$$\lim_{n\to\infty} \|\hat{z}_n - x_n\| = 0. \tag{4.12}$$

Utilizing Lemma 2.5 and the definition of $\hat{z}_n$, we have

$$\begin{aligned}
\|\hat{z}_n - p\|^2 &= \left\| \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n} - p \right\|^2 \\
&= \left\| \frac{\gamma_n}{\gamma_n + \delta_n}(S_n x_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G x_n - p) \right\|^2 \\
&\le \frac{\gamma_n}{\gamma_n + \delta_n}\|S_n x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G x_n - p\|^2 \\
&\quad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2\big(\|J_{r_n} G x_n - S_n x_n\|\big) \\
&\le \|x_n - p\|^2 - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2\big(\|J_{r_n} G x_n - S_n x_n\|\big),
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2\big(\|J_{r_n} G x_n - S_n x_n\|\big) &\le \|x_n - p\|^2 - \|\hat{z}_n - p\|^2 \\
&\le \big(\|x_n - p\| + \|\hat{z}_n - p\|\big)\|x_n - \hat{z}_n\|.
\end{aligned}$$

Since $\{x_n\}$ and $\{\hat{z}_n\}$ are bounded, we deduce from (4.12) and condition (ii) that

$$\lim_{n\to\infty} g_2\big(\|S_n x_n - J_{r_n} G x_n\|\big) = 0.$$

From the properties of g_2, we have

$$\lim_{n\to\infty} \|S_n x_n - J_{r_n} G x_n\| = 0. \tag{4.13}$$

Furthermore, y_n can also be rewritten as follows:

$$\begin{aligned}
y_n &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n \\
&= \beta_n x_n + \gamma_n S_n x_n + (\alpha_n + \delta_n)\frac{\alpha_n f(x_n) + \delta_n J_{r_n} G x_n}{\alpha_n + \delta_n} \\
&= \beta_n x_n + \gamma_n S_n x_n + d_n \tilde{z}_n,
\end{aligned}$$

where $d_n = \alpha_n + \delta_n$ and $\tilde{z}_n = \frac{\alpha_n f(x_n) + \delta_n J_{r_n} G x_n}{\alpha_n + \delta_n}$. Utilizing Lemma 2.11 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \big\|\beta_n(x_n - p) + \gamma_n(S_n x_n - p) + d_n(\tilde{z}_n - p)\big\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + d_n\|\tilde{z}_n - p\|^2 - \beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big) \\
&\quad = \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + d_n\left\|\frac{\alpha_n f(x_n) + \delta_n J_{r_n} G x_n}{\alpha_n + \delta_n} - p\right\|^2 \\
&\qquad - \beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big)
\end{aligned}$$

$$\begin{aligned}
&= \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + d_n\left\|\frac{\alpha_n}{\alpha_n+\delta_n}\big(f(x_n)-p\big) + \frac{\delta_n}{\alpha_n+\delta_n}(J_{r_n}Gx_n - p)\right\|^2 \\
&\quad - \beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big) \\
&\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + d_n\left[\frac{\alpha_n}{\alpha_n+\delta_n}\big\|f(x_n)-p\big\|^2 + \frac{\delta_n}{\alpha_n+\delta_n}\|J_{r_n}Gx_n - p\|^2\right] \\
&\quad - \beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big) \\
&\le \alpha_n\big\|f(x_n)-p\big\|^2 + (\beta_n+\gamma_n)\|x_n - p\|^2 + \delta_n\|x_n - p\|^2 - \beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big) \\
&= \alpha_n\big\|f(x_n)-p\big\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big) \\
&\le \alpha_n\big\|f(x_n)-p\big\|^2 + \|x_n - p\|^2 - \beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big),
\end{aligned}$$

which hence implies that

$$\begin{aligned}
\beta_n\gamma_n g_3\big(\|x_n - S_n x_n\|\big) &\le \alpha_n\big\|f(x_n)-p\big\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\le \alpha_n\big\|f(x_n)-p\big\|^2 + \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|.
\end{aligned}$$

Utilizing (4.10), conditions (i), (ii), (v) and the boundedness of $\{x_n\}$, $\{y_n\}$ and $\{f(x_n)\}$, we get

$$\lim_{n\to\infty} g_3\big(\|x_n - S_n x_n\|\big) = 0.$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{4.14}$$

Thus, from (4.13) and (4.14), we get

$$\|x_n - J_{r_n}Gx_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - J_{r_n}Gx_n\| \to 0 \quad \text{as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} \|x_n - J_{r_n}Gx_n\| = 0. \tag{4.15}$$

In terms of (4.14) and Lemma 2.6, we have

$$\|x_n - Sx_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad \text{as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.16}$$

Furthermore, repeating the same arguments as those of (3.30) in the proof of Theorem 3.1, we can conclude that

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0 \tag{4.17}$$

for a fixed number r such that $\varepsilon > r > 0$.

