Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for N-generalized hybrid mappings in uniformly convex metric spaces

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Abstract

In this paper, we prove some fixed point theorems for *N*-generalized hybrid mappings in both uniformly convex metric spaces and CAT(0) spaces. We also introduce a new iteration method for approximating a fixed point of *N*-generalized hybrid mappings in CAT(0) spaces and obtain Δ-convergence to a fixed point of *N*-generalized hybrid mappings in such spaces. Our results improve and extend the corresponding results existing in the literature.

MSC: 47H09; 47H10

Keywords: fixed point; uniformly convex metric spaces; CAT(0) spaces; generalized hybrid mappings

1 Introduction and preliminaries

Let C be a nonempty closed subset of a metric space (X,d) and let T be a mapping of C into itself. The set of all fixed points of T is denoted by $F(T) = \{x \in C : x = Tx\}$. In 1970, Takahashi [1] introduced the concept of convex metric spaces by using the convex structure as follows.

Definition 1.1 Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a *convex structure* on X if for each $x, y \in X$ and $\lambda \in [0,1]$,

$$d\bigl(z, W(x,y,\lambda)\bigr) \leq \lambda d(z,x) + (1-\lambda)d(z,y)$$

for all $z \in X$. A metric space (X,d) together with a convex structure W is called a *convex metric space* which will be denoted by (X,d,W).

A nonempty subset C of X is said to be *convex* if $W(x,y,\lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$. Clearly, a normed space and each of its convex subsets are convex metric spaces, but the converse does not hold. For each $x, y \in X$ and $\lambda \in [0,1]$, it is known that a convex metric space has the following properties [1, 2]:

(i) $W(x,x,\lambda) = x$, $W(x,y,0) = y$ and $W(x,y,1) = x$;

(ii) $d(x, W(x,y,\lambda)) = (1-\lambda)d(x,y)$ and $d(y, W(x,y,\lambda)) = \lambda d(x,y)$.

In 1996, Shimizu and Takahashi [3] introduced the concept of uniform convexity in convex metric spaces and studied some properties of these spaces. A convex metric space

(X, d, W) is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \le r$, $d(z, y) \le r$ and $d(x, y) \ge r\varepsilon$ imply that $d(z, W(x, y, \frac{1}{2})) \le (1 - \delta_\varepsilon)r$. Obviously, uniformly convex Banach spaces are uniformly convex metric spaces.

Let C be a nonempty closed and convex subset of a convex metric space (X, d, W) and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r\big(x, \{x_n\}\big) = \limsup_{n \to \infty} d(x, x_n).$$

Clearly, $r(\cdot, \{x_n\})$ is a continuous and convex function. The *asymptotic radius* of $\{x_n\}$ relative to C is given by

$$r\big(C, \{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in C\big\},$$

and the *asymptotic center* of $\{x_n\}$ relative to C is the set

$$A\big(C, \{x_n\}\big) = \big\{x \in C : r\big(x, \{x_n\}\big) = r\big(C, \{x_n\}\big)\big\}.$$

It is clear that the asymptotic center $A(C, \{x_n\})$ is always closed and convex. It may either be empty or consist of one or many points. The asymptotic center $A(C, \{x_n\})$ is singleton for uniformly convex Banach spaces [4, 5] or CAT(0) spaces [6]. The following lemma obtained by Phuengrattana and Suantai [7] is useful for our results.

Lemma 1.2 *Let C be a nonempty closed and convex subset of a complete uniformly convex metric space (X, d, W) and let $\{x_n\}$ be a bounded sequence in X. Then $A(C, \{x_n\})$ is a singleton set.*

One of the special spaces of uniformly convex metric spaces is a CAT(0) space; see [8]. It was noted in [9] that any CAT(κ) space ($\kappa > 0$) is uniformly convex in a certain sense but it is not a CAT(0) space. Fixed point theory in CAT(0) spaces was first studied by Kirk [9, 10]. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (*e.g.*, see [11–27]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. Write $c(\alpha 0 + (1 - \alpha)l) = \alpha x \oplus (1 - \alpha)y$ for $\alpha \in (0, 1)$. The space (X, d) is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset Y of X is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a CAT(0) *space* if all geodesic triangles satisfy the following comparison axiom: Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces [8], $\mathbb{R}$-trees [16], the complex Hilbert ball with a hyperbolic metric [5], and many others.

If z, x, y are points in a CAT(0) space and if $m[x, y]$ is the midpoint of the segment $[x, y]$, then the CAT(0) inequality implies

$$d\big(z, m[x, y]\big)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [28], which is equivalent to

$$d\big(z, \lambda x \oplus (1-\lambda)y\big)^2 \leq \lambda d(z, x)^2 + (1-\lambda)d(z, y)^2 - \lambda(1-\lambda)d(x, y)^2 \tag{CN*}$$

for any $\lambda \in [0, 1]$. The (CN*) inequality has appeared in [29]. Moreover, if X is a CAT(0) space and $x, y \in X$, then for any $\lambda \in [0, 1]$, there exists a unique point $\lambda x \oplus (1-\lambda)y \in [x, y]$ such that

$$d\big(z, \lambda x \oplus (1-\lambda)y\big) \leq \lambda d(z, x) + (1-\lambda)d(z, y)$$

for any $z \in X$. It follows that CAT(0) spaces have a convex structure $W(x, y, \lambda) = \lambda x \oplus (1-\lambda)y$.

Remark 1.3

(i) By using the (CN) inequality, it is easy to see that CAT(0) spaces are uniformly convex.
(ii) A geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality; see [8].

