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Equivalence results between Nash equilibrium theorem and some fixed point theorems

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Abstract

We show that the Kakutani and Brouwer fixed point theorems can be obtained by directly using the Nash equilibrium theorem. The corresponding set-valued problems, such as the Kakutani fixed point theorem, Walras equilibrium theorem (set-valued excess demand function), and generalized variational inequality, can be derived from the Nash equilibrium theorem, with the aid of an inverse of the Berge maximum theorem. For the single-valued situation, we derive the Brouwer fixed point theorem, Walras equilibrium theorem (single-valued excess demand function), KKM lemma, and variational inequality from the Nash equilibrium theorem directly, without any recourse.

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Keywords: Brouwer fixed point theorem; Kakutani fixed point theorem; Nash equilibrium theorem; Walras equilibrium theorem; KKM lemma; variational inequality

1 Introduction

It is well known that fixed point theorems play an important role in game theory and mathematical economics [1–3]. Nash [4] firstly defined the best response correspondence and applied the Berge maximum theorem and Kakutani fixed point theorem to prove the existence of Nash equilibrium points in finite games, where finitely many players may choose from a finite number of pure strategies in finite-dimensional Euclidean spaces. Later, Debreu [5] extended finite games to noncooperative games with nonlinear payoff functions and obtained the following equilibrium theorem.

Theorem 1.1 (see [6, 7]) *Let $N = \{1,\dots,n\}$ be a finite set of players. For each $i \in N$, X_i is a nonempty, convex, and compact subset of the n_i-dimensional Euclidean space, $f_i : X := \prod_{i\in N} X_i \longrightarrow \mathbb{R}$ is continuous, and $f_i(x_i,x_{-i})$ is quasi-concave in x_i for any x_{-i}, where $-i = N\backslash\{i\}$. Then, there exists $x^* \in X$ such that*

$$f_i\bigl(x_i^*,x_{-i}^*\bigr) = \max_{u_i\in X_i} f_i\bigl(u_i,x_{-i}^*\bigr), \quad \forall i \in N.$$

Such x^* is called an equilibrium of the game $\Gamma = (X_1,\dots,X_n;f_1,\dots,f_n)$.

In recent years, a great deal of mathematical effort has been devoted to prove the equivalence between the KKM principle and several fixed point theorems or minimax inequalities. Park [8] showed a sequence of equivalent formulations for the KKM principle in abstract convex spaces. From the statements of [8, 9] we know that the fixed point theorem, minimax inequility, and Nash equilibrium theorem can be derived from the KKM principle. However, to the best of our knowledge, there is no proof for the Kakutani and Brouwer fixed point theorems via the Nash equilibrium theorem, although we can find in the previous literature many proofs or equivalent results for these two theorems [2, 8, 9]. In this paper, we fill these gaps. In Section 2, we show that the Kakutani fixed point theorem, Walras equilibrium theorem (set-valued excess demand function), and generalized variational inequality can be derived from the Nash equilibrium theorem with the aid of an inverse of the Berge maximum theorem [10, 11]. In Section 3, for the single-valued situation, we derive the Brouwer fixed point theorem, Walras equilibrium theorem (single-valued excess demand function), KKM lemma, and variational inequality from the Nash equilibrium theorem directly, without any recourse.

2 Kakutani fixed point theorem via Nash equilibrium theorem

To obtain the Kakutani fixed point theorem from the Nash equilibrium theorem, we need an inverse of the Berge maximum theorem.

Theorem 2.1 (Berge maximum theorem) (see [2, 6]) *Let X be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$, and Y be a subset of the m-dimensional Euclidean space $\mathbb{R}^m$. Let $u: X \times Y \longrightarrow \mathbb{R}$ be continuous, and let $S: X \rightrightarrows Y$ be continuous and nonempty compact-valued. Then, the correspondence $K: X \rightrightarrows Y$ defined by*

$$K(x) = \Big\{ y \in S(x) : u(x,y) = \max_{z \in S(x)} u(x,z) \Big\}, \quad \forall x \in X,$$

is upper semicontinuous and compact-valued.

In 1997, Komiya [10] considered an inverse of the Berge maximum theorem, and Zhou [11] gave a simple alternative proof.

