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A new multi-step iteration for solving a fixed point problem of nonexpansive mappings

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Abstract

We introduce a new nonlinear mapping generated by a finite family of nonexpansive mappings. Weak and strong convergence theorems are also established in the setting of a Banach space.

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1 Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E . Let $T : C \rightarrow C$ be a nonlinear mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and a mapping $f : C \rightarrow C$ is called a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in C$. We use Σ_C to denote a class of contractions with constant α .

Fixed point problems are now arising in a wide range of applications such as optimization, physics, engineering, economics and applied sciences. Many related problems can be cast as the problem of finding fixed points for nonlinear mappings. The interdisciplinary nature of fixed point problems is evident through a vast literature which includes a large body of mathematical and algorithmic developments.

In the literature, several types of iterations have been constructed and proposed in order to get convergence results for nonexpansive mappings in various settings. One classical iteration process is defined as follows: $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$. This method was introduced in 1953 by Mann [1] and is known as the *Mann iteration process*. However, we note that it has only weak convergence in general; for instance, see [2].

In 1974, Ishikawa [3] proposed the following two-step iteration: $x_1 \in C$ and

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. This method is often called the *Ishikawa iteration process*.

Very recently, Agarwal *et al.* [4] introduced a new iteration process as follows: $x_1 \in C$ and

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \\ x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_nTy_n, \quad \forall n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. This method is called the *S-iteration process*. The weak convergence was studied in [5] for nonexpansive mappings. It was also shown in [5] that the convergence rate of the *S-iteration process* is faster than the Picard and Mann iteration processes for contractive mappings.

Firstly, motivated by Agarwal *et al.* [4], we have the aim to introduce and study a new mapping defined by the following definition.

Definition 1.1 Let C be a nonempty and convex subset of a real Banach space E . Let T_1, T_2, \dots, T_N be a finite family of nonexpansive mappings of C into itself, and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for all $i = 1, 2, \dots, N$. Define the mapping $B : C \rightarrow C$ as follows:

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1)I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2)T_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3)T_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})T_{N-2}, \\ B = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N)T_{N-1}. \end{aligned} \tag{1.1}$$

Such a mapping B is called the *B-mapping* generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$. See [6–9] for the corresponding concept.

Secondly, using the definition above, we study weak convergence of the following Mann-type iteration process in a uniformly convex Banach space with a Fréchet differentiable norm or that satisfies Opial’s condition: $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n B_n x_n, \quad \forall n \geq 1, \tag{1.2}$$

where B_n is a *B-mapping* generated by T_1, T_2, \dots, T_N and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ (see Section 3).

Finally, we discuss strong convergence of the iteration scheme involving the modified viscosity approximation method [10] defined as follows: $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n, \quad \forall n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $f \in \Sigma_C$.

More references on earlier works promoting the theory of fixed points and common fixed points for nonexpansive mappings can be found in [11–21].

Throughout this paper, we use the notation:

- \rightharpoonup for weak convergence and \rightarrow for strong convergence.
- $\omega_\omega(x_n) = \{x : x_{n_i} \rightharpoonup x\}$ denotes the weak ω -limit set of $\{x_n\}$.

2 Preliminaries and lemmas

In this section, we begin by recalling some basic facts and lemmas which will be used in the sequel.

Let E be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *strictly convex* if for any $x, y \in U$,

$$x \neq y \text{ implies } \left\| \frac{x+y}{2} \right\| < 1.$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0, 2] \rightarrow [0, 1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. It is said to be *Fréchet differentiable* if for $x \in U$, the limit is attained uniformly for $y \in U$. It is said to be *uniformly smooth* or *uniformly Fréchet differentiable* if the limit (2.1) is attained uniformly for $x, y \in U$. The *normalized duality mapping* $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. It is known that E is smooth if and only if the duality mapping J is single valued, and that if E has a uniformly Gâteaux differentiable norm, J is uniformly norm-to-weak* continuous on each bounded subset of E . A Banach space E is said to satisfy Opial's condition [22]. If $x \in E$ and $x_n \rightharpoonup x$, then

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in E, x \neq y.$$

Let $T : C \rightarrow C$. Then $I - T$ is demiclosed at 0 if for all sequence $\{x_n\}$ in C , $x_n \rightharpoonup q$ and $\|x_n - Tx_n\| \rightarrow 0$ imply $q = Tq$. It is known that if E is uniformly convex, C is nonempty closed and convex, and T is nonexpansive, then $I - T$ is demiclosed at 0 [23]. For more details, we refer the reader to [5, 24].