In 2012, Dhompongsa *et al.* [12] introduced the following notation in CAT(0) spaces: Let $x_1, \ldots, x_N$ be points in a CAT(0) space X and $\lambda_1, \ldots, \lambda_N \in (0, 1)$ with $\sum_{i=1}^N \lambda_i = 1$, we write

$$\bigoplus_{i=1}^N \lambda_i x_i := (1-\lambda_N)\left(\frac{\lambda_1}{1-\lambda_N}x_1 \oplus \frac{\lambda_2}{1-\lambda_N}x_2 \oplus \cdots \oplus \frac{\lambda_{N-1}}{1-\lambda_N}x_{N-1}\right) \oplus \lambda_N x_N. \tag{1.1}$$

The definition of $\bigoplus$ is an ordered one in the sense that it depends on the order of points $x_1, \ldots, x_N$. Under (1.1) we obtain that

$$d\left(\bigoplus_{i=1}^N \lambda_i x_i, y\right) \leq \sum_{i=1}^N \lambda_i d(x_i, y) \quad \text{for each } y \in X.$$

In 1976, Lim [30] introduced the concept of Δ-convergence in a general metric space. Later in 2008, Kirk and Panyanak [15] extended the concept of Lim to a CAT(0) space.

Definition 1.4 [15] A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-*limit* of $\{x_n\}$.

Lemma 1.5 [15] *Every bounded sequence in a complete* CAT(0) *space has a* Δ-*convergent subsequence*.

For any nonempty subset C of a CAT(0) space X, let $\pi := \pi_C$ be the nearest point projection mapping from X to a subset C of X. In [8], it is known that if C is closed and convex, the mapping π is well defined, nonexpansive, and the following inequality holds:

$$d(x,y)^2 \geq d(x,\pi x)^2 + d(\pi x,y)^2$$

for all $x \in X$ and $y \in C$. By using the same argument as in [31, Lemma 3.2], we can prove the following result for nearest point projection mappings in CAT(0) spaces.

Lemma 1.6 *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X, let $\pi : X \to C$ be the nearest point projection mapping, and let $\{x_n\}$ be a sequence in X. If $d(x_{n+1},p) \leq d(x_n,p)$ for all $p \in C$ and $n \in \mathbb{N}$, then $\{\pi x_n\}$ converges strongly to some element in C.*

Proof Let $m > n$. By the (CN) inequality and the property of π, it follows that

$$\begin{aligned} d(\pi x_m,\pi x_n)^2 &\leq 2d(x_m,\pi x_m)^2 + 2d(x_m,\pi x_n)^2 - 4d\left(x_m, \frac{\pi x_m \oplus \pi x_n}{2}\right)^2 \\ &\leq 2d(x_m,\pi x_m)^2 + 2d(x_m,\pi x_n)^2 - 4d(x_m,\pi x_m)^2 \\ &= 2d(x_m,\pi x_n)^2 - 2d(x_m,\pi x_m)^2 \\ &\leq 2d(x_n,\pi x_n)^2 - 2d(x_m,\pi x_m)^2. \end{aligned} \tag{1.2}$$

This implies that

$$d(x_m,\pi x_m)^2 \leq d(x_n,\pi x_n)^2 \quad \text{for } m > n.$$

Then $\lim_{n\to\infty} d(x_n,\pi x_n)^2$ exists. Letting $m,n \to \infty$ in (1.2), we have that $\{\pi x_n\}$ is a Cauchy sequence in a closed subset C of a complete CAT(0) space X, hence it converges to some element in C. □

Let C be a nonempty closed and convex subset of a Hilbert space H. A mapping $T : C \to C$ is called *generalized hybrid* if there exist $\alpha,\beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x,y \in C$. We note that the generalized hybrid mappings generalize several well-known mappings. For example, a generalized hybrid mapping is nonexpansive for $\alpha = 1$

and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. In 2010, Kocourek *et al.* [32] proved the fixed point theorems for generalized hybrid mappings in Hilbert spaces. Later in 2011, Takahashi and Yao [33] extended the results of Kocourek *et al.* to uniformly convex Banach spaces.

Recently, Maruyama *et al.* [34] introduced a new nonlinear mapping in a Hilbert space as follows. Let $N \in \mathbb{N}$. A mapping $T : C \to C$ is called *N-generalized hybrid* if there are $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N \in \mathbb{R}$ such that

$$\sum_{k=1}^{N} \alpha_k \left\| T^{N+1-k}x - Ty \right\|^2 + \left(1 - \sum_{k=1}^{N} \alpha_k\right) \|x - Ty\|^2$$
$$\leq \sum_{k=1}^{N} \beta_k \left\| T^{N+1-k}x - y \right\|^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) \|x - y\|^2$$

for all $x, y \in C$. They obtained the existence and weak convergence theorems for N-generalized hybrid mappings in Hilbert spaces. Hojo *et al.* [35] also studied the fixed point theorems for N-generalized hybrid mappings in Hilbert spaces and provided an example of N-generalized hybrid mappings which are not generalized hybrid mappings as follows.

Example 1.7 Let H be a Hilbert space, $A = \{x \in H : \|x\| \leq 1\}$ and define a mapping $T : H \to H$ as follows:

$$Tx = \begin{cases} 0 & \text{for all } x \in A; \\ \frac{x}{\|x\|} & \text{for all } x \notin A. \end{cases}$$

We observe that the N-generalized hybrid mappings generalize several well-known mappings, for instance, nonexpansive mappings, nonspreading mappings, hybrid mappings, λ-hybrid mappings, generalized hybrid mappings, and 2-generalized hybrid mappings. Many researchers have studied the fixed point theorems of those mappings in both Hilbert spaces and Banach spaces (*e.g.*, see [32, 33, 36–38]). However, no researcher has studied the fixed point theorems for N-generalized hybrid mappings in more general spaces. So, in this paper, we are interested in studying and extending those mappings to both uniformly convex metric spaces and CAT(0) spaces.

2 Fixed point theorems in uniformly convex metric spaces

We first define N-generalized hybrid mappings in convex metric spaces. Let C be a nonempty subset of a convex metric space (X, d, W). Let $N \in \mathbb{N}$. A mapping $T : C \to C$ is called *N-generalized hybrid* if there are $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N \in \mathbb{R}$ such that

$$\sum_{k=1}^{N} \alpha_k d\left(T^{N+1-k}x, Ty\right)^2 + \left(1 - \sum_{k=1}^{N} \alpha_k\right) d(x, Ty)^2$$
$$\leq \sum_{k=1}^{N} \beta_k d\left(T^{N+1-k}x, y\right)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) d(x, y)^2$$

for all $x, y \in C$. Now, we prove a fixed point theorem for N-generalized hybrid mappings in complete uniformly convex metric spaces.

