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The split common fixed point problem for asymptotically nonexpansive semigroups in Banach spaces

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Abstract

In this paper, we propose an iteration method for finding a split common fixed point of asymptotically nonexpansive semigroups in the setting of two Banach spaces, and we obtain some weak and strong convergence theorems of the iteration scheme proposed. The results presented in the paper are new and improve and extend some recent corresponding results.

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1 Introduction

Let E be a real normed linear space and C be a nonempty closed convex subset of E . The mapping $T : C \rightarrow C$ is said to be nonexpansive if for all $x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\|. \quad (1.1)$$

The mapping $T : C \rightarrow C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that for all $x, y \in C$ and each $n \geq 1$

$$\|T^n x - T^n y\| \leq k_n \|x - y\|. \quad (1.2)$$

The class of nonexpansive mappings is one of the most important classes of mappings in nonlinear science. The class of asymptotically nonexpansive mappings is an important generalization of the class of nonexpansive mappings, which was introduced by Goebel and Kirk [1] in 1972. They proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping, then T has a fixed point.

Example 1.1 ([2]) Let C be a unit ball in a real Hilbert space l^2 and let $S : C \rightarrow C$ be a mapping defined by

$$S : (x_1, x_2, \dots) \rightarrow (0, x_1^2, a_2 x_2, a_3 x_3, \dots),$$

It is proved in Goebal and Kirk [1] that:

- (i) $\|Sx - Sy\| \leq 2\|x - y\|$, for all $x, y \in C$;
- (ii) $\|S^n x - S^n y\| \leq 2 \prod_{j=2}^n a_j \|x - y\|$, for all $x, y \in C, \forall n \geq 2$.

Taking $a_j = 2^{-\frac{1}{2^{j-1}}}, j \geq 2$, it is easy to see that $\prod_{j=2}^\infty a_j = \frac{1}{2}$. So we can take $k_1 = 2$ and $k_n = 2 \prod_{j=2}^n a_j, n \geq 2$, then

$$\lim_{n \rightarrow \infty} k_n = \lim_{n \rightarrow \infty} 2 \prod_{j=2}^n 2^{-\frac{1}{2^{j-1}}} = 1. \tag{1.3}$$

Therefore S is an asymptotically nonexpansive mapping from C into itself with $F(S) = \{(0, 0, \dots, 0, \dots)\}$.

Definition 1.2 ([3]) A one-parameter family $\mathcal{F} := \{T(t) : t \geq 0\}$ of E into itself is called a strongly continuous semigroup of Lipschitzian mappings on E if it satisfies the following conditions:

- (i) $T(0)x = x$, for all $x \in E$;
- (ii) $T(s + t) = T(s)T(t)$, for all $s, t \geq 0$;
- (iii) for each $x \in E$, the mapping $t \mapsto T(t)x$ is continuous;
- (iv) for each $t > 0$, there exists a bounded measurable function $L(t) : [0, \infty) \rightarrow [0, \infty)$ such that

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \quad \text{for all } x, y \in E. \tag{1.4}$$

If the bounded measurable function $L(t)$ in (1.4) is such that $L(t) \geq 1$ for each $t > 0$, $L(t)$ is nonincreasing in t , and $\lim_{t \rightarrow \infty} L(t) = 1$, then the strong continuous semigroup of Lipschitzian mappings is said to be an asymptotically nonexpansive semigroup. We denote by $F(\mathcal{F})$ the set of all common fixed points of \mathcal{F} , that is,

$$F(\mathcal{F}) := \{x \in E : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t \geq 0} F(T(t)). \tag{1.5}$$

If \mathcal{F} satisfies (i)-(iii) and

$$\limsup_{t \rightarrow \infty, x \in D} \|T(t)x - T(s)T(t)x\| = 0, \quad \text{for all } s > 0 \text{ and bounded } D \subseteq C, \tag{1.6}$$

then \mathcal{F} is called uniformly asymptotically regular on C .