Lemma 2.1 [5] *Let E be a smooth Banach space. Then the following hold:*

- (i) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, J(x) \rangle$ for all $x, y \in E$;
- (ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle$ for all $x, y \in E$.

Lemma 2.2 [24] *In a strictly convex Banach space E , if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|$$

for all $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Lemma 2.3 [25] *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a Banach space E such that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, \quad n \geq 1,$$

where $\{\beta_n\}$ satisfies the conditions $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

If $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\|z_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2.4 [26] *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a closed and convex subset of E , and let $\{S_n\}_{n=1}^\infty$ be a family of L_n -Lipschitzian self-mappings on C such that $\sum_{n=1}^\infty (L_n - 1) < \infty$ and $F = \bigcap_{n=1}^\infty F(S_n) \neq \emptyset$. For arbitrary $x_1 \in C$, define $x_{n+1} = S_n x_n$ for all $n \geq 1$. Then, for every $p, q \in F$, $\lim_{n \rightarrow \infty} \langle x_n, J(p - q) \rangle$ exists, in particular, for all $u, v \in \omega_\omega(x_n)$ and $p, q \in F$, $\langle u - v, J(p - q) \rangle = 0$.*

Lemma 2.5 [15] *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E , let $A : C \rightarrow C$ be a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$, and let $T : C \rightarrow E$ be a continuous pseudocontractive mapping satisfying the weakly inward condition. If T has a fixed point in C , then the path $\{x_t\}$ defined by*

$$x_t = tAx_t + (1 - t)Tx_t$$

converges strongly to a fixed point q of T as $t \rightarrow 0$, which is a unique solution of the variational inequality

$$\langle (I - A)q, J(q - p) \rangle \leq 0, \quad \forall p \in F(T).$$

Remark 2.6 Lemma 2.5 holds if $T : C \rightarrow C$ is a nonexpansive mapping and $A = f$ is a contraction.

The following lemma gives us a nice property of real sequences.

Lemma 2.7 [18] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - c_n)a_n + b_n, \quad \forall n \geq 1,$$

where $\{c_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

- (a) $\sum_{n=1}^\infty c_n = \infty$;
- (b) $\limsup_{n \rightarrow \infty} \frac{b_n}{c_n} \leq 0$ or $\sum_{n=1}^\infty |b_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3 Weak convergence theorem

In this section, we give some properties concerning the B -mapping and then prove a weak convergence theorem for nonexpansive mappings.

Lemma 3.1 *Let C be a nonempty, closed and convex subset of a strictly convex Banach space E . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for all $i = 1, 2, \dots, N - 1$ and $0 < \lambda_N \leq 1$. Let B be the B -mapping generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$. Then the following hold:*

- (i) $F(B) = \bigcap_{i=1}^N F(T_i)$;
- (ii) B is nonexpansive.

Proof (i) Since $\bigcap_{i=1}^N F(T_i) \subset F(B)$ is trivial, it suffices to show that $F(B) \subset \bigcap_{i=1}^N F(T_i)$. To this end, let $p \in F(B)$ and $x^* \in \bigcap_{i=1}^N F(T_i)$. Then we have