Theorem 2.1 *Let C be a nonempty closed and convex subset of a complete uniformly convex metric space (X, d, W) and let $T : C \to C$ be an N-generalized hybrid mapping with $\sum_{k=1}^{N} \alpha_k \in (-\infty, 0] \cup [1, \infty)$ and $\sum_{k=1}^{N} \beta_k \in [0, 1]$. Then T has a fixed point if and only if there exists an $x \in C$ such that $\{T^n x\}$ is bounded.*

Proof The necessity is obvious. Conversely, we assume that there exists an $x \in C$ such that $\{T^n x\}$ is bounded. We will show that $F(T)$ is nonempty. From Lemma 1.2, $A(C, \{T^n x\})$ is a singleton set. Let $A(C, \{T^n x\}) = \{z\}$. Since T is N-generalized hybrid, there are $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N \in \mathbb{R}$ such that

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha_k d\bigl(T^{n+N+1-k} x, Tz\bigr)^2 + \left(1 - \sum_{k=1}^{N} \alpha_k\right) d\bigl(T^n x, Tz\bigr)^2 \\
&\quad \le \sum_{k=1}^{N} \beta_k d\bigl(T^{n+N+1-k} x, z\bigr)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) d\bigl(T^n x, z\bigr)^2. \qquad (2.1)
\end{aligned}$$

If $\sum_{k=1}^{N} \alpha_k \in [1, \infty)$ and $\sum_{k=1}^{N} \beta_k \in [0, 1]$, then (2.1) becomes

$$\begin{aligned}
\sum_{k=1}^{N} \alpha_k d\bigl(T^{n+N+1-k} x, Tz\bigr)^2 \le{}& \sum_{k=1}^{N} \beta_k d\bigl(T^{n+N+1-k} x, z\bigr)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) d\bigl(T^n x, z\bigr)^2 \\
&+ \left(\sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(T^n x, Tz\bigr)^2.
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} d\bigl(T^n x, Tz\bigr)^2 \le \limsup_{n\to\infty} d\bigl(T^n x, z\bigr)^2.$$

If $\sum_{k=1}^{N} \alpha_k \in (-\infty, 0]$ and $\sum_{k=1}^{N} \beta_k \in [0, 1]$, then (2.1) becomes

$$\begin{aligned}
\left(1 - \sum_{k=1}^{N} \alpha_k\right) d\bigl(T^n x, Tz\bigr)^2 \le{}& \sum_{k=1}^{N} \beta_k d\bigl(T^{n+N+1-k} x, z\bigr)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) d\bigl(T^n x, z\bigr)^2 \\
&- \sum_{k=1}^{N} \alpha_k d\bigl(T^{n+N+1-k} x, Tz\bigr)^2.
\end{aligned}$$

This implies again that

$$\limsup_{n\to\infty} d\bigl(T^n x, Tz\bigr)^2 \le \limsup_{n\to\infty} d\bigl(T^n x, z\bigr)^2.$$

Therefore, we have

$$r\bigl(Tz, \{T^n x\}\bigr) \le r\bigl(z, \{T^n x\}\bigr).$$

Since $Tz \in C$ and $r(z, \{T^n x\}) = \inf\{r(y, \{T^n x\}) : y \in C\}$, it implies that $Tz = z$. Hence, $F(T)$ is nonempty. □

As a direct consequence of Theorem 2.1, we obtain a fixed point theorem for N-generalized hybrid mappings in uniformly convex metric spaces as follows.

Theorem 2.2 *Let C be a nonempty bounded closed and convex subset of a complete uniformly convex metric space (X,d,W) and let $T:C\to C$ be an N-generalized hybrid mapping with $\sum_{k=1}^{N}\alpha_k\in(-\infty,0]\cup[1,\infty)$ and $\sum_{k=1}^{N}\beta_k\in[0,1]$. Then T has a fixed point.*

We can show that if T is an N-generalized hybrid mapping and $x = Tx$, then for any $y\in C$, we get

$$\sum_{k=1}^{N}\alpha_k d(x,Ty)^2+\left(1-\sum_{k=1}^{N}\alpha_k\right)d(x,Ty)^2\leq\sum_{k=1}^{N}\beta_k d(x,y)^2+\left(1-\sum_{k=1}^{N}\beta_k\right)d(x,y)^2$$

and hence $d(x,Ty)\leq d(x,y)$. This means that an N-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Then, using the methods of the proof of Theorem 1.3 in [13], we can prove the following.

Corollary 2.3 *Let C be a nonempty convex subset of a complete uniformly convex metric space (X,d,W). Suppose that $T:C\to C$ is an N-generalized hybrid mapping and has a fixed point. Then $F(T)$ is closed and convex.*

Remark 2.4

(i) Theorems 2.1 and 2.2 extend and generalize the corresponding results in [17, 32–34, 36–38] to N-generalized hybrid mappings on uniformly convex metric spaces.

(ii) In CAT(0) spaces, if we set $W(x,y,\lambda):=\lambda x\oplus(1-\lambda)y$, then Theorems 2.1 and 2.2 can be applied to these spaces under the assumption that $\sum_{k=1}^{N}\alpha_k\in(-\infty,0]\cup[1,\infty)$ and $\sum_{k=1}^{N}\beta_k\in[0,1]$.

3 Fixed point theorems in CAT(0) spaces

In this section, we study the existence and Δ-convergence theorems for N-generalized hybrid mappings in complete CAT(0) spaces.