Theorem 2.2 (Inverse of Berge maximum theorem) *Let X be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$, and $K: X \rightrightarrows \mathbb{R}^m$ be a nonempty convex compact-valued and upper semicontinuous correspondence. Then there exists a continuous function $v: X \times \mathbb{R}^m \longrightarrow [0,1]$ such that*

(i) $K(x) = \{ y \in \mathbb{R}^m : v(x,y) = \max_{z \in \mathbb{R}^m} v(x,z) \}$, $\forall x \in X$;
(ii) $v(x,y)$ *is quasi-concave in y for any $x \in X$.*

We begin by proving the following results.

2.1 Kakutani fixed point theorem

Komiya [10] showed that the Kakutani fixed point theorem can be derived from the existence theorem of maximal elements with the aid of Theorem 2.2. However, in this section, by using different methods, we derive the Kakutani fixed point theorem.

Theorem 2.3 (Kakutani fixed point theorem) *Let X be a nonempty, convex, bounded, and closed subset of $\mathbb{R}^n$, and $F : X \rightrightarrows X$ be a nonempty convex compact-valued and upper semicontinuous correspondence. Then, there exists $x^* \in X$ such that $x^* \in F(x^*)$.*

Proof We apply Theorem 2.2 to find a continuous function $f : X \times \mathbb{R}^n \longrightarrow [0,1]$ such that

$$F(x) = \Bigl\{ y \in \mathbb{R}^n : f(x,y) = \max_{z \in \mathbb{R}^n} f(x,z) \Bigr\}, \quad \forall x \in X,$$

and $f(x,y)$ is quasi-concave in y for any $x \in X$. Since $F(x) \subset X$,

$$F(x) = \Bigl\{ y \in X : f(x,y) = \max_{z \in X} f(x,z) \Bigr\}.$$

Next, define the mapping $g : X \times X \longrightarrow \mathbb{R}$ by

$$g(x,y) = -\|x - y\|.$$

Obviously, g is continuous on $X \times X$, and $g(x,\cdot)$ is concave on X for any $x \in X$.

For the game $\Gamma = (X,X;f,g)$, by Theorem 1.1 there exists $(x^*,y^*) \in X \times X$ such that

$$f\bigl(x^*,y^*\bigr) = \max_{y \in X} f\bigl(x^*,y\bigr),$$

$$g\bigl(x^*,y^*\bigr) = -\bigl\|x^* - y^*\bigr\| = \max_{x \in X}\bigl[-\bigl\|x - y^*\bigr\|\bigr] = -\min_{x \in X}\bigl\|x - y^*\bigr\| = 0.$$

Therefore, $y^* \in F(x^*)$ and $x^* = y^*$, which implies $x^* \in F(x^*)$. This completes the proof. □

2.2 Walras equilibrium theorem (set-valued excess demand function)

Walras equilibrium may be formulated as follows. Let there be n commodities, and $P \subset \mathbb{R}^n$ be the set of all price vectors,

$$P = \left\{ (p_1,\dots,p_n) \in \mathbb{R}^n : p_i \ge 0, \sum_{i=1}^{n} p_i = 1 \right\}.$$

The excess demand function $\zeta(p) = (\zeta_1(p),\dots,\zeta_n(p))$ is a correspondence from P to $\mathbb{R}^n$. A price vector $p^* \in P$ is an equilibrium if there exists $z^* \in \zeta(p^*)$ such that

$$z_i^* \le 0, \quad \forall i = 1,\dots,n.$$

Theorem 2.4 (Walras equilibrium theorem) *Let an excess demand function $\zeta(p)$ satisfy the following conditions*:

(i) *$\zeta : P \rightrightarrows \mathbb{R}^n$ is a nonempty convex compact-valued and upper semicontinuous correspondence*;

(ii) *the weak Walras law holds*:

$$\langle p,z \rangle \le 0, \quad \forall p \in P, \forall z \in \zeta(p).$$

Then there exists at least one equilibrium p^*, that is, there exists $z^* \in \zeta(p^*)$ such that

$$z_i^* \le 0, \quad \forall i = 1,\dots,n.$$

Proof Let $Z = co\zeta(P)$, where $co\zeta(P)$ is the convex hull. Corollary 5.33 and Lemma 17.84 of [12] yield that Z is a nonempty, convex, and compact subset of $\mathbb{R}^n$. We apply Theorem 2.2 to find a continuous function $f : P \times Z \longrightarrow [0,1]$ such that

$$\zeta(p) = \Bigl\{z \in Z : f(p,z) = \max_{y\in Z} f(p,y)\Bigr\}, \quad \forall p \in P,$$

and $f(p,z)$ is quasi-concave in z for any $p \in P$.