$$\begin{aligned}
 \|p - x^*\| &= \|Bp - x^*\| = \|\lambda_N(T_N U_{N-1}p - x^*) + (1 - \lambda_N)(T_{N-1}p - x^*)\| \\
 &\leq \lambda_N \|U_{N-1}p - x^*\| + (1 - \lambda_N) \|p - x^*\| \\
 &= \lambda_N \|\lambda_{N-1}(T_{N-1} U_{N-2}p - x^*) + (1 - \lambda_{N-1})(T_{N-2}p - x^*)\| + (1 - \lambda_N) \|p - x^*\| \\
 &\leq \lambda_N \lambda_{N-1} \|U_{N-2}p - x^*\| + (1 - \lambda_N \lambda_{N-1}) \|p - x^*\| \\
 &= \lambda_N \lambda_{N-1} \|\lambda_{N-2}(T_{N-2} U_{N-3}p - x^*) + (1 - \lambda_{N-2})(T_{N-3}p - x^*)\| \\
 &\quad + (1 - \lambda_N \lambda_{N-1}) \|p - x^*\| \\
 &\leq \lambda_N \lambda_{N-1} \lambda_{N-2} \|U_{N-3}p - x^*\| + (1 - \lambda_N \lambda_{N-1} \lambda_{N-2}) \|p - x^*\| \\
 &\quad \vdots \\
 &= \lambda_N \lambda_{N-1} \cdots \lambda_3 \|\lambda_2(T_2 U_1 p - x^*) + (1 - \lambda_2)(T_1 p - x^*)\| \\
 &\quad + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_3) \|p - x^*\| \\
 &\leq \lambda_N \lambda_{N-1} \cdots \lambda_2 \|T_2 U_1 p - x^*\| + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_2) \|p - x^*\| \\
 &\leq \lambda_N \lambda_{N-1} \cdots \lambda_2 \|U_1 p - x^*\| + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_2) \|p - x^*\| \\
 &= \lambda_N \lambda_{N-1} \cdots \lambda_2 \|\lambda_1(T_1 p - x^*) + (1 - \lambda_1)(p - x^*)\| \\
 &\quad + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_2) \|p - x^*\| \\
 &\leq \lambda_N \lambda_{N-1} \cdots \lambda_2 \lambda_1 \|T_1 p - x^*\| + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_2 \lambda_1) \|p - x^*\| \\
 &\leq \lambda_N \lambda_{N-1} \cdots \lambda_2 \lambda_1 \|p - x^*\| + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_2 \lambda_1) \|p - x^*\| \\
 &= \|p - x^*\|. \tag{3.1}
 \end{aligned}$$

This shows that

$$\|p - x^*\| = \lambda_N \lambda_{N-1} \cdots \lambda_2 \|\lambda_1(T_1 p - x^*) + (1 - \lambda_1)(p - x^*)\| + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_2) \|p - x^*\|,$$

which turns out to be

$$\|p - x^*\| = \|\lambda_1(T_1 p - x^*) + (1 - \lambda_1)(p - x^*)\|.$$

Again by (3.1), we see that $\|p - x^*\| = \|T_1p - x^*\|$ and thus

$$\|p - x^*\| = \|T_1p - x^*\| = \|\lambda_1(T_1p - x^*) + (1 - \lambda_1)(p - x^*)\|.$$

Using Lemma 2.2, we get that $T_1p = p$ and hence $U_1p = p$. Again by (3.1), we have

$$\begin{aligned} \|p - x^*\| &= \lambda_N \lambda_{N-1} \cdots \lambda_3 \|\lambda_2(T_2U_1p - x^*) + (1 - \lambda_2)(T_1p - x^*)\| \\ &\quad + (1 - \lambda_N \lambda_{N-1} \cdots \lambda_3) \|p - x^*\|, \end{aligned}$$

which implies that

$$\|p - x^*\| = \|\lambda_2(T_2U_1p - x^*) + (1 - \lambda_2)(T_1p - x^*)\|.$$

From (3.1) we see that $\|U_1p - x^*\| = \|T_2U_1p - x^*\|$. Since $U_1p = p$ and $T_1p = p$,

$$\|p - x^*\| = \|T_2p - x^*\| = \|\lambda_2(T_2p - x^*) + (1 - \lambda_2)(p - x^*)\|.$$

Using Lemma 2.2, we get that $T_2p = p$ and hence $U_2p = p$.