We first recall the definition of a Banach limit. Let μ be a continuous linear functional on l^∞, the Banach space of bounded real sequences, and $(a_1,a_2,\dots)\in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_1,a_2,\dots))$. We call μ a *Banach limit* if μ satisfies $\|\mu\|=\mu(1,1,\dots)=1$ and $\mu_n(a_n)=\mu_n(a_{n+1})$ for each $(a_1,a_2,\dots)\in l^\infty$. For a Banach limit μ, we know that $\liminf_{n\to\infty}a_n\leq\mu_n(a_n)\leq\limsup_{n\to\infty}a_n$ for all $(a_1,a_2,\dots)\in l^\infty$. So if $(a_1,a_2,\dots)\in l^\infty$ with $\lim_{n\to\infty}a_n=c$, then $\mu_n(a_n)=c$; see [39] for more details.

Now, we obtain the following lemma in CAT(0) spaces.

Lemma 3.1 *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X, let $\{x_n\}$ be a bounded sequence in X, and let μ be a Banach limit. If a function $f:C\to\mathbb{R}$ is defined by*

$$f(z)=\mu_n d(x_n,z)^2\quad\textit{for all } z\in C,$$

then there exists a unique $z_0\in C$ such that

$$f(z_0)=\min\{f(z):z\in C\}.$$

Proof It is easy to show that f is continuous. By (CN) inequality, we obtain that

$$f\left(\frac{x \oplus y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - \frac{1}{4}d(x,y)^2 \quad \text{for all } x, y \in C.$$

This implies by Proposition 1.7 in [40] that there exists a unique $z_0 \in C$ such that $f(z_0) = \min\{f(z) : z \in C\}$. □

By using Lemma 3.1, we can prove the following fixed point theorem for N-generalized hybrid mappings in CAT(0) spaces without the assumptions that $\sum_{k=1}^{N} \alpha_k \in (-\infty, 0] \cup [1, \infty)$ and $\sum_{k=1}^{N} \beta_k \in [0, 1]$.

Theorem 3.2 *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X and let $T : C \to C$ be an N-generalized hybrid mapping. Then T has a fixed point if and only if there exists an $x \in C$ such that $\{T^n x\}$ is bounded.*

Proof The necessity is obvious. Conversely, we assume that there exists an $x \in C$ such that $\{T^n x\}$ is bounded. Let μ be a Banach limit. Since T is N-generalized hybrid, there are $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \in \mathbb{R}$ such that

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha_k d\bigl(T^{n+N+1-k}x, Tz\bigr)^2 + \left(1 - \sum_{k=1}^{N} \alpha_k\right) d\bigl(T^n x, Tz\bigr)^2 \\
&\quad \leq \sum_{k=1}^{N} \beta_k d\bigl(T^{n+N+1-k}x, z\bigr)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) d\bigl(T^n x, z\bigr)^2
\end{aligned}$$

for any $z \in C$ and $n \in \mathbb{N} \cup \{0\}$. Since $\{T^n x\}$ is bounded, we have

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha_k \mu_n d\bigl(T^{n+N+1-k}x, Tz\bigr)^2 + \left(1 - \sum_{k=1}^{N} \alpha_k\right) \mu_n d\bigl(T^n x, Tz\bigr)^2 \\
&\quad \leq \sum_{k=1}^{N} \beta_k \mu_n d\bigl(T^{n+N+1-k}x, z\bigr)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) \mu_n d\bigl(T^n x, z\bigr)^2.
\end{aligned}$$

This implies that

$$\mu_n d\bigl(T^n x, Tz\bigr)^2 \leq \mu_n d\bigl(T^n x, z\bigr)^2$$

for all $z \in C$. It follows by Lemma 3.1 that $Tz = z$. Hence, $F(T)$ is nonempty. □

As a direct consequence of Theorem 3.2, we obtain a fixed point theorem for N-generalized hybrid mappings in CAT(0) spaces as follows.

Theorem 3.3 *Let C be a nonempty bounded closed and convex subset of a complete* CAT(0) *space X and let $T : C \to C$ be an N-generalized hybrid mapping. Then T has a fixed point.*

Remark 3.4 Theorems 3.2 and 3.3 extend and generalize the corresponding results in [17, 32–34, 36–38] to N-generalized hybrid mappings on CAT(0) spaces.

Next, we study the Δ-convergence theorem for N-generalized hybrid mappings in CAT(0) spaces. Before proving the theorem, we need the following lemma.

Lemma 3.5 *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X and let $T : C \to C$ be an N-generalized hybrid mapping with $\sum_{k=1}^{N} \alpha_k \in (-\infty, 0] \cup [1, \infty)$ and $\sum_{k=1}^{N} \beta_k \in [0, \infty)$. If $\{x_n\}$ is a bounded sequence in C with $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, T^i x_n) = 0$ for all $i = 1, 2, \dots, N$, then $x \in F(T)$.*

Proof Since T is an N-generalized hybrid mapping, there are $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N \in \mathbb{R}$ such that

$$\begin{aligned}&\sum_{k=1}^{N} \alpha_k d\bigl(T^{N+1-k} x_n, Tx\bigr)^2 + \left(1 - \sum_{k=1}^{N} \alpha_k\right) d(x_n, Tx)^2 \\ &\quad\le \sum_{k=1}^{N} \beta_k d\bigl(T^{N+1-k} x_n, x\bigr)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) d(x_n, x)^2. \end{aligned} \tag{3.1}$$