Example 1.3 ([4] (Example of asymptotically nonexpansive semigroup)) Let E be a uniformly convex Banach space which admits a weakly continuous duality mapping. Let $L(E)$ be the space of all bounded linear operators on E . For $\Psi \in L(E)$, define $\mathcal{F} := \{T(t) : t \in R^+\}$ of bounded linear operators by using the following exponential expression:

$$T(t) = e^{-t\Psi} := \sum_{k=0}^\infty \frac{(-1)^k}{k!} t^k \Psi^k.$$

Then the family $\mathcal{F} := \{T(t) : t \in R^+\}$ satisfies the semigroup properties. Moreover, this family forms a one-parameter semigroup of self-mappings of E because $e^{t\Psi} = [e^{-t\Psi}]^{-1} : E \rightarrow E$ exists for each $t \in R^+$.

In 1994, in finite dimensional Hilbert spaces, Censor and Elfving [5] introduced the split feasibility problem for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [6]. It has been found that split feasibility problems can be used in various disciplines, such as image restoration, computer tomograph and radiation therapy treatment planning [7–9].

Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2 , respectively. The split feasibility problem is formulated as finding a point $q \in H_1$ such that

$$q \in C \quad \text{and} \quad Aq \in Q, \quad (1.7)$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator.

If (1.7) has a solution, it is not hard to see that $x \in C$ solves (1.7) if and only if it solves the following fixed point equation:

$$x = P_C(I - \gamma A^*(I - P_Q)A)x, \quad x \in C, \quad (1.8)$$

where P_C and P_Q are the projections onto C and Q , respectively, γ is a positive constant, and A^* denotes the adjoint of A .

When C and Q in (1.7) are the sets of fixed points of two nonlinear mappings, and C and Q are nonempty closed convex subsets of H_1 and H_2 , respectively, then the split feasibility problem (1.7) is also said to be split common fixed point problem [10]. It is well known that each nonempty closed convex subset of a Hilbert space is the set of fixed points of its projection, therefore, the split common fixed point problem may be considered as a generalization of split feasibility problem.

In the setting of two Hilbert spaces, for demicontractive mappings, Moudafi [10] proposed an iteration scheme and obtained a weak convergence theorem of the split common fixed point problem. Since then, the split common fixed point problems of other nonlinear mappings in the setting of two Hilbert spaces have been studied by some authors; see, for instance, [2, 11–14]. Especially, Choleamjiak *et al.* [15] obtained a strong convergence theorem of split common fixed point problem involving a uniformly asymptotically regular nonexpansive semigroup and a total asymptotically strict pseudo-contractive mapping in Hilbert spaces.

In 2015, in the setting of one Hilbert space and one Banach space, Takahashi [16] investigated the split feasibility problem and split common null point problem, and obtained some strong and weak convergence theorems under some mild control conditions.

Recently, in the setting of two Banach spaces, Tang *et al.* [17] obtained a weak convergence theorem and a strong convergence theorem of the split common fixed point problem involving a quasi-strict pseudo-contractive mapping and an asymptotically nonexpansive mapping under the following assumptions:

- (1) E_1 is a real uniformly convex and 2-uniformly smooth Banach space having the Opial property and the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$.
- (2) E_2 is a real Banach space.
- (3) $A : E_1 \rightarrow E_2$ be a bounded linear operator and A^* is the adjoint of A .
- (4) $S : E_1 \rightarrow E_1$ is an $\{l_n\}$ -asymptotically nonexpansive mapping with $\{l_n\} \subset (1, \infty)$ and $l_n \rightarrow 1$. $T : E_2 \rightarrow E_2$ is a τ -quasi-strict pseudo-contractive mapping with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, and T is demiclosed at zero.

Theorem 1.4 ([17]) *Let E_1, E_2, A, S, T , and $\{l_n\}$ be the same as above. For each $x_1 \in E_1$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2 (T - I) A x_n, \\ x_{n+1} = (1 - \alpha_n) z_n + \alpha_n S^n(z_n), \quad \forall n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, γ is a positive constant satisfying $0 < \gamma < \min\{\frac{1-2k^2}{\|A\|^2}, \frac{1-\tau}{\|A\|^2}\}$, $\{l_n\}$ is a sequence in $[0, \infty)$ with $L = \sup_{n \geq 1} \{l_n\}$ and $\sum_{n=1}^{\infty} (l_n - 1) < \infty$.