By continuing this process, we can show that $T_ip = p$ and $U_ip = p$ for all $i = 1, 2, \dots, N - 1$. Finally, we obtain

$$\begin{aligned} \|p - T_Np\| &\leq \|p - Bp\| + \|Bp - T_Np\| \\ &= \|p - Bp\| + (1 - \lambda_N) \|p - T_Np\|, \end{aligned}$$

which yields that $p = T_Np$ since $p \in F(B)$. Hence $p = T_1p = T_2p = \cdots = T_Np$ and thus $p \in \bigcap_{i=1}^N F(T_i)$.

(ii) The proof follows directly from (i). □

Lemma 3.2 *Let C be a nonempty closed convex subset of a real Banach space E . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let B be the B -mapping generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be a real sequence in $(0, 1)$. For every $n \in \mathbb{N}$, let B_n be the B -mapping generated by T_1, T_2, \dots, T_N and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ as follows:*

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_1 + (1 - \lambda_{n,2})T_1, \\ U_{n,3} &= \lambda_{n,3}T_3U_2 + (1 - \lambda_{n,3})T_2, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{N-2} + (1 - \lambda_{n,N-1})T_{N-2}, \\ B_n &= U_{n,N} = \lambda_{n,N}T_NU_{N-1} + (1 - \lambda_{n,N})T_{N-1}. \end{aligned}$$

If $\lambda_{n,i} \rightarrow \lambda_i \in (0, 1)$ for all $i = 1, 2, \dots, N$, then $\lim_{n \rightarrow \infty} B_nx = Bx$ for all $x \in C$.

Proof Let $x \in C$ and U_k and $U_{n,k}$ be generated by T_1, T_2, \dots, T_k and $\lambda_1, \lambda_2, \dots, \lambda_k$, and T_1, T_2, \dots, T_k and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,k}$, respectively. Then

$$\|U_{n,1}x - U_1x\| = \|(\lambda_{n,1} - \lambda_1)(T_1x - x)\| \leq |\lambda_{n,1} - \lambda_1| \|T_1x - x\|.$$

Let $k \in \{2, 3, \dots, N\}$ and $M = \max\{\|T_k U_{k-1}x\| + \|T_{k-1}x\| : k = 2, 3, \dots, N\}$. Then

$$\begin{aligned} \|U_{n,k}x - U_kx\| &= \|\lambda_{n,k}T_k U_{n,k-1}x + (1 - \lambda_{n,k})T_{k-1}x - \lambda_k T_k U_{k-1}x - (1 - \lambda_k)T_{k-1}x\| \\ &= \|\lambda_{n,k}T_k U_{n,k-1}x - \lambda_{n,k}T_{k-1}x - \lambda_k T_k U_{k-1}x + \lambda_k T_{k-1}x\| \\ &\leq \lambda_{n,k} \|T_k U_{n,k-1}x - T_k U_{k-1}x\| + |\lambda_{n,k} - \lambda_k| \|T_k U_{k-1}x\| \\ &\quad + |\lambda_{n,k} - \lambda_k| \|T_{k-1}x\| \\ &\leq \|U_{n,k-1}x - U_{k-1}x\| + |\lambda_{n,k} - \lambda_k| M. \end{aligned}$$

It follows that

$$\begin{aligned} \|B_nx - Bx\| &= \|U_{n,N}x - U_Nx\| \\ &\leq \|U_{n,N-1}x - U_{N-1}x\| + |\lambda_{n,N} - \lambda_N| M \\ &\leq \|U_{n,N-2}x - U_{N-2}x\| + |\lambda_{n,N-1} - \lambda_{N-1}| M + |\lambda_{n,N} - \lambda_N| M \\ &\quad \vdots \\ &\leq \|U_{n,1}x - U_1x\| + |\lambda_{n,2} - \lambda_2| M + \dots + |\lambda_{n,N-1} - \lambda_{N-1}| M + |\lambda_{n,N} - \lambda_N| M \\ &\leq |\lambda_{n,1} - \lambda_1| \|T_1x - x\| + |\lambda_{n,2} - \lambda_2| M + \dots + |\lambda_{n,N-1} - \lambda_{N-1}| M \\ &\quad + |\lambda_{n,N} - \lambda_N| M. \end{aligned}$$

Since $\lambda_{n,i} \rightarrow \lambda_i$ as $n \rightarrow \infty$ ($i = 1, 2, \dots, N$), we thus complete the proof. \square

Remark 3.3 It is easily seen that for all $n \in \mathbb{N}$, B_n is nonexpansive.