Case 1: $\sum_{k=1}^{N} \alpha_k \in [1, \infty)$ and $\sum_{k=1}^{N} \beta_k \in [0, \infty)$. It follows by (3.1) that

$$\begin{aligned}&\sum_{k=1}^{N} \alpha_k d\bigl(T^{N+1-k} x_n, Tx\bigr)^2 \\ &\quad\le \sum_{k=1}^{N} \beta_k d\bigl(T^{N+1-k} x_n, x\bigr)^2 + \left(1 - \sum_{k=1}^{N} \beta_k\right) d(x_n, x)^2 + \left(\sum_{k=1}^{N} \alpha_k - 1\right) d(x_n, Tx)^2 \\ &\quad\le \sum_{k=1}^{N} \beta_k \bigl(d\bigl(T^{N+1-k} x_n, x_n\bigr)^2 + 2d\bigl(T^{N+1-k} x_n, x_n\bigr) d(x_n, x) + d(x_n, x)^2\bigr) \\ &\qquad + \left(1 - \sum_{k=1}^{N} \beta_k\right) d(x_n, x)^2 + \left(\sum_{k=1}^{N} \alpha_k - 1\right) \bigl(d\bigl(x_n, T^{N+1-k} x_n\bigr)^2 \\ &\qquad + 2d\bigl(x_n, T^{N+1-k} x_n\bigr) d\bigl(T^{N+1-k} x_n, Tx\bigr) + d\bigl(T^{N+1-k} x_n, Tx\bigr)^2\bigr) \\ &\quad= d(x_n, x)^2 + \left(\sum_{k=1}^{N} \beta_k + \sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(T^{N+1-k} x_n, x_n\bigr)^2 \\ &\qquad + 2\sum_{k=1}^{N} \beta_k d\bigl(T^{N+1-k} x_n, x_n\bigr) d(x_n, x) \\ &\qquad + 2\left(\sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(x_n, T^{N+1-k} x_n\bigr) d\bigl(T^{N+1-k} x_n, Tx\bigr) \\ &\qquad + \left(\sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(T^{N+1-k} x_n, Tx\bigr)^2. \end{aligned}$$

This implies that

$$\begin{aligned}&d\bigl(T^{N+1-k} x_n, Tx\bigr)^2 \\ &\quad\le d(x_n, x)^2 + \left(\sum_{k=1}^{N} \beta_k + \sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(T^{N+1-k} x_n, x_n\bigr)^2 \end{aligned}$$

$$+ 2\sum_{k=1}^{N} \beta_k d\bigl(T^{N+1-k}x_n, x_n\bigr) d(x_n, x)$$
$$+ 2\left(\sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(x_n, T^{N+1-k}x_n\bigr) d\bigl(T^{N+1-k}x_n, Tx\bigr).$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, T^i x_n) = 0$ for all $i = 1, 2, \ldots, N$, we have that $\{Tx_n\}, \{T^2x_n\}, \ldots, \{T^N x_n\}$ are bounded. So, we have

$$\begin{aligned}
&d\bigl(T^{N+1-k}x_n, Tx\bigr)^2 \\
&\quad \le d(x_n, x)^2 + \left(\sum_{k=1}^{N} \beta_k + \sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(T^{N+1-k}x_n, x_n\bigr)^2 \\
&\qquad + 2\sum_{k=1}^{N} \beta_k d\bigl(T^{N+1-k}x_n, x_n\bigr) M + 2\left(\sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(x_n, T^{N+1-k}x_n\bigr) M \\
&\quad = d(x_n, x)^2 + \left(\sum_{k=1}^{N} \beta_k + \sum_{k=1}^{N} \alpha_k - 1\right) d\bigl(T^{N+1-k}x_n, x_n\bigr)\bigl(d\bigl(T^{N+1-k}x_n, x_n\bigr) + 2M\bigr),
\end{aligned}$$

where $M = \max_{1\le k\le N} \sup\{d(x_n, x), d(T^{N+1-k}x_n, Tx) : n \in \mathbb{N}\}$.

Case 2: $\sum_{k=1}^{N} \alpha_k \in (-\infty, 0]$ and $\sum_{k=1}^{N} \beta_k \in [0, \infty)$. In the same way as Case 1, we can show that

$$\begin{aligned}
&d\bigl(T^{N+1-k}x_n, Tx\bigr)^2 \\
&\quad \le d(x_n, x)^2 + \left(\sum_{k=1}^{N} \beta_k - \sum_{k=1}^{N} \alpha_k\right) d\bigl(T^{N+1-k}x_n, x_n\bigr)\bigl(d\bigl(T^{N+1-k}x_n, x_n\bigr) + 2M\bigr).
\end{aligned}$$

By Case 1, Case 2, and the assumption $\lim_{n\to\infty} d(x_n, T^i x_n) = 0$ for all $i = 1, 2, \ldots, N$, we obtain

$$\limsup_{n\to\infty} d(x_n, Tx) \le \limsup_{n\to\infty} d(x_n, x).$$

Since $\Delta\text{-}\lim_{n\to\infty} x_n = x$, it follows by the uniqueness of asymptotic centers that $Tx = x$. Hence, $x \in F(T)$. □

Fixed point iteration methods are very useful for approximating a fixed point of various nonlinear mappings such as Mann iteration, Ishikawa iteration, Noor iteration and so on. We now introduce a new iteration method for approximating a fixed point of mappings in a CAT(0) space X as follows: Let C be a nonempty closed and convex subset of X, let $T : C \to C$ be a mapping and $N \in \mathbb{N}$. For $x_1 \in C$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \bigoplus_{i=0}^{N} \lambda_n^{(i)} T^i x_n \quad \text{for all } n \in \mathbb{N}, \tag{3.2}$$

where $\{\lambda_n^{(i)}\}$ is a sequence in $[0, 1]$ for all $i = 0, 1, \ldots, N$ with $\sum_{i=0}^{N} \lambda_n^{(i)} = 1$.