Next, define the mapping $g : P \times Z \longrightarrow \mathbb{R}$ by

$$g(p,z) = \langle p,z\rangle.$$

Obviously, g is continuous on $P \times Z$, and $g(p,\cdot)$ is concave for any $p \in P$.

For the game $\Gamma = (Z,P;f,g)$, by Theorem 1.1 there exists $(z^*,p^*) \in Z \times P$ such that

$$f\bigl(p^*,z^*\bigr) = \max_{z\in Z} f\bigl(p^*,z\bigr),$$

$$g\bigl(p^*,z^*\bigr) = \bigl\langle p^*,z^*\bigr\rangle = \max_{p\in P}\bigl\langle p,z^*\bigr\rangle.$$

Therefore, $z^* \in \zeta(p^*)$. From the weak Walras law we have

$$0 \geq \bigl\langle p^*,z^*\bigr\rangle = \max_{p\in P}\bigl\langle p,z^*\bigr\rangle,$$

that is,

$$\bigl\langle p,z^*\bigr\rangle \leq 0, \quad \forall p \in P.$$

We conclude that

$$z_i^* \leq 0, \quad \forall i = 1,\ldots,n.$$

Otherwise, there is $i_0 \in \{1,\ldots,n\}$ such that $z_{i_0}^* > 0$. Let $\overline{q} \in P$ with $\overline{q}_{i_0} = 1$ and $\overline{q}_i = 0$ for any $i \neq i_0$. Then

$$\bigl\langle \overline{q},z^*\bigr\rangle = z_{i_0}^* > 0,$$

which is a contradiction. □

2.3 Generalized variational inequality

In 1968, Browder [13] first gave the generalized variational inequality, which plays a very important role in game theory and nonlinear analysis (see, for example, [6] and the references therein). Here we show that the generalized variational inequality can be derived from the Nash equilibrium theorem with the aid of Theorem 2.2 as follows.

Theorem 2.5 (Generalized variational inequality) *Let X be a nonempty, convex, bounded, and closed subset of $\mathbb{R}^n$, and $F : X \rightrightarrows \mathbb{R}^n$ be a nonempty convex compact-valued and upper semicontinuous correspondence. Then, there exist $x^* \in X$ and $u^* \in F(x^*)$ such that*

$$\bigl\langle u^*,y - x^*\bigr\rangle \geq 0, \quad \forall y \in X.$$

Proof Let $U = coF(X)$, where $coF(X)$ is the convex hull. Corollary 5.33 and Lemma 17.8 of [12] yield that U is a nonempty, convex, and compact subset of $\mathbb{R}^n$. We apply Theorem 2.2 to find a continuous function $f : X \times U \longrightarrow [0,1]$ such that

$$F(x) = \left\{u \in U : f(x,u) = \max_{z\in U} f(x,z)\right\}, \quad \forall x \in X,$$

and $f(x,u)$ is quasi-concave in u for any $x \in X$.

Next, define two mappings $g, h : U \times X \times X \longrightarrow \mathbb{R}$ by

$$g(x,y) = -\|x-y\|, \quad \forall (x,y) \in X \times X,$$

$$h(u,x,y) = \langle u, x-y\rangle, \quad \forall (u,x,y) \in U \times X \times X.$$

Obviously, g, h are continuous on $U \times X \times X$, and $g(\cdot,y)$ and $h(u,x,\cdot)$ are concave for any $x \in X$ and any $u \in U$.

For the game $\Gamma = (U,X,X;f,g,h)$, by Theorem 1.1 there exists $(u^*,x^*,y^*) \in U \times X \times X$ such that

$$f(x^*,u^*) = \max_{z\in U} f(x^*,z),$$

$$g(x^*,y^*) = -\|x^*-y^*\| = \max_{x\in X}\left[-\|x-y^*\|\right] = -\min_{x\in X}\|x-y^*\| = 0,$$

$$h(u^*,x^*,y^*) = \langle u^*,x^*-y^*\rangle = \max_{y\in X}\langle u^*,x^*-y\rangle.$$

Therefore, $u^* \in F(x^*)$, $x^* = y^*$ and

$$0 = \langle u^*,x^*-x^*\rangle = \langle u^*,x^*-y^*\rangle = \max_{y\in X}\langle u^*,x^*-y\rangle,$$

that is, $u^* \in F(x^*)$ and

$$\langle u^*,y-x^*\rangle \geq 0, \quad \forall y \in X.$$

This completes the proof. □

3 Brouwer fixed point theorem via Nash equilibrium theorem

In this section, we apply only the Nash equilibrium theorem to conclude the Brouwer fixed point theorem and related problems, without recourse to the inverse of the Berge maximum theorem.