Lemma 3.4 Let C be a nonempty closed convex subset of a real Banach space E . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be a real sequence in $(0, 1)$. For every $n \in \mathbb{N}$, let B_n be the B -mapping generated by T_1, T_2, \dots, T_N and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$.

If $\lim_{n \rightarrow \infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ for all $i = 1, 2, \dots, N$, then

$$\lim_{n \rightarrow \infty} \|B_{n+1}z_n - B_n z_n\| = 0$$

for each bounded sequence $\{z_n\}$ in C .

Proof Let $\{z_n\}$ be a bounded sequence in C . For $j \in \{0, 1, \dots, N - 2\}$ and for some $M > 0$, we have

$$\begin{aligned} &\|U_{n+1,N-j}z_n - U_{n,N-j}z_n\| \\ &= \|\lambda_{n+1,N-j}T_{N-j}U_{n+1,N-j-1}z_n + (1 - \lambda_{n+1,N-j})T_{N-j-1}z_n \end{aligned}$$

$$\begin{aligned}
 & - \lambda_{n,N-j} T_{N-j} U_{n,N-j-1} z_n - (1 - \lambda_{n,N-j}) T_{N-j-1} z_n \| \\
 \leq & \lambda_{n+1,N-j} \| T_{N-j} U_{n+1,N-j-1} z_n - T_{N-j} U_{n,N-j-1} z_n \| \\
 & + |\lambda_{n+1,N-j} - \lambda_{n,N-j}| \| T_{N-j} U_{n,N-j-1} z_n \| \\
 & + |\lambda_{n+1,N-j} - \lambda_{n,N-j}| \| T_{N-j-1} z_n \| \\
 \leq & \| U_{n+1,N-j-1} z_n - U_{n,N-j-1} z_n \| + |\lambda_{n+1,N-j} - \lambda_{n,N-j}| M.
 \end{aligned}$$

Using the relation above, we can show that

$$\begin{aligned}
 \| B_{n+1} z_n - B_n z_n \| & = \| U_{n+1,N} z_n - U_{n,N} z_n \| \\
 & \leq M \sum_{j=2}^N |\lambda_{n+1,j} - \lambda_{n,j}| + |\lambda_{n+1,1} - \lambda_{n,1}| (\| z_n \| + \| T_1 z_n \|).
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ for all $i = 1, 2, \dots, N$, we obtain the desired result. \square

Using the concept of B -mapping, we study weak convergence of the sequence generated by Mann-type iteration process (1.2).

Theorem 3.5 *Let E be a uniformly convex Banach space having a Fréchet differentiable norm or that satisfies Opial's condition. Let C be a nonempty, closed and convex subset of E . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be a real sequence in $(0, 1)$ such that $\lambda_{n,i} \rightarrow \lambda_i$ ($i = 1, 2, \dots, N$). For every $n \in \mathbb{N}$, let B_n be the B -mapping generated by T_1, T_2, \dots, T_N and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Let $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n B_n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges weakly to $x^* \in \bigcap_{i=1}^N F(T_i)$.

Proof Let $p \in \bigcap_{i=1}^N F(T_i)$. Then $p = B_n p$ for all $n \geq 1$ and hence

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n \|B_n x_n - p\| \leq \|x_n - p\|.$$