- (I) *If $\Gamma = \{p \in F(S) : Ap \in F(T)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.*
- (II) *In addition, if $\Gamma = \{p \in F(S) : Ap \in F(T)\} \neq \emptyset$ and S is semi-compact, then $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$.*

This naturally brings about the following question:

Question Can we obtain the convergence results of split common fixed point problem for asymptotically nonexpansive semigroups in the setting of two Banach spaces?

In this paper, motivated and inspired by the recent research going on in the direction of split feasibility problems and split common fixed point problems, we construct an iteration scheme to approximate a split common fixed point of two asymptotically nonexpansive semigroups in the setting of two Banach spaces. Under some suitable conditions on parameters, the iteration scheme proposed is shown to converge strongly and weakly to a split common fixed point of asymptotically nonexpansive semigroups in two Banach spaces.

2 Preliminaries

We now recall some definitions and elementary facts which will be used in the proofs of our main results.

Let E be a real Banach space with the dual E^* . The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^* .

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x+y) \right\| : \|x\|, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\},$$

for all $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. A Hilbert space is 2-uniformly convex, while L^p is $\max\{p, 2\}$ -uniformly convex for every $p > 1$.

Let $\rho_E : [0, \infty) \rightarrow [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2} (\|x+y\| + \|x-y\|) - 1 : x \in U, \|y\| \leq t \right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \rightarrow 0$ as $t \rightarrow 0$. A typical example of uniformly smooth Banach space is L^p , where $p > 1$. More precisely, L^p is $\min\{p, 2\}$ -uniformly smooth for every $p > 1$. Let q be a fixed real number with $q > 1$, then a Banach space E is said to be q -uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$. It is well known that every q -uniformly smooth Banach space is uniformly smooth.

Lemma 2.1 ([18]) *Given a number $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|),$$

for all $x, y \in E, t \in [0, 1]$, with $\|x\| \leq r$ and $\|y\| \leq r$.

Let E_1 and E_2 be two real Banach spaces, C and Q be nonempty closed convex subsets of E_1 and E_2 , respectively. $A : E_1 \rightarrow E_2$ is a bounded linear operator such that $A \neq 0$. Then the split common fixed point problem is to find a point $q \in E_1$ with the property:

$$q \in F(S) \quad \text{and} \quad Aq \in F(T), \quad \text{i.e., } q \in F(S) \cap A^{-1}F(T), \tag{2.2}$$

where $F(S)$ and $F(T)$ denote the sets of fixed points of S and T , respectively.

Let $T : C \rightarrow C$ be a mapping with $F(T) \neq \emptyset$. Then T is said to be demiclosed at zero if for any $\{x_n\} \subset C$ with $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \rightarrow 0, x = Tx$.

A mapping $T : C \rightarrow C$ is said to be semi-compact, if for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \rightarrow 0 (n \rightarrow \infty)$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$.

A Banach space E is said to satisfy Opial property if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$, for any $y \in E$ with $y \neq x$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|. \tag{2.3}$$

Lemma 2.2 ([19]) *Let E be a real uniformly convex Banach space, C be a nonempty closed subset of E , and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\} \subset C$ converges weakly to a point $p \in C$ and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then $p = Tp$.*

Lemma 2.3 ([20]) *Let $\{a_n\}$ and $\{\alpha_n\}$ be two nonnegative real number sequences and satisfy*

$$a_{n+1} \leq (1 + \alpha_n)a_n, \quad \forall n \geq 1,$$

where $a_n \geq 0, \alpha_n \geq 0$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then

- (1) $\lim_{n \rightarrow \infty} a_n$ exists;
- (2) if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4 ([18]) *Let E be a 2-uniformly smooth Banach space with the best smoothness constants $K > 0$. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + 2\|Ky\|^2, \quad \forall x, y \in E. \tag{2.4}$$