3.1 Brouwer fixed point theorem

Theorem 3.1 (Brouwer fixed point theorem) *Let X be a nonempty, convex, bounded, and closed subset of $\mathbb{R}^n$, and φ be a continuous function from X to itself. Then, there exists $x^* \in X$ such that $x^* = \varphi(x^*)$.*

Proof Define two mappings $f, g : X \times X \longrightarrow \mathbb{R}$ by

$$f(x,y) = -\|x-y\|,$$

$$g(x,y) = -\|y - \varphi(x)\|.$$

Obviously, f, g are continuous on $X \times X$, and $f(\cdot,y)$ and $g(x,\cdot)$ are concave for any $x \in X$ and any $y \in X$.

For the game $\Gamma = (X,X;f,g)$, by Theorem 1.1 there exists $(x^*,y^*) \in X \times X$ such that

$$f(x^*,y^*) = -\|x^* - y^*\| = \max_{x\in X}[-\|x - y^*\|] = -\min_{x\in X}\|x - y^*\| = 0,$$

$$g(x^*,y^*) = -\|y^* - \varphi(x^*)\| = \max_{y\in X}[-\|y - \varphi(x^*)\|] = -\min_{y\in X}\|y - \varphi(x^*)\| = 0.$$

Therefore, $x^* = y^*$ and $y^* = \varphi(x^*)$, that is, $x^* = \varphi(x^*)$. This completes the proof. □

3.2 Walras equilibrium theorem (single-valued excess demand function)

Following the statement of Section 2.2, the Walras equilibrium theorem for a single-valued excess demand function can be obtained from the Nash equilibrium theorem. The excess demand function $\zeta(p) = (\zeta_1(p),\ldots,\zeta_n(p))$ is a function from P to R^n. A price vector $p^* \in P$ is an equilibrium if

$$\zeta_i(p^*) \leq 0, \quad \forall i = 1,\ldots,n.$$

Theorem 3.2 (Walras equilibrium theorem) *Let an excess demand function $\zeta(p)$ satisfy the following conditions*:

(i) *$\zeta(p)$ is a continuous function from P to R^n*;

(ii) *The Weak Walras law holds*:

$$\langle\zeta(p),p\rangle \leq 0, \quad \forall p \in P.$$

Then there exists at least one equilibrium p^*, that is, there exists $p^* \in P$ such that

$$\zeta_i(p^*) \leq 0, \quad \forall i = 1,\ldots,n.$$

Proof Define two mappings $f,g : P \times P \longrightarrow \mathbb{R}$ by

$$f(p,q) = -\|p - q\|,$$

$$g(p,q) = \langle q,\zeta(p)\rangle.$$

Obviously, f, g are continuous on $P \times P$, and $f(\cdot,q)$ and $g(p,\cdot)$ are concave for any $p \in P$ and any $q \in P$.

For the game $\Gamma = (P,P;f,g)$, by Theorem 1.1 there exists $(p^*,q^*) \in P \times P$ such that

$$f(p^*,q^*) = -\|p^* - q^*\| = \max_{p\in P}[-\|p - q^*\|] = -\min_{p\in P}\|p - q^*\| = 0,$$

$$g(p^*,q^*) = \langle q^*,\zeta(p^*)\rangle = \max_{q\in P}\langle q,\zeta(p^*)\rangle.$$

Therefore, $p^* = q^*$, and by the weak Walras law we have

$$0 \geq \langle p^*,\zeta(p^*)\rangle = \langle q^*,\zeta(p^*)\rangle = \max_{q\in P}\langle q,\zeta(p^*)\rangle,$$

that is,

$$\langle q, \zeta(p^*)\rangle \le 0, \quad \forall q \in P.$$

We conclude

$$\zeta_i(p^*) \le 0, \quad \forall i = 1, \dots, n.$$

Otherwise, there is $i_0 \in \{1, \dots, n\}$ such that $\zeta_{i_0}(p^*) > 0$. Let $\overline{q} \in P$ with $\overline{q}_{i_0} = 1$ and $\overline{q}_i = 0$ for any $i \neq i_0$. Then

$$\langle \overline{q}, \zeta(p^*)\rangle = \zeta_{i_0}(p^*) > 0,$$

which is a contradiction. □

3.3 KKM lemma

The KKM lemma is a very basic theorem, and the Brouwer fixed point theorem can be obtained by this lemma. The proof can be found in [6, 7]. We still derive the KKM lemma from the Nash equilibrium theorem.