It follows that $\{\|x_n - p\|\}$ is nonincreasing; consequently, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Assume $\|x_n - p\| > 0$. Since E is uniformly convex, it follows (see, for example, [27]) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| \left\{ 1 - 2 \min\{\alpha_n, 1 - \alpha_n\} \delta_E \left(\frac{\|x_n - B_n x_n\|}{\|x_n - p\|} \right) \right\},$$

which implies that

$$\begin{aligned}
 \alpha_n(1 - \alpha_n)\|x_n - p\| \delta_E \left(\frac{\|x_n - B_n x_n\|}{\|x_n - p\|} \right) & \leq \min\{\alpha_n, 1 - \alpha_n\} \|x_n - p\| \delta_E \left(\frac{\|x_n - B_n x_n\|}{\|x_n - p\|} \right) \\
 & \leq \frac{1}{2} \{ \|x_n - p\| - \|x_{n+1} - p\| \}.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, by the continuity of δ_E , we have $\lim_{n \rightarrow \infty} \|x_n - B_n x_n\| = 0$. Since $\lambda_{n,i} \rightarrow \lambda_i$ ($i = 1, 2, \dots, N$), let the mapping $B : C \rightarrow C$ be generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$. Then, by Lemma 3.2, we have $\lim_{n \rightarrow \infty} \|B_n x - Bx\| = 0$ for all $x \in C$. So we have

$$\begin{aligned} \|x_n - Bx_n\| &\leq \|x_n - B_n x_n\| + \|B_n x_n - Bx_n\| \\ &\leq \|x_n - B_n x_n\| + \sup_{z \in \{x_n\}} \|B_n z - Bz\| \\ &\rightarrow 0. \end{aligned}$$

Since B is nonexpansive and E is uniformly convex, by the demiclosedness principle, $\omega_\omega(x_n) \subset F(B)$. Moreover, $F(B) = \bigcap_{i=1}^N F(T_i)$ by Lemma 3.1(i).

We next show that $\omega_\omega(x_n)$ is a singleton. Indeed, suppose that $x^*, y^* \in \omega_\omega(x_n) \subset \bigcap_{i=1}^N F(T_i)$. Define $S_n : C \rightarrow C$ by

$$S_n x = (1 - \alpha_n)x + \alpha_n B_n x, \quad x \in C.$$

Then S_n is nonexpansive and $x^*, y^* \in \bigcap_{n=1}^\infty F(S_n)$. Using Lemma 2.4, we have $\lim_{n \rightarrow \infty} \langle x_n, J(x^* - y^*) \rangle$ exists. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. Then

$$\|x^* - y^*\|^2 = \langle x^* - y^*, J(x^* - y^*) \rangle = \lim_{k \rightarrow \infty} \langle x_{n_k} - x_{m_k}, J(x^* - y^*) \rangle = 0.$$

This shows that $x^* = y^*$.

Assume that E satisfies Opial's condition. Let $x^*, y^* \in \omega_\omega(x_n)$ and $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. If $x^* \neq y^*$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - x^*\| &= \lim_{k \rightarrow \infty} \|x_{n_k} - x^*\| < \lim_{k \rightarrow \infty} \|x_{n_k} - y^*\| = \lim_{k \rightarrow \infty} \|x_{m_k} - y^*\| \\ &< \lim_{k \rightarrow \infty} \|x_{m_k} - x^*\| = \lim_{n \rightarrow \infty} \|x_n - x^*\|, \end{aligned}$$

which is a contradiction. It follows that $x^* = y^*$. Therefore $x_n \rightharpoonup x^* \in \bigcap_{i=1}^N F(T_i)$ as $n \rightarrow \infty$. This completes the proof. \square

4 Strong convergence theorem

In this section, we prove a strong convergence theorem for a finite family of nonexpansive mappings in Banach spaces.

Theorem 4.1 *Let E be a strictly convex and reflexive Banach space having a uniformly Gâteaux differentiable norm. Let C be a nonempty, closed and convex subset of E . Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be a real sequence in $(0, 1)$ such that $\lambda_{n,i} \rightarrow \lambda_i$ ($i = 1, 2, \dots, N$). For every $n \in \mathbb{N}$, let B_n be the B -mapping generated by T_1, T_2, \dots, T_N and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ which satisfy the conditions:*

- (C1) $\alpha_n + \beta_n + \gamma_n = 1$;
- (C2) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

$$(C3) \quad 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

Let $f \in \Sigma_C$ and define the sequence $\{x_n\}$ by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to $q \in \bigcap_{i=1}^N F(T_i)$, where q is also the unique solution of the variational inequality

$$\langle (I-f)(q), J(q-p) \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i). \quad (4.1)$$

Proof We divide the proof into the following steps.