3 Main results

Theorem 3.1 *Let E_1 be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial’s condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, E_2 be a real Banach space, and $A : E_1 \rightarrow E_2$ be a bounded linear operator and A^* be the adjoint of A . Let $\{S(t) : t \geq 0\} : E_1 \rightarrow E_1$ be a uniformly asymptotically regular asymptotically nonexpansive semigroup with a bounded measurable function $L^{(1)}(t) : [0, \infty) \rightarrow [1, \infty)$ satisfying $\lim_{t \rightarrow \infty} L^{(1)}(t) = 1$ and $C := \bigcap_{t \geq 0} F(S(t)) \neq \emptyset$, $\{T(t) : t \geq 0\} : E_2 \rightarrow E_2$ be a uniformly asymptotically regular family of asymptotically nonexpansive semigroup with a bounded measurable function $L^{(2)}(t) : [0, \infty) \rightarrow [1, \infty)$ satisfying $\lim_{t \rightarrow \infty} L^{(2)}(t) = 1$ and $Q := \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$, respectively. Let $\{x_n\}$ be a sequence generated by $x_1 \in E_1$,*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2 (T(t_n) - I) A x_n, & \forall n \geq 1, \\ x_{n+1} = (1 - \alpha_n) z_n + \alpha_n S(t_n)(z_n), \end{cases} \tag{3.1}$$

where $\{t_n\}$ is a sequence of real numbers, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and γ is a positive constant satisfying

- (1) $t_n > 0$ and $\lim_{n \rightarrow \infty} t_n = \infty$;
- (2) $L(t) = \max\{L^{(1)}(t), L^{(2)}(t)\}$ and $\sum_{n=1}^{\infty} (L^2(t_n) - 1) < \infty$;
- (3) $M = \sup_n L^2(t_n)$, $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $0 < \gamma < \min\{\frac{1-2k^2}{\|A\|^2 M}, \frac{1}{\|A\|^2}\}$.
- (I) If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$.
- (II) In addition, if $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$ and there exists at least one $S(t) \in \{S(t) : t \geq 0\}$ that is semi-compact, then $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

Proof Now we prove the conclusion (I).

We shall divide the proof into four steps.

Step 1. We first show that the limit $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for any $p \in \Gamma$.

For any given $p \in \Gamma$, then $p \in C$ and $Ap \in Q$. It follows from (1.4), (3.1), and Lemma 2.4 that

$$\begin{aligned} \|z_n - p\|^2 &= \|(x_n - p) + \gamma J_1^{-1} A^* J_2 (T(t_n) - I) A x_n\|^2 \\ &\leq \|\gamma J_1^{-1} A^* J_2 (T(t_n) - I) A x_n\|^2 + 2\gamma \langle x_n - p, A^* J_2 (T(t_n) - I) A x_n \rangle \\ &\quad + 2k^2 \|x_n - p\|^2 \\ &\leq \gamma^2 \|A\|^2 \|(T(t_n) - I) A x_n\|^2 + 2\gamma \langle A x_n - Ap, J_2 (T(t_n) - I) A x_n \rangle \\ &\quad + 2k^2 \|x_n - p\|^2 \\ &= \gamma^2 \|A\|^2 \|(T(t_n) - I) A x_n\|^2 + 2k^2 \|x_n - p\|^2 \\ &\quad + 2\gamma \langle A x_n - T(t_n) A x_n + T(t_n) A x_n - T(t_n) Ap, J_2 (T(t_n) - I) A x_n \rangle \end{aligned}$$

$$\begin{aligned}
 &= \gamma^2 \|A\|^2 \|(T(t_n) - I)Ax_n\|^2 - 2\gamma \|Ax_n - T(t_n)Ax_n\|^2 \\
 &\quad + 2\gamma \langle T(t_n)Ax_n - T(t_n)Ap, J_2(T(t_n) - I)Ax_n \rangle + 2k^2 \|x_n - p\|^2 \\
 &\leq \gamma^2 \|A\|^2 \|(T(t_n) - I)Ax_n\|^2 - 2\gamma \|(T(t_n) - I)Ax_n\|^2 \\
 &\quad + \gamma [\|T(t_n)Ax_n - T(t_n)Ap\|^2 + \|(T(t_n) - I)Ax_n\|^2] + 2k^2 \|x_n - p\|^2 \\
 &\leq \gamma(\gamma \|A\|^2 - 1) \|(T(t_n) - I)Ax_n\|^2 + \gamma \|A\|^2 L^2(t_n) \|x_n - p\|^2 + 2k^2 \|x_n - p\|^2 \\
 &= (\gamma \|A\|^2 L^2(t_n) + 2k^2) \|x_n - p\|^2 - \gamma(1 - \gamma \|A\|^2) \|(T(t_n) - I)Ax_n\|^2. \tag{3.2}
 \end{aligned}$$