Define a mapping $Wx = (1-\theta_1-\theta_2)J_r x+\theta_1 Sx+\theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then by Lemma 2.8, we have that $\mathrm{Fix}(W) = \mathrm{Fix}(J_r) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned}\|x_n - Wx_n\| &= \big\|(1-\theta_1-\theta_2)(x_n - J_r x_n) + \theta_1(x_n - Sx_n) + \theta_2(x_n - Gx_n)\big\| \\ &\le (1-\theta_1-\theta_2)\|x_n - J_r x_n\| + \theta_1\|x_n - Sx_n\| + \theta_2\|x_n - Gx_n\|.\end{aligned}$$

From (4.11), (4.16) and (4.17), we obtain

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{4.18}$$

Now, we claim that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \le 0, \tag{4.19}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Repeating the same arguments as those of (3.37) in the proof of Theorem 3.1, we can obtain that

$$\begin{aligned}\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \\ &\le \limsup_{t\to 0}\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle.\end{aligned}$$

Since X has a uniformly Gâteaux differentiable norm, the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable, and hence (4.19) holds. From (4.10), we get $(y_n - q) - (x_n - q) \to 0$. Noticing the norm-to-weak* uniform continuity of J on bounded subsets of X, we deduce from (4.19) that

$$\begin{aligned}&\limsup_{n\to\infty} \langle f(q) - q, J(y_n - q)\rangle \\ &\quad= \limsup_{n\to\infty} \big(\langle f(q) - q, J(x_n - q)\rangle + \langle f(q) - q, J(y_n - q) - J(x_n - q)\rangle\big) \\ &\quad= \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \le 0.\end{aligned} \tag{4.20}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. Indeed, observe that

$$\begin{aligned}&\|y_n - q\|^2 \\ &\quad= \big\|\alpha_n\big(f(x_n) - f(q)\big) + \beta_n(x_n - q) + \gamma_n(S_n x_n - q) + \delta_n(J_{r_n} Gx_n - q) + \alpha_n\big(f(q) - q\big)\big\|^2 \\ &\quad\le \big\|\alpha_n\big(f(x_n) - f(q)\big) + \beta_n(x_n - q) + \gamma_n(S_n x_n - q) + \delta_n(J_{r_n} Gx_n - q)\big\|^2 \\ &\qquad + 2\alpha_n\langle f(q) - q, J(y_n - q)\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \big\|f(x_n) - f(q)\big\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\|S_n x_n - q\|^2 + \delta_n\|J_{r_n} G x_n - q\|^2 \\
&\quad + 2\alpha_n\big\langle f(q) - q, J(y_n - q)\big\rangle \\
&\leq \alpha_n\rho\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\|x_n - q\| + \delta_n\|x_n - q\|^2 + 2\alpha_n\big\langle f(q) - q, J(y_n - q)\big\rangle \\
&= (\alpha_n\rho + \beta_n + \gamma_n + \delta_n)\|x_n - q\|^2 + 2\alpha_n\big\langle f(q) - q, J(y_n - q)\big\rangle \\
&= \big(1 - \alpha_n(1-\rho)\big)\|x_n - q\|^2 + 2\alpha_n\big\langle f(q) - q, J(y_n - q)\big\rangle,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \leq \sigma_n\|Gx_n - q\|^2 + (1-\sigma_n)\|y_n - q\|^2 \\
&\quad \leq \sigma_n\|x_n - q\|^2 + (1-\sigma_n)\big[\big(1 - \alpha_n(1-\rho)\big)\|x_n - q\|^2 + 2\alpha_n\big\langle f(q) - q, J(y_n - q)\big\rangle\big] \\
&\quad = \big(1 - (1-\sigma_n)\alpha_n(1-\rho)\big)\|x_n - q\|^2 + 2(1-\sigma_n)\alpha_n\big\langle f(q) - q, J(y_n - q)\big\rangle \\
&\quad = \big(1 - (1-\sigma_n)\alpha_n(1-\rho)\big)\|x_n - q\|^2 + (1-\sigma_n)\alpha_n(1-\rho)\frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}. \qquad (4.21)
\end{aligned}
$$

Applying Lemma 2.1 to (4.21), we conclude from conditions (i), (v) and (4.20) that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 4.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $A \subset X \times X$ be an accretive operator in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $V : C \to C$ be a self-mapping such that $I - V : C \to X$ is λ-strictly pseudocontractive and α-strongly accretive with $\alpha + \lambda \geq 1$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(V) \cap A^{-1}0 \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}
y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n}((1-l)I + lV)x_n, \\
x_{n+1} = \sigma_n((1-l)I + lV)x_n + (1-\sigma_n)y_n, \quad \forall n \geq 0,
\end{cases}
\qquad (4.22)
$$

where $1 - \frac{\lambda}{1+\lambda}(1 - \sqrt{\frac{1-\alpha}{\lambda}}) \leq l \leq 1$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\sigma_n\}$ are the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c,d]$ *for some* $c, d \in (0,1)$;