Step 1. We show that $\{x_n\}$ is bounded. Let $p \in \bigcap_{i=1}^N F(T_i)$. Then $p = B_n p$ for all $n \geq 1$ and hence, by the nonexpansiveness of $\{B_n\}_{n=1}^\infty$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(B_n x_n - p)\| \\ &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &= (1 - \alpha_n(1 - \alpha)) \|x_n - p\| + \alpha_n \|f(p) - p\| \\ &\leq \max \left\{ \|x_n - p\|, \frac{1}{1 - \alpha} \|f(p) - p\| \right\}. \end{aligned}$$

By induction, we can conclude that $\{x_n\}$ is bounded. So are $\{f(x_n)\}$ and $\{B_n x_n\}$.

Step 2. We show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. To this end, we define $z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$. From (1.3) we have

$$\begin{aligned} \|z_{n+1} - z_n\| &= \left\| \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} B_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n B_n x_n}{1 - \beta_n} \right\| \\ &= \left\| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (f(x_{n+1}) - B_n x_n) + \frac{\alpha_n}{1 - \beta_n} (B_n x_n - f(x_n)) \right. \\ &\quad \left. + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (B_{n+1} x_{n+1} - B_n x_n) \right\| \\ &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} M + \frac{\alpha_n}{1 - \beta_n} M + \|B_{n+1} x_{n+1} - B_n x_n\| \\ &\leq \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} + \frac{\alpha_n}{1 - \beta_n} \right) M + \|B_{n+1} x_{n+1} - B_{n+1} x_n\| \\ &\quad + \|B_{n+1} x_n - B_n x_n\| \\ &\leq \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} + \frac{\alpha_n}{1 - \beta_n} \right) M + \|x_{n+1} - x_n\| + \|B_{n+1} x_n - B_n x_n\| \end{aligned}$$

for some $M > 0$. It turns out that

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} + \frac{\alpha_n}{1 - \beta_n} \right) M + \|B_{n+1} x_n - B_n x_n\|.$$

From conditions (C2), (C3) and Lemma 3.4, we have

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Lemma 2.3 yields that $\|z_n - x_n\| \rightarrow 0$ and hence

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|z_n - x_n\| \rightarrow 0.$$

Step 3. We show that $\lim_{n \rightarrow \infty} \|Bx_n - x_n\| = 0$. Indeed, noting that

$$B_n x_n - x_n = \frac{1}{\gamma_n} \{ (x_{n+1} - x_n) + \alpha_n (x_n - f(x_n)) \},$$

we have, by (C2) and (C3),

$$\lim_{n \rightarrow \infty} \|B_n x_n - x_n\| = 0.$$

Let $B : C \rightarrow C$ be the B -mapping generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$. So, by Lemma 3.2, we have $B_n x \rightarrow Bx$ for all $x \in C$. It also follows that

$$\begin{aligned} \|Bx_n - x_n\| &\leq \|Bx_n - B_n x_n\| + \|B_n x_n - x_n\| \\ &\leq \sup_{z \in \{x_n\}} \|Bz - B_n z\| + \|B_n x_n - x_n\| \\ &\rightarrow 0. \end{aligned}$$

For $t \in (0, 1)$, we define a contraction as follows:

$$S_t x = tf(x) + (1 - t)Bx.$$

Then there exists a unique path $x_t \in C$ such that

$$x_t = tf(x_t) + (1 - t)Bx_t.$$

From Lemma 2.5, we know that $x_t \rightarrow q$ as $t \rightarrow 0$, where $q \in F(B)$. Lemma 3.1(i) also yields that $q \in F(B) = \bigcap_{i=1}^N F(T_i)$. Moreover, q is the unique solution of variational inequality (4.1).