It follows from (3.1), (3.2), and Lemma 2.1 that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|z_n - p + \alpha_n(S(t_n)z_n - z_n)\|^2 \\
 &\leq (1 - \alpha_n) \|z_n - p\|^2 + \alpha_n \|S(t_n)z_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|z_n - S(t_n)z_n\|) \\
 &\leq (1 - \alpha_n) \|z_n - p\|^2 + \alpha_n L^2(t_n) \|z_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|z_n - S(t_n)z_n\|) \\
 &= (1 + \alpha_n(L^2(t_n) - 1)) \|z_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|z_n - S(t_n)z_n\|) \\
 &\leq (1 + \alpha_n(L^2(t_n) - 1)) [(\gamma \|A\|^2 L^2(t_n) + 2k^2) \|x_n - p\|^2 \\
 &\quad - \gamma(1 - \gamma \|A\|^2) \|(T(t_n) - I)Ax_n\|^2] - \alpha_n(1 - \alpha_n)g(\|z_n - S(t_n)z_n\|) \\
 &\leq (1 + \alpha_n(L^2(t_n) - 1)) (\gamma \|A\|^2 M + 2k^2) \|x_n - p\|^2 \\
 &\quad - (1 + \alpha_n(L^2(t_n) - 1)) \gamma(1 - \gamma \|A\|^2) \|(T(t_n) - I)Ax_n\|^2 \\
 &\quad - \alpha_n(1 - \alpha_n)g(\|z_n - S(t_n)z_n\|). \tag{3.3}
 \end{aligned}$$

Since $\lim_{t_n \rightarrow \infty} L(t_n) = 1$, $\sum_{n=1}^{\infty} (L^2(t_n) - 1) < \infty$, $0 < k < \frac{1}{\sqrt{2}}$, $0 < \gamma < \frac{1-2k^2}{\|A\|^2 M}$, so $0 < \gamma \|A\|^2 M + 2k^2 < 1$, and from (3.3) and Lemma 2.3 we see that the $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded. Further, it follows from (3.2) that $\{z_n\}$ is bounded, too.

Step 2. We prove that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0$.

It follows from (3.3) that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq (1 + \alpha_n(L^2(t_n) - 1)) (\gamma \|A\|^2 M + 2k^2) \|x_n - p\|^2 \\
 &\quad - \gamma(1 - \gamma \|A\|^2) \|(T(t_n) - I)Ax_n\|^2 - \alpha_n(1 - \alpha_n)g(\|z_n - S(t_n)z_n\|). \tag{3.4}
 \end{aligned}$$

From (3.4), we have

$$\begin{aligned}
 &\gamma(1 - \gamma \|A\|^2) \|(T(t_n) - I)Ax_n\|^2 + \alpha_n(1 - \alpha_n)g(\|z_n - S(t_n)z_n\|) \\
 &\leq (1 + \alpha_n(L^2(t_n) - 1)) (\gamma \|A\|^2 L^2(t_n) + 2k^2) \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
 &= \|x_n - p\|^2 + \alpha_n(L^2(t_n) - 1) (\gamma \|A\|^2 L^2(t_n) + 2k^2) \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{3.5}
 \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \|(T(t_n) - I)Ax_n\| = 0 \tag{3.6}$$