(iii) $\sum_{n=1}^{\infty}(|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;

(iv) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ *and* $r_n \geq \varepsilon > 0$ *for all* $n \geq 0$;

(v) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$
\big\langle q - f(q), J(q - p)\big\rangle \leq 0, \quad \forall p \in F.
$$

Proof In Theorem 4.1, we put $B_1 = I - V$, $B_2 = 0$ and $\mu_1 = l$, where $1 - \frac{\lambda}{1+\lambda}(1 - \sqrt{\frac{1-\alpha}{\lambda}}) \leq l \leq 1$. Then GSVI (1.1) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1 x^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C.$$

In this case, $B_1 : C \to X$ is λ-strictly pseudocontractive and α-strongly accretive. Repeating the same arguments as those in the proof of Corollary 3.1, we can infer that $\mathrm{Fix}(V) = \mathrm{VI}(C, B_1)$. Accordingly, $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \cap A^{-1}0 = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(V) \cap A^{-1}0$, and

$$Gx_n = \big((1-l)I + lV\big)x_n, \quad \forall n \geq 0.$$

So, the scheme (4.1) reduces to (4.22). Therefore, the desired result follows from Theorem 4.1. □

Remark 4.1 Theorem 4.1 improves, extends, supplements and develops Jung [22, Theorem 3.1], Ceng and Yao [2, Theorem 3.1] and Cai and Bu [3, Theorem 3.1] in the following aspects.

(i) The problem of finding a point $q \in \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \cap A^{-1}0$ in our Theorem 4.1 is more general and more subtle than any of the problems of finding a point $q \in A^{-1}0$ in [22, Theorem 3.1], the problem of finding a point $q \in \mathrm{Fix}(T)$ in [2, Theorem 3.1], and the problem of finding a point $q \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \Omega$ in [3, Theorem 3.1].

(ii) The iterative scheme in [22, Theorem 3.1] is extended to develop the iterative scheme (4.1) of Theorem 4.1 by virtue of the iterative schemes of [2, Theorems 3.1] and [3, Theorem 3.1]. The iterative scheme (4.1) of Theorem 4.1 is more advantageous and more flexible than the iterative scheme of [2, Theorem 3.1], because it can be applied to solving three problems (*i.e.*, GSVI (1.1), fixed point problem and zero point problem) and involves several parameter sequences $\{\sigma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$.

(iii) Theorem 4.1 extends and generalizes Ceng and Yao [2, Theorem 3.1] from a nonexpansive mapping to a countable family of nonexpansive mappings, and Jung [22, Theorem 3.1] to the setting of a countable family of nonexpansive mappings and GSVI (1.1) for two strictly pseudocontractive and strongly accretive mappings. In the meantime, Theorem 4.1 extends and generalizes Cai and Bu [3, Theorem 3.1] to the setting of an accretive operator.

(iv) The iterative scheme (4.1) in Theorem 4.1 is very different from any in [22, Theorem 3.1], [2, Theorem 3.1] and [3, Theorem 3.1] because the mapping J_{r_n} in [22, Theorem 3.1] and the mapping G in [3, Theorem 3.1] are replaced by the same composite mapping $J_{r_n}G$ in the iterative scheme (4.1) of Theorem 4.1.

(v) Cai and Bu's proof in [3, Theorem 3.1] depends on the argument techniques in [6], the inequality in 2-uniformly smooth Banach spaces (see Proposition 2.1) and the inequality in smooth and uniform convex Banach spaces (see Proposition 2.2). However, the proof of Theorem 4.1 does not depend on the argument techniques in [6], the inequality in 2-uniformly smooth Banach spaces (see Proposition 2.1), and the inequality in smooth and uniform convex Banach spaces (see Proposition 2.2). It depends on only the inequalities in uniform convex Banach spaces (see Lemmas 2.5 and 2.11 in Section 2 of this paper) and the resolvent identity for accretive operators (see Proposition 1.1).

(vi) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [3, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X having a uniformly Gâteaux differentiable norm in Theorem 4.1. Moreover, the assumption of the uniformly smooth Banach space X in [2, Theorem 3.1] is replaced by the one of the uniformly convex Banach space X having a uniformly Gâteaux differentiable norm in Theorem 4.1. It is worth emphasizing that the assumption of asymptotic regularity on $\{x_n\}$ in [22, Theorem 3.1] is dropped by Theorem 4.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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