Step 4. We show that $\limsup_{n \rightarrow \infty} \langle f(q) - q, J(x_n - q) \rangle \leq 0$. We see that

$$x_t - x_n = (1 - t)(Bx_t - x_n) + t(f(x_t) - x_n).$$

It follows, by Lemma 2.1(ii) that

$$\begin{aligned} \|x_t - x_n\|^2 &\leq (1 - t)^2 \|Bx_t - x_n\|^2 + 2t \langle f(x_t) - x_n, J(x_t - x_n) \rangle \\ &\leq (1 - 2t + t^2) (\|x_t - x_n\| + \|Bx_n - x_n\|)^2 \\ &\quad + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle + 2t \|x_t - x_n\|^2, \end{aligned}$$

which gives

$$\langle f(x_t) - x_t, J(x_n - x_t) \rangle \leq \frac{(1 + t^2)\|x_n - Bx_n\|}{2t} (2\|x_t - x_n\| + \|x_n - Bx_n\|) + \frac{t}{2}\|x_t - x_n\|^2.$$

So we have

$$\limsup_{n \rightarrow \infty} \langle f(x_t) - x_t, J(x_n - x_t) \rangle \leq \frac{t}{2}M \tag{4.2}$$

for some $M > 0$. Since E has a uniformly Gâteaux differentiable norm, J is norm-to-weak* uniformly continuous on bounded subsets of E . So we have

$$\langle f(q) - q, J(x_n - q) - J(x_n - x_t) \rangle \rightarrow 0 \tag{4.3}$$

and

$$\langle f(q) - f(x_t) + x_t - q, J(x_n - x_t) \rangle \rightarrow 0 \tag{4.4}$$

as $t \rightarrow 0$. On the other hand, we have

$$\begin{aligned} \langle f(q) - q, J(x_n - q) \rangle &= \langle f(x_t) - x_t, J(x_n - x_t) \rangle \\ &\quad + \langle f(q) - f(x_t) + x_t - q, J(x_n - x_t) \rangle \\ &\quad + \langle f(q) - q, J(x_n - q) - J(x_n - x_t) \rangle. \end{aligned} \tag{4.5}$$

Since $\limsup_{n \rightarrow \infty}$ and $\limsup_{t \rightarrow 0}$ are interchangeable, using (4.2)-(4.5), we obtain

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(x_n - q) \rangle \leq 0.$$

Step 5. We show that $x_n \rightarrow q$ as $n \rightarrow \infty$. In fact, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \alpha_n \langle f(x_n) - q, J(x_{n+1} - q) \rangle + \beta_n \langle x_n - q, J(x_{n+1} - q) \rangle \\ &\quad + \gamma_n \langle B_n x_n - q, J(x_{n+1} - q) \rangle \\ &\leq \alpha_n \alpha \|x_n - q\| \|x_{n+1} - q\| + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\ &\quad + \beta_n \|x_n - q\| \|x_{n+1} - q\| + \gamma_n \|x_n - q\| \|x_{n+1} - q\| \\ &= (1 - \alpha_n(1 - \alpha)) \|x_n - q\| \|x_{n+1} - q\| + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\ &\leq \frac{1}{2} (1 - \alpha_n(1 - \alpha)) (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \frac{1 - \alpha_n(1 - \alpha)}{1 + \alpha_n(1 - \alpha)} \|x_n - q\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1 - \alpha)} \langle f(q) - q, J(x_{n+1} - q) \rangle \\ &= \left(1 - \frac{2\alpha_n(1 - \alpha)}{1 + \alpha_n(1 - \alpha)} \right) \|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 + \alpha_n(1 - \alpha)} \langle f(q) - q, J(x_{n+1} - q) \rangle. \end{aligned}$$

Put $c_n = \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}$ and $b_n = \frac{2\alpha_n}{1+\alpha_n(1-\alpha)} \langle f(q) - q, J(x_{n+1} - q) \rangle$. So it is easy to check that $\{c_n\}$ is a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} c_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{b_n}{c_n} \leq 0$. Hence, by Lemma 2.7, we conclude that $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. \square

Competing interests

The author declares that he has no competing interests.

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