and

$$\lim_{n \rightarrow \infty} g(\|z_n - S(t_n)z_n\|) = 0. \tag{3.7}$$

By virtue of Lemma 2.1 and the property of g , we may get

$$\lim_{n \rightarrow \infty} \|z_n - S(t_n)z_n\| = 0. \tag{3.8}$$

It follows from (3.1) that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1 - \alpha_n)z_n + \alpha_n S(t_n)z_n - x_n\| \\ &= \|(1 - \alpha_n)[x_n + \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n] + \alpha_n S(t_n)z_n - x_n\| \\ &= \|(1 - \alpha_n)\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n + \alpha_n(S(t_n)z_n - x_n)\| \\ &= \|(1 - \alpha_n)\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n + \alpha_n(S(t_n)z_n - z_n) + \alpha_n(z_n - x_n)\| \\ &= \|(1 - \alpha_n)\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n + \alpha_n(S(t_n)z_n - z_n) \\ &\quad + \alpha_n \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n\| \\ &= \|\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n + \alpha_n(S(t_n)z_n - z_n)\| \\ &\leq \|\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n\| + \alpha_n \|S(t_n)z_n - z_n\|. \end{aligned} \tag{3.9}$$

It follows from (3.6), (3.8), and (3.9) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

In addition, since

$$\begin{aligned} \|(T(t_n) - I)Ax_{n+1}\| &= \|(T(t_n) - I)Ax_{n+1} - (T(t_n) - I)Ax_n + (T(t_n) - I)Ax_n\| \\ &\leq (L(t_n) + 1)\|A\|\|x_{n+1} - x_n\| + \|(T(t_n) - I)Ax_n\|, \end{aligned}$$

it follows from (3.6) and (3.10) that

$$\lim_{n \rightarrow \infty} \|(T(t_n) - I)Ax_{n+1}\| = 0. \tag{3.11}$$

Similarly,

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|x_{n+1} + \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_{n+1} - x_n - \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n\| \\ &\leq \|(x_{n+1} - x_n) + \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_{n+1} - \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_{n+1}\| + \|\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n\|. \end{aligned} \tag{3.12}$$

In view of (3.6), (3.10), (3.11), and (3.12), we have

$$\lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0. \tag{3.13}$$

In addition,

$$\|x_n - z_n\| = \|J_1(x_n - z_n)\| = \|\gamma A^* J_2(T(t_n) - I)Ax_n\| \leq \gamma \|A\| \|(T(t_n) - I)Ax_n\|, \tag{3.14}$$

from (3.6), we have

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \tag{3.15}$$

Step 3. We prove that $\lim_{n \rightarrow \infty} \|z_n - S(t)z_n\| = 0$ and $\lim_{n \rightarrow \infty} \|(T(t) - I)Az_n\| = 0$ for all $t \geq 0$.

Since $\{S(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ are uniformly asymptotically regular, and $\lim_{n \rightarrow \infty} t_n = \infty$, for all $t \geq 0$,

$$\lim_{n \rightarrow \infty} \|S(t)S(t_n)z_n - S(t_n)z_n\| \leq \limsup_{n \rightarrow \infty, x \in C} \|S(t)S(t_n)x - S(t_n)x\| = 0 \tag{3.16}$$

and

$$\begin{aligned} &\lim_{n \rightarrow \infty} \|T(t)(T(t_n) - I)Az_n - (T(t) - I)Az_n\| \\ &\leq \limsup_{n \rightarrow \infty, x \in C} \|T(t)(T(t_n) - I)Ax - (T(t) - I)Ax\| = 0. \end{aligned} \tag{3.17}$$

Since $\{S(t)x\}$ is continuous on t for all $x \in E_1$, and

$$\begin{aligned} \|z_n - S(t)z_n\| &\leq \|z_n - S(t_n)z_n\| + \|S(t_n)z_n - S(t)S(t_n)z_n\| \\ &\quad + \|S(t)S(t_n)z_n - S(t)z_n\|, \end{aligned} \tag{3.18}$$

it follows from (3.6) and (3.16) that

$$\lim_{n \rightarrow \infty} \|z_n - S(t)z_n\| = 0. \tag{3.19}$$

Similarly,

$$\lim_{n \rightarrow \infty} \|(T(t) - I)Ax_n\| = 0. \tag{3.20}$$

Step 4. We prove that $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.