Remark 3.6 If we put

$$W_n^{(N)} = \bigoplus_{i=0}^{N} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{N} \lambda_n^{(j)}} T^i x_n,$$

then by (1.1) we get

$$W_n^{(N)} = \frac{\sum_{j=0}^{N-1} \lambda_n^{(j)}}{\sum_{j=0}^{N} \lambda_n^{(j)}} W_n^{(N-1)} \oplus \frac{\lambda_n^{(N)}}{\sum_{j=0}^{N} \lambda_n^{(j)}} T^N x_n. \tag{3.3}$$

Indeed, we put $\delta_n^{(i,N)} = \frac{\lambda_n^{(i)}}{\sum_{j=0}^{N} \lambda_n^{(j)}}$ for $i = 0, 1, \dots, N$. Thus

$$\begin{aligned}
W_n^{(N)} &= \bigoplus_{i=0}^{N} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{N} \lambda_n^{(j)}} T^i x_n = \bigoplus_{i=0}^{N} \delta_n^{(i,N)} T^i x_n \\
&= \left(1 - \delta_n^{(N,N)}\right)\left(\frac{\delta_n^{(0,N)}}{1 - \delta_n^{(N,N)}} x_n \oplus \frac{\delta_n^{(1,N)}}{1 - \delta_n^{(N,N)}} T x_n \oplus \cdots \oplus \frac{\delta_n^{(N-1,N)}}{1 - \delta_n^{(N,N)}} T^{N-1} x_n \right) \\
&\quad \oplus \delta_n^{(N,N)} T^N x_n \\
&= \left(1 - \delta_n^{(N,N)}\right)\left(\delta_n^{(0,N-1)} x_n \oplus \delta_n^{(1,N-1)} T x_n \oplus \cdots \oplus \delta_n^{(N-1,N-1)} T^{N-1} x_n\right) \oplus \delta_n^{(N,N)} T^N x_n \\
&= \left(1 - \delta_n^{(N,N)}\right)\left(\frac{\lambda_n^{(0)}}{\sum_{j=0}^{N-1} \lambda_n^{(j)}} x_n \oplus \frac{\lambda_n^{(1)}}{\sum_{j=0}^{N-1} \lambda_n^{(j)}} T x_n \oplus \cdots \oplus \frac{\lambda_n^{(N-1)}}{\sum_{j=0}^{N-1} \lambda_n^{(j)}} T^{N-1} x_n \right) \\
&\quad \oplus \delta_n^{(N,N)} T^N x_n \\
&= \left(1 - \delta_n^{(N,N)}\right) W_n^{(N-1)} \oplus \delta_n^{(N,N)} T^N x_n \\
&= \frac{\sum_{j=0}^{N-1} \lambda_n^{(j)}}{\sum_{j=0}^{N} \lambda_n^{(j)}} W_n^{(N-1)} \oplus \frac{\lambda_n^{(N)}}{\sum_{j=0}^{N} \lambda_n^{(j)}} T^N x_n.
\end{aligned}$$

Therefore, (3.3) is justified.

Using Lemma 3.5, we can prove the Δ-convergence theorem for N-generalized hybrid mappings in complete CAT(0) spaces as follows.

Theorem 3.7 *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X and let $T : C \to C$ be an N-generalized hybrid mapping with $F(T) \neq \emptyset$ and $\sum_{k=1}^{N} \alpha_k \in (-\infty, 0] \cup [1, \infty)$ and $\sum_{k=1}^{N} \beta_k \in [0, \infty)$. Let $\pi : C \to F(T)$ be the nearest point projection mapping. Suppose that $\{x_n\}$ is a sequence in C defined by* (3.2) *with $0 < a \leq \lambda_n^{(i)} \leq b < 1$ for all $i = 0, 1, \dots, N$. Then $\{x_n\}$ Δ-converges to a fixed point u of T, where $u = \lim_{n\to\infty} \pi x_n$.*

Proof Since T is N-generalized hybrid and $F(T) \neq \emptyset$, we get T is quasi-nonexpansive. Then, for $p \in F(T)$, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(\bigoplus_{i=0}^{N} \lambda_n^{(i)} T^i x_n, p \right) \\
&\leq \sum_{i=0}^{N} \lambda_n^{(i)} d\left(T^i x_n, p\right)
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=0}^{N} \lambda_n^{(i)} d(x_n,p) \\
&= d(x_n,p).
\end{aligned}$$

Therefore, $\lim_{n\to\infty} d(x_n,p)$ exists and hence $\{x_n\}$ is bounded.

For each $p \in F(T)$, we obtain, by (3.2), (3.3), and the (CN*) inequality, that