Theorem 3.3 (KKM lemma) *Let*

$$\Delta = co\{e^0, \dots, e^m\} \subset \mathbb{R}^{m+1},$$

and let $\{F_0, \dots, F_m\}$ be a family of closed subsets of Δ such that, for any $A \subset \{0, \dots, m\}$,

$$co\{e^i : i \in A\} \subset \bigcup_{i\in A} F_i.$$

Then

$$\bigcap_{i=0}^{m} F_i \neq \emptyset.$$

Proof For any $x = \sum_{i=0}^m x_i e^i \in \Delta$, $y = \sum_{i=0}^m y_i e^i \in \Delta$, where $x_i \ge 0$, $\sum_{i=0}^m x_i = 1$, $y_i \ge 0$, $\sum_{i=0}^m y_i = 1$, define two mappings $f, g : \Delta \times \Delta \longrightarrow \mathbb{R}$ by

$$f(x,y) = -\|x-y\|,$$

$$g(x,y) = \sum_{i=0}^{m} y_i d(x, F_i),$$

where $d(x, F_i)$ is the distance from a point x to the set F_i. Obviously, f, g are continuous on $\Delta \times \Delta$, and $f(\cdot, y)$ and $g(x, \cdot)$ are concave for any $x \in \Delta$ and any $y \in \Delta$.

For the game $\Gamma = (\Delta, \Delta; f, g)$, by Theorem 1.1 there exists $(x^*, y^*) \in \Delta \times \Delta$ such that

$$f(x^*, y^*) = -\|x^* - y^*\| = \max_{x\in X}\left[-\|x - y^*\|\right] = -\min_{x\in X}\|x - y^*\| = 0,$$

$$g(x^*,y^*) = \sum_{i=0}^{m} y_i^* d(x^*,F_i) = \max_{y\in\Delta} \sum_{i=0}^{m} y_i d(x^*,F_i) = \max_{i=0,\ldots,m} d(x^*,F_i).$$

Therefore, $x^* = y^*$ and

$$\sum_{i=0}^{m} x_i^* d(x^*,F_i) = \sum_{i=0}^{m} y_i^* d(x^*,F_i) = \max_{i=0,\ldots,m} d(x^*,F_i).$$

Let $I(x^*) = \{i : x_i^* > 0\}$. Then $I(x^*) \neq \emptyset$ and

$$\sum_{i\in I(x^*)} x_i^* d(x^*,F_i) = \sum_{i=0}^{m} x_i^* d(x^*,F_i) = \max_{i=0,\ldots,m} d(x^*,F_i).$$

It must be

$$d(x^*,F_i) = \max_{i=0,\ldots,m} d(x^*,F_i), \quad \forall i \in I(x^*).$$

Additionally, since

$$x^* \in co\{e^i : i \in I(x^*)\} \subset \bigcup_{i\in I(x^*)} F_i,$$

there exists $i_0 \in I(x^*)$ such that $x^* \in F_{i_0}$, which implies

$$\max_{i=0,\ldots,m} d(x^*,F_i) = d(x^*,F_{i_0}) = 0,$$

that is, $d(x^*,F_i) = 0$ for all $i = 0,\ldots,m$. Since F_i is a closed set, it follows that $x^* \in F_i$. Therefore,

$$x^* \in \bigcap_{i=0}^{m} F_i.$$

This completes the proof. □

3.4 Variational inequality

The variational inequality is an important tool in the study of optimization theory and game theory [6]; we also refer to early celebrated works [14] and [15]. Here, we deduced the variational inequality by Nash equilibrium theorem directly.

Theorem 3.4 (Variational inequality) *Let X be a nonempty, convex, bounded, and closed subset of $\mathbb{R}^n$, and $\varphi : X \longrightarrow \mathbb{R}^n$ be a continuous function. Then, there exists $x^* \in X$ such that*

$$\langle \varphi(x^*), y - x^* \rangle \geq 0, \quad \forall y \in X.$$

Proof Define two mappings $f, g : X \times X \longrightarrow \mathbb{R}$ by

$$f(x,y) = -\|x - y\|,$$

$$g(x,y) = \langle \varphi(x), x - y \rangle.$$

Obviously, f, g are continuous on $X \times X$, and $f(\cdot, y)$ and $g(x, \cdot)$ are concave for any $x \in X$ and any $y \in X$.