By the reflexivity of Banach space E_1 and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to x^* . By using (3.15) this implies that $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to x^* , too. Since $S(t)$ is asymptotically nonexpansive for all $t \geq 0$, it is demiclosed at zero, we know from Lemma 2.2 that $x^* \in F(S(t))$.

On the other hand, since A is a bounded linear operator, we know that $\{Ax_{n_i}\}$ converges weakly to Ax^* . It follows from (3.20) that $\lim_{n_i \rightarrow \infty} \|(T(t) - I)Ax_{n_i}\| = 0$. Since $T(t)$ is demiclosed at zero for all $t \geq 0$, we have $Ax^* \in F(T(t))$. This means that $x^* \in \Gamma$.

We now prove that $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

In fact, if there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $y^* \in E_1$, we also know that $y^* \in F(T(t))$. By the assumption that E_1 satisfies Opial's condition, we have

$$\begin{aligned} \liminf_{n_i \rightarrow \infty} \|x_{n_i} - x^*\| &< \liminf_{n_i \rightarrow \infty} \|x_{n_i} - y^*\| \\ &= \liminf_{n \rightarrow \infty} \|x_n - y^*\| \\ &= \liminf_{n_j \rightarrow \infty} \|x_{n_j} - y^*\| \\ &< \liminf_{n_i \rightarrow \infty} \|x_{n_i} - x^*\| \\ &= \liminf_{n \rightarrow \infty} \|x_n - x^*\| = \liminf_{n_i \rightarrow \infty} \|x_{n_i} - x^*\|. \end{aligned} \tag{3.21}$$

This is a contradiction. Therefore $\{x_n\}$ converges weakly to $x^* \in \Gamma$. The proof of conclusion (I) is completed.

Next, we prove conclusion (II).

Since there exists at least one $S(t) \in \{S(t) : t \geq 0\}$ that is semi-compact and $\lim_{n \rightarrow \infty} \|z_n - S(t)z_n\| = 0$ for all $t \geq 0$, there exists subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $\{z_{n_j}\}$ converges strongly to $\mu^* \in E_1$. By using (3.15) again, we know that the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges strongly to μ^* . Due to $\{x_n\}$ converging weakly to x^* , we obtain $\mu^* = x^*$. Since $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists and $\lim_{n \rightarrow \infty} \|x_{n_j} - x^*\| = 0$, we know that $\{x_n\}$ converges strongly to $x^* \in \Gamma$. This completes the proof of conclusion (II). \square

Corollary 3.2 *Let E_1 be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, E_2 be a real Banach space, and $A : E_1 \rightarrow E_2$ be a bounded linear operator and A^* is the adjoint of A , $\{S(t) : t \geq 0\} : E_1 \rightarrow E_1$ and $\{T(t) : t \geq 0\} : E_2 \rightarrow E_2$ be two-parameter nonexpansive semigroups satisfying $C := \bigcap_{t \geq 0} F(S(t)) \neq \emptyset$ and $Q := \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$, respectively. Let $\{x_n\}$ be a sequence generated by $x_1 \in E_1$,*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n, & \forall n \geq 1, \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n S(t_n)(z_n), \end{cases} \tag{3.22}$$

where $\{t_n\}$ is a sequence of real numbers satisfying $t_n > 0$ and $\lim_{n \rightarrow \infty} t_n = \infty$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, γ is a positive constant satisfying $0 < \gamma < \frac{1-2k^2}{\|A\|^2}$.

- (I) If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$.
- (II) In addition, if $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$ and there exists at least one $S(t) \in \{S(t) : t \geq 0\}$ that is semi-compact, then $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

4 Application to hierarchical variational inequality problem in Banach spaces

Let E be a strictly convex and real reflexive Banach space and K be a nonempty closed and convex subset of E . Then, for any $x \in E$, there exists a unique element $z \in K$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in K$. Putting $z = P_K x$, we call P_K the metric projection of E onto K .