$$\begin{aligned}
&d(x_{n+1},p)^2 \\
&\quad= d\left(\bigoplus_{i=0}^{N} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{N}\lambda_n^{(j)}} T^i x_n, p\right)^2 = d\bigl(W_n^{(N)},p\bigr)^2 \\
&\quad= d\left(\frac{\sum_{j=0}^{N-1}\lambda_n^{(j)}}{\sum_{j=0}^{N}\lambda_n^{(j)}} W_n^{(N-1)} \oplus \frac{\lambda_n^{(N)}}{\sum_{j=0}^{N}\lambda_n^{(j)}} T^N x_n, p\right)^2 \\
&\quad\le \frac{\sum_{j=0}^{N-1}\lambda_n^{(j)}}{\sum_{j=0}^{N}\lambda_n^{(j)}} d\bigl(W_n^{(N-1)},p\bigr)^2 + \frac{\lambda_n^{(N)}}{\sum_{j=0}^{N}\lambda_n^{(j)}} d\bigl(T^N x_n,p\bigr)^2 \\
&\qquad - \frac{\lambda_n^{(N)}}{\sum_{j=0}^{N}\lambda_n^{(j)}} \frac{\sum_{j=0}^{N-1}\lambda_n^{(j)}}{\sum_{j=0}^{N}\lambda_n^{(j)}} d\bigl(W_n^{(N-1)},T^N x_n\bigr)^2 \\
&\quad= \sum_{j=0}^{N-1}\lambda_n^{(j)} d\bigl(W_n^{(N-1)},p\bigr)^2 + \lambda_n^{(N)} d\bigl(T^N x_n,p\bigr)^2 - \lambda_n^{(N)} \sum_{j=0}^{N-1}\lambda_n^{(j)} d\bigl(W_n^{(N-1)},T^N x_n\bigr)^2 \\
&\quad= \sum_{j=0}^{N-1}\lambda_n^{(j)} d\left(\frac{\sum_{j=0}^{N-2}\lambda_n^{(j)}}{\sum_{j=0}^{N-1}\lambda_n^{(j)}} W_n^{(N-2)} \oplus \frac{\lambda_n^{(N-1)}}{\sum_{j=0}^{N-1}\lambda_n^{(j)}} T^{N-1} x_n, p\right)^2 + \lambda_n^{(N)} d\bigl(T^N x_n,p\bigr)^2 \\
&\qquad - \lambda_n^{(N)} \sum_{j=0}^{N-1}\lambda_n^{(j)} d\bigl(W_n^{(N-1)},T^N x_n\bigr)^2 \\
&\quad\le \sum_{j=0}^{N-1}\lambda_n^{(j)} \left(\frac{\sum_{j=0}^{N-2}\lambda_n^{(j)}}{\sum_{j=0}^{N-1}\lambda_n^{(j)}} d\bigl(W_n^{(N-2)},p\bigr)^2 + \frac{\lambda_n^{(N-1)}}{\sum_{j=0}^{N-1}\lambda_n^{(j)}} d\bigl(T^{N-1}x_n,p\bigr)^2\right. \\
&\qquad \left. - \frac{\sum_{j=0}^{N-2}\lambda_n^{(j)}}{\sum_{j=0}^{N-1}\lambda_n^{(j)}} \frac{\lambda_n^{(N-1)}}{\sum_{j=0}^{N-1}\lambda_n^{(j)}} d\bigl(W_n^{(N-2)},T^{N-1}x_n\bigr)^2\right) + \lambda_n^{(N)} d\bigl(T^N x_n,p\bigr)^2 \\
&\qquad - \lambda_n^{(N)} \sum_{j=0}^{N-1}\lambda_n^{(j)} d\bigl(W_n^{(N-1)},T^N x_n\bigr)^2 \\
&\quad= \sum_{j=0}^{N-2}\lambda_n^{(j)} d\bigl(W_n^{(N-2)},p\bigr)^2 + \lambda_n^{(N-1)} d\bigl(T^{N-1}x_n,p\bigr)^2 + \lambda_n^{(N)} d\bigl(T^N x_n,p\bigr)^2 \\
&\qquad - \frac{\lambda_n^{(N-1)}\sum_{j=0}^{N-2}\lambda_n^{(j)}}{\sum_{j=0}^{N-1}\lambda_n^{(j)}} d\bigl(W_n^{(N-2)},T^{N-1}x_n\bigr)^2 - \lambda_n^{(N)} \sum_{j=0}^{N-1}\lambda_n^{(j)} d\bigl(W_n^{(N-1)},T^N x_n\bigr)^2 \\
&\quad\le \sum_{j=0}^{N-3}\lambda_n^{(j)} d\bigl(W_n^{(N-3)},p\bigr)^2 + \lambda_n^{(N-2)} d\bigl(T^{N-2}x_n,p\bigr)^2 + \lambda_n^{(N-1)} d\bigl(T^{N-1}x_n,p\bigr)^2 \\
&\qquad + \lambda_n^{(N)} d\bigl(T^N x_n,p\bigr)^2 - \frac{\lambda_n^{(N-2)}\sum_{j=0}^{N-3}\lambda_n^{(j)}}{\sum_{j=0}^{N-2}\lambda_n^{(j)}} d\bigl(W_n^{(N-3)},T^{N-2}x_n\bigr)^2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda_n^{(N-1)} \sum_{j=0}^{N-2} \lambda_n^{(j)}}{\sum_{j=0}^{N-1} \lambda_n^{(j)}} d\big(W_n^{(N-2)}, T^{N-1}x_n\big)^2 - \lambda_n^{(N)} \sum_{j=0}^{N-1} \lambda_n^{(j)} d\big(W_n^{(N-1)}, T^N x_n\big)^2 \\
&\vdots \\
&\leq \lambda_n^{(0)} d\big(W_n^{(0)}, p\big)^2 + \sum_{k=1}^{N} \lambda_n^{(k)} d\big(T^k x_n, p\big)^2 - \sum_{k=1}^{N} \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}} d\big(W_n^{(k-1)}, T^k x_n\big)^2 \\
&\leq \sum_{k=0}^{N} \lambda_n^{(k)} d(x_n, p)^2 - \sum_{k=1}^{N} \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}} d\big(W_n^{(k-1)}, T^k x_n\big)^2 \\
&= d(x_n, p)^2 - \sum_{k=1}^{N} \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}} d\big(W_n^{(k-1)}, T^k x_n\big)^2.
\end{aligned}
$$

This implies that

$$
\sum_{k=1}^{N} \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}} d\big(W_n^{(k-1)}, T^k x_n\big)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.
$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists and $0 < a \leq \lambda_n^{(i)} \leq b < 1$ for all $i = 0, 1, \ldots, N$, we get that

$$
\lim_{n\to\infty} d(x_n, Tx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(W_n^{(k-1)}, T^k x_n\big) = 0 \quad \text{for all } k = 2, 3, \ldots, N. \tag{3.4}
$$

For $k = 2, 3, \ldots, N$, we have

$$
\begin{aligned}
d\big(x_n, T^k x_n\big) &\leq d\big(x_n, W_n^{(k-1)}\big) + d\big(W_n^{(k-1)}, T^k x_n\big) \\
&= d\left(x_n, \bigoplus_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{k-1} \lambda_n^{(j)}} T^i x_n\right) + d\big(W_n^{(k-1)}, T^k x_n\big) \\
&\leq \sum_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{k-1} \lambda_n^{(j)}} d\big(x_n, T^i x_n\big) + d\big(W_n^{(k-1)}, T^k x_n\big) \\
&= \sum_{i=1}^{k-1} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{k-1} \lambda_n^{(j)}} d\big(x_n, T^i x_n\big) + d\big(W_n^{(k-1)}, T^k x_n\big).
\end{aligned}
$$

This implies by (3.4) that

$$
\lim_{n\to\infty} d\big(x_n, T^k x_n\big) = 0 \quad \text{for all } k = 1, 2, \ldots, N. \tag{3.5}
$$

