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Coupled fixed point theorems in C^*-algebra-valued b-metric spaces with application

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Abstract

Based on the concept of a C^*-algebra-valued b-metric space, this paper establishes some coupled fixed point theorems for mapping satisfying different contractive conditions on such space. As applications, we obtain the existence and uniqueness of a solution for an integral equation.

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1 Introduction and preliminaries

In 1989, Bakhtin [1] introduced b-metric space as a generalization of metric space. Since then, more other generalized b-metric spaces such as b-metric-like spaces [2], quasi-b-metric spaces [3] and quasi-b-metric-like spaces [4] were introduced. Recently, Ma and Jiang [5] initially introduced the concept of a C^*-algebra-valued b-metric space which generalized the concept of b-metric spaces, and they established certain basic fixed point theorems for self-map with contractive condition in this new setting. In 2016, Kamran *et al.* [6] also introduced the concept of C^*-algebra-valued b-metric space, and they generalized the Banach contraction principle on such spaces.

The notion of coupled fixed point was introduced by Guo and Lakshmikantham [7] in 1987. Since then, many researchers investigated coupled fixed point theorems in ordered metric spaces and have given some applications [8–12]. Recently, Cao [13] first studied some coupled fixed point theorems in the context of complete C^*-algebra-valued metric spaces.

Motivated by the work in [5, 6, 13–17], in this paper, we will establish coupled fixed point theorems in C^*-algebra-valued b-metric space. More precisely, we will prove some coupled fixed point theorems for the mapping with different contractive conditions on such spaces.

For convenience, we now recall some basic definitions, notations, and results of C^*-algebra. The details of C^*-algebras can be found in [18].

Let $\mathbb{A}$ be an algebra. An involution on $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. If $\mathbb{A}$ contains the identity element $1_{\mathbb{A}}$, then $(\mathbb{A}, *)$ is called a unital $*$-algebra. A $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ is said to be a Banach $*$-algebra. Moreover, if for all $a \in \mathbb{A}$, we have $\|a^*a\| = \|a\|^2$ in a Banach $*$-algebra, then $\mathbb{A}$ is known as a C^*-algebra. An element a of a C^*-algebra $\mathbb{A}$ is positive if $a = a^*$ and its spectrum $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}}\text{-}a \text{ is not invertible}\}$. Each positive element a of C^*-algebra $\mathbb{A}$ has a unique positive square root. The set of all positive elements will be denoted by $\mathbb{A}_+$. There is a natural partial ordering on the elements of $\mathbb{A}$ given by

$$a \preceq b \quad \Longleftrightarrow \quad b - a \in \mathbb{A}_+.$$

If $a \in \mathbb{A}_+$, then we write $a \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of $\mathbb{A}$. In the following, we always assume that $\mathbb{A}$ is a unital C^*-algebra with identity element $1_{\mathbb{A}}$.

Let $\mathbb{A}' = \{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$, and $\mathbb{A}'_+ = \mathbb{A}_+ \cap \mathbb{A}'$. From [5, 6], we now give the definition of C^*-algebra-valued b-metric as follows.

Definition 1.1 Let $\mathbb{A}$ be a C^*-algebra, and X be a nonempty set. Let $b \in \mathbb{A}'_+$ be such that $\|b\| \geq 1$. A mapping $d_b : X \times X \to \mathbb{A}_+$ is said to be a C^*-algebra-valued b-metric on X if the following conditions hold for all $x, y, z \in \mathbb{A}$:
1. $d_b(x,y) = 0_{\mathbb{A}}$ if and only if $x = y$;
2. $d_b(x,y) = d_b(y,x)$;
3. $d_b(x,y) \preceq b[d_b(x,z) + d_b(z,y)]$.

The triplet $(X, \mathbb{A}, d_b)$ is called a C^*-algebra-valued b-metric space with coefficient b.

Remark 1.1 From Example 2.1 in [6], we know that a C^*-algebra-valued metric space is C^*-algebra-valued b-metric space, but the converse is not true.

Definition 1.2 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space, $x \in X$, and $\{x_n\}$ a sequence in X. Then:
1. $\{x_n\}$ converges to x with respect to $\mathbb{A}$ whenever for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\|d_b(x_n, x)\| < \varepsilon$ for all $n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$ if for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\|d_b(x_n, x_m)\| < \varepsilon$ for all $n, m > N$.
3. $(X, \mathbb{A}, d_b)$ is complete if every Cauchy sequence in X is convergent with respect to $\mathbb{A}$.

Lemma 1.1 [18, 19] *Assume that $\mathbb{A}$ is a unital C^*-algebra with a unit $1_{\mathbb{A}}$.*

(1) *For any $x \in \mathbb{A}_+$, we have $x \preceq 1_{\mathbb{A}} \Leftrightarrow \|x\| \leq 1$;*
(2) *if $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $1_{\mathbb{A}} - a$ is invertible and $\|a(1_{\mathbb{A}} - a)^{-1}\| < 1$;*
(3) *assume that $a, b \in \mathbb{A}$ with $a, b \succeq 0_{\mathbb{A}}$ and $ab = ba$, then $ab \succeq 0_{\mathbb{A}}$;*
(4) *let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq 0_{\mathbb{A}}$, and $1_{\mathbb{A}} - a \in \mathbb{A}'_+$ is an invertible operator, then*

$$(1_{\mathbb{A}} - a)^{-1} b \succeq (1_{\mathbb{A}} - a)^{-1} c;$$

(5) *if $b, c \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $a \in \mathbb{A}$, then $b \preceq c \Longrightarrow a^*ba \preceq a^*ca$;*
(6) *if $0_{\mathbb{A}} \preceq a \preceq b$, then $\|a\| \leq \|b\|$.*

Lemma 1.2 [18] *The sum of two positive elements in a C^*-algebra is a positive element.*

Remark 1.2 From Lemmas 1.1(3) and 1.2, we know that the condition $b \in \mathbb{A}'_+$ in Definition 1.1 is necessary, in this case, we see that $b[d_b(x,z) + d_b(z,y)]$ is a positive element.