Lemma 4.1 ([21]) *Let E be a smooth, strictly convex, and reflexive Banach space, and K be a nonempty, closed, and convex subset of E . Let $x \in E$ and $z \in K$. Then the following conditions are equivalent:*

- (i) $z = P_Kx$;
- (ii) $\langle z - y, J(x - z) \rangle \geq 0, \forall y \in K$, where J is the normalized duality mapping on E .

Definition 4.2 ([17]) *Let E be a smooth, strictly convex, and reflexive Banach space, and K be a nonempty, closed and convex subset of E . Let $S : K \rightarrow K$ be a nonlinear mapping with $F(S)$ being a nonempty, closed, and convex subset of K and $V : K \rightarrow K$ be a nonlinear mapping. The so-called hierarchical variational inequality problem for a mapping S with respect to a mapping V in Banach spaces is to find $x \in F(S)$ such that*

$$\langle x^* - x, J(Vx^* - x^*) \rangle \geq 0, \quad \forall x \in F(S). \tag{4.1}$$

By Lemma 4.1, the hierarchical variational inequality problem in Banach space (4.1) is equivalent to the following fixed point equation:

$$x^* = P_{F(S)}V(x^*). \tag{4.2}$$

Letting $C = F(S)$ and $Q = F(P_{F(S)}V)$ (the fixed point set of $P_{F(S)}V$) and $A = I$ (the identity mapping on E), then the hierarchical variational inequality problem (4.1) for a mapping S with respect to a mapping V in Banach space is equivalent to the following split common fixed point problem in Banach space:

$$\text{to find } x \in C \text{ such that } x \in Q. \tag{4.3}$$

Therefore the set of solutions of hierarchical variational inequality problem (4.1) is just the set of solutions of split common fixed point problem (4.3).

In Theorem 3.1, we take $E_1 = E_2 = E$, $A = I$, $T(t) = P_{F(S(t))}V(t)$, $J_1 = J_2 = J$ (where J is the normalized duality mapping on E), the following conclusion can be obtained from Theorem 3.1 immediately.

Theorem 4.3 *Let E be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$. Let $\{S(t) : t \geq 0\} : E \rightarrow E$ be uniformly asymptotically regular family of asymptotically nonexpansive semigroup with a bounded measurable function $L^{(1)}(t) : [0, \infty) \rightarrow [1, \infty)$ satisfying $\lim_{t \rightarrow \infty} L^{(1)}(t) = 1$ and $C := \bigcap_{t \geq 0} F(S(t)) \neq \emptyset$. Let $\{V(t) : t \geq 0\} : E \rightarrow E$ be a semigroup and $\{T(t) : t \geq 0\} := \{P_{F(S(t))}V(t) : t \geq 0\}$ be uniformly asymptotically regular family of asymptotically nonexpansive semigroup with a bounded measurable function $L^{(2)}(t) : [0, \infty) \rightarrow [1, \infty)$ satisfying $\lim_{t \rightarrow \infty} L^{(2)}(t) = 1$ and $Q := \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in E$,*

$$\begin{cases} z_n = x_n + \gamma(T(t_n) - I)x_n, & \forall n \geq 1, \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n S(t_n)(z_n), \end{cases} \tag{4.4}$$

where $\{t_n\}$ is a sequence of real numbers, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and γ is a positive constant satisfying

- (1) $t_n > 0$ and $\lim_{n \rightarrow \infty} t_n = \infty$;
- (2) $L(t) = \max\{L^{(1)}(t), L^{(2)}(t)\}$ and $\sum_{n=1}^{\infty} (L^2(t_n) - 1) < \infty$;
- (3) $M = \sup_n L^2(t_n)$, $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, and $0 < \gamma < \min\{\frac{1-2k^2}{\|A\|^2 M}, \frac{1}{\|A\|^2}\}$.
- (I) If $\Gamma_1 \neq \emptyset$ (the set of solutions of hierarchical variational inequality problem (4.1) is nonempty), then the sequence $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma_1$.
- (II) In addition, if $\Gamma_1 = \{p \in C : Ap \in Q\} \neq \emptyset$ and there exists at least one $S(t) \in \{S(t) : t \geq 0\}$ that is semi-compact, then $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma_1$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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