For the game $\Gamma = (X, X; f, g)$, by Theorem 1.1 there exists $(x^*, y^*) \in X \times X$ such that

$$f(x^*, y^*) = -\|x^* - y^*\| = \max_{x \in X}\left[-\|x - y^*\|\right] = -\min_{x \in X}\|x - y^*\| = 0,$$
$$g(x^*, y^*) = \langle \varphi(x^*), x^* - y^* \rangle = \max_{y \in X}\langle \varphi(x^*), x^* - y \rangle.$$

Therefore, $x^* = y^*$ and

$$0 = \langle \varphi(x^*), x^* - x^* \rangle = \langle \varphi(x^*), x^* - y^* \rangle = \max_{y \in X}\langle \varphi(x^*), x^* - y \rangle,$$

which implies

$$\langle \varphi(x^*), x^* - y \rangle \leq 0, \quad \forall y \in X,$$

that is,

$$\langle \varphi(x^*), y - x^* \rangle \geq 0, \quad \forall y \in X.$$

This completes the proof. □

4 Concluding remarks

Nash equilibrium is a very important notion in the game theory. In general, the Nash equilibrium theorem can be derived from the Brouwer and Kakutani fixed point theorems. However, there is no proof for the Kakutani and Brouwer fixed point theorems via the Nash equilibrium theorem. In this paper, we fill these gaps. We show that the Kakutani fixed point theorem, Walras equilibrium theorem (set-valued excess demand function), and generalized variational inequality can be derived from the Nash equilibrium theorem with the aid of an inverse of the Berge maximum theorem. For the single-valued situation, we derive the Brouwer fixed point theorem, Walras equilibrium theorem (single-valued excess demand function), KKM lemma, and variational inequality from the Nash equilibrium theorem directly, without any recourse.

Moreover, it is known that the Nash equilibrium theorem has been extended by Ky Fan to Hausdorff topological vector spaces (see Theorem 4 in [16]). We next apply the Fan extension of the Nash equilibrium theorem to give an infinite-dimensional extension of the Brouwer fixed point theorem (*i.e.*, the Tychonoff fixed point theorem).

Theorem 4.1 (see Theorem 4 in [16]) *Let $N = \{1, \dots, n\}$ be a finite set of players. Suppose that, for each $i \in N$, X_i is a nonempty, convex, and compact set in a locally convex Hausdorff topological vector space E_i, $f_i : X := \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ is continuous, and $f_i(x_i, x_{-i})$ is quasi-concave in x_i for any x_{-i}, where $-i = N \setminus \{i\}$. Then, there exists $x^* \in X$ such that*

$$f_i(x_i^*, x_{-i}^*) = \max_{u_i \in X_i} f_i(u_i, x_{-i}^*), \quad \forall i \in N.$$

Theorem 4.2 (Tychonoff fixed point theorem)[a] *Let X be a compact convex subset of a locally convex Hausdorff topological vector space E, and $\varphi : X \to X$ be a continuous function. Then, there exists $x^* \in X$ such that $x^* = \varphi(x^*)$.*

Proof Let X be a compact convex subset of a locally convex Hausdorff topological vector space E, $\varphi : X \to X$ be a continuous function, and $\mathbb{P}$ be a separating family of seminorms that generates the topology of E. For every $p \in \mathbb{P}$, set $F_p = \{x \in X : p(x - \varphi(x)) = 0\}$. We have to prove that $\bigcap_{p\in\mathbb{P}} F_p \neq \emptyset$. Since X is compact and the sets F_p are closed, it suffices to show that, for any finite set $\{p_1, \dots, p_n\} \subseteq \mathbb{P}$, $\bigcap_{i=1}^n F(p_i) \neq \emptyset$. To this end, apply the Nash equilibrium theorem (Ky Fan's version) to the functions $f(x, y) = -\sum_{i=1}^n p_i(x - y)$ and $g(x, y) = -\sum_{i=1}^n p_i(\varphi(x) - y)$. The following proof is similar to that given in Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Endnote
[a] This result and its proof has been suggested by an anonymous referee.

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