We now let $\omega_\Delta(x_n) := \bigcup A(C, \{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_\Delta(x_n) \subset F(T)$. Let $u \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(C, \{u_n\}) = \{u\}$. By Lemma 1.5, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} u_{n_k} = y \in C$. It implies by (3.5) and Lemma 3.5 that $y \in F(T)$. Then, $\lim_{n\to\infty} d(x_n, y)$ exists. Suppose that $u \neq y$. By the uniqueness of asymptotic centers, we get

$$
\begin{aligned}
\limsup_{k\to\infty} d(u_{n_k}, y) &< \limsup_{k\to\infty} d(u_{n_k}, u) \\
&\leq \limsup_{n\to\infty} d(u_n, u)
\end{aligned}
$$

$$\begin{aligned} &< \limsup_{n\to\infty} d(u_n, y) \\ &= \limsup_{n\to\infty} d(x_n, y) \\ &= \limsup_{k\to\infty} d(u_{n_k}, y). \end{aligned}$$

This is a contradiction, hence $u = y \in F(T)$. This shows that $\omega_\Delta(x_n) \subset F(T)$.

Next, we show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(C, \{u_n\}) = \{u\}$ and let $A(C, \{x_n\}) = \{z\}$. Since $u \in \omega_\Delta(x_n) \subset F(T)$, it follows that $\lim_{n\to\infty} d(x_n, u)$ exists. We will show that $z = u$. To show this, suppose not. By the uniqueness of asymptotic centers, we get

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, z) \\ &\le \limsup_{n\to\infty} d(x_n, z) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

which is a contradiction, and so $z = u$. Hence, $\{x_n\}$ Δ-converges to a fixed point u of T. Since $F(T)$ is a closed convex subset of X and $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in F(T)$ and $n \in \mathbb{N}$, we obtain by Lemma 1.6 that $\{\pi x_n\}$ converges strongly to some element in $F(T)$, say q. Thus, by the property of π, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, q) &\le \limsup_{n\to\infty} \bigl(d(x_n, \pi x_n) + d(\pi x_n, q)\bigr) \\ &= \limsup_{n\to\infty} d(x_n, \pi x_n) \\ &\le \limsup_{n\to\infty} d(x_n, u). \end{aligned}$$

This implies, by the uniqueness of asymptotic centers, that $q = u$. This means $u = \lim_{n\to\infty} \pi x_n$. □

Taking $N = 2$ in Theorem 3.7, we obtain the following Δ-convergence theorem of a 2-generalized hybrid mapping in CAT(0) spaces.

Theorem 3.8 *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X. Let $T : C \to C$ be a* 2*-generalized hybrid mapping, i.e., there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that*

$$\begin{aligned} &\alpha_1 d\bigl(T^2x, Ty\bigr)^2 + \alpha_2 d(Tx, Ty)^2 + (1 - \alpha_1 - \alpha_2) d(x, Ty)^2 \\ &\quad \le \beta_1 d\bigl(T^2x, y\bigr)^2 + \beta_2 d(Tx, y)^2 + (1 - \beta_1 - \beta_2) d(x, y)^2 \end{aligned}$$

for all $x, y \in C$. Assume that $F(T) \neq \emptyset$ and $\alpha_1 + \alpha_2 \in (-\infty, 0] \cup [1, \infty)$ and $\beta_1 + \beta_2 \in [0, \infty)$. Let $\pi : C \to F(T)$ be the nearest point projection mapping. For $x_1 \in C$, let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \bigoplus_{i=0}^{2} \lambda_n^{(i)} T^i x_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\lambda_n^{(i)}\}$ *is a sequence in* $[0,1]$ *with* $0 < a \leq \lambda_n^{(i)} \leq b < 1$ *for all* $i = 0,1,2$ *and* $\sum_{i=0}^{2} \lambda_n^{(i)} = 1$. *Then* $\{x_n\}$ Δ*-converges to a fixed point* u *of* T, *where* $u = \lim_{n\to\infty} \pi x_n$.

Taking $N = 1$ in Theorem 3.7, we obtain the following Δ-convergence theorem of a generalized hybrid mapping in CAT(0) spaces.

Theorem 3.9 *Let* C *be a nonempty closed and convex subset of a complete* CAT(0) *space* X. *Let* $T : C \to C$ *be a generalized hybrid mapping, i.e., there are* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\alpha d(Tx, Ty)^2 + (1-\alpha)d(x, Ty)^2 \leq \beta d(Tx, y)^2 + (1-\beta)d(x, y)^2$$

for all $x, y \in C$. *Assume that* $F(T) \neq \emptyset$ *and* $\alpha \in (-\infty, 0] \cup [1, \infty)$ *and* $\beta \in [0, \infty)$. *Let* $\pi : C \to F(T)$ *be the nearest point projection mapping. For* $x_1 \in C$, *let* $\{x_n\}$ *be a sequence defined by*

$$x_{n+1} = \lambda_n^{(0)} x_n \oplus \lambda_n^{(1)} Tx_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\lambda_n^{(0)}\}$ *and* $\{\lambda_n^{(1)}\}$ *are sequences in* $[0,1]$ *with* $0 < a \leq \lambda_n^{(0)}, \lambda_n^{(1)} \leq b < 1$ *and* $\lambda_n^{(0)} + \lambda_n^{(1)} = 1$. *Then* $\{x_n\}$ Δ*-converges to a fixed point* u *of* T, *where* $u = \lim_{n\to\infty} \pi x_n$.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
The author would like to thank the referees for valuable suggestions on the paper.

Received: 22 March 2013 Accepted: 4 July 2013 Published: 18 July 2013

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doi:10.1186/1687-1812-2013-188
Cite this article as: Phuengrattana: **Fixed point theorems for *N*-generalized hybrid mappings in uniformly convex metric spaces.** *Fixed Point Theory and Applications* 2013 **2013**:188.