Definition 1.3 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = x$.

2 Main results

In this section, we will prove some coupled fixed point theorems for mappings with contractive conditions in the setting of C^*-algebra-valued b-metric space.

Theorem 2.1 *Let $(X, \mathbb{A}, d_b)$ be a complete C^*-valued b-metric space. Assume that the mapping $T : X \times X \to X$ satisfies the following condition*:

$$d_b\bigl(T(x,y), T(u,v)\bigr) \preceq a^* d_b(x,u)a + a^* d_b(y,v)a, \quad \forall x, y, u, v \in X, \tag{2.1}$$

where $a \in \mathbb{A}$ with $2\|a\|^2\|b\| < 1$. Then T has a unique coupled fixed point in X. Moreover, T has a unique fixed point in X.

Proof Let $x_0, y_0 \in X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X by the iterative scheme as

$$x_{n+1} = T(x_n, y_n), \qquad y_{n+1} = T(y_n, x_n).$$

By using the condition (2.1), for $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b\bigl(T(x_{n-1}, y_{n-1}), T(x_n, y_n)\bigr) \\ &\preceq a^* d_b(x_{n-1}, x_n)a + a^* d_b(y_{n-1}, y_n)a = a^* M_n a, \end{aligned} \tag{2.2}$$

where

$$M_n = d_b(x_{n-1}, x_n) + d_b(y_{n-1}, y_n). \tag{2.3}$$

Similarly, we get

$$d_b(y_n, y_{n+1}) = d_b\bigl(T(y_{n-1}, x_{n-1}), T(y_n, x_n)\bigr) \preceq a^* M_n a. \tag{2.4}$$

By (2.2), (2.3), and (2.4), we have

$$\begin{aligned} M_{n+1} &= d_b(x_n, x_{n+1}) + d_b(y_n, y_{n+1}) \\ &\preceq a^*\bigl[d_b(x_{n-1}, x_n) + d_b(y_{n-1}, y_n)\bigr]a + a^*\bigl[d_b(y_{n-1}, y_n) + d_b(x_{n-1}, x_n)\bigr]a \\ &\preceq (\sqrt{2}a)^*\bigl[d_b(x_{n-1}, x_n) + d_b(y_{n-1}, y_n)\bigr](\sqrt{2}a) \\ &\preceq (\sqrt{2}a)^* M_n (\sqrt{2}a). \end{aligned} \tag{2.5}$$

Thus, from (2.5) and Lemma 1.1(5), we have

$$0_{\mathbb{A}} \preceq M_{n+1} \preceq (\sqrt{2}a)^* M_n (\sqrt{2}a) \preceq \cdots \preceq \bigl[(\sqrt{2}a)^*\bigr]^n M_1 (\sqrt{2}a)^n.$$

If $M_1 = 0_{\mathbb{A}}$, then from Definition 1.3 we easily know that (x_0, y_0) is a coupled fixed point of the mapping T. Now, let $0_{\mathbb{A}} \preceq M_1$. Let $n, m \in \mathbb{N}$ with $m > n$, by using Definition 1.1, it follows that

$$\begin{aligned} d_b(x_n, x_m) &\preceq b\big[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_m)\big] \\ &\preceq bd_b(x_n, x_{n+1}) + b^2\big[d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_m)\big] \\ &= bd_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) + b^2 d_b(x_{n+2}, x_m) \\ &\preceq bd_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + b^{m-n-1} d_b(x_{m-2}, x_{m-1}) + b^{m-n-1} d_b(x_{m-1}, x_m). \end{aligned}$$

Similarly, we have

$$\begin{aligned} d_b(y_n, y_m) &\preceq bd_b(y_n, y_{n+1}) + b^2 d_b(y_{n+1}, y_{n+2}) + \cdots \\ &\quad + b^{m-n-1} d_b(y_{m-2}, y_{m-1}) + b^{m-n-1} d_b(y_{m-1}, y_m). \end{aligned}$$

Hence,

$$\begin{aligned} &d_b(x_n, x_m) + d_b(y_n, y_m) \\ &\quad \preceq bM_{n+1} + b^2 M_{n+2} + \cdots + b^{m-n-1} M_{m-1} + b^{m-n-1} M_m \\ &\quad \preceq b\big[(\sqrt{2}a)^*\big]^n M_1 (\sqrt{2}a)^n + b^2\big[(\sqrt{2}a)^*\big]^{n+1} M_1 (\sqrt{2}a)^{n+1} + \cdots \\ &\qquad + b^{m-n-1}\big[(\sqrt{2}a)^*\big]^{m-2} M_1 (\sqrt{2}a)^{m-2} + b^{m-n-1}\big[(\sqrt{2}a)^*\big]^{m-1} M_1 (\sqrt{2}a)^{m-1} \\ &\quad = b\sum_{i=n}^{m-2} b^{i-n}\big[(\sqrt{2}a)^*\big]^i M_1 (\sqrt{2}a)^i + b^{m-n-1}\big[(\sqrt{2}a)^*\big]^{m-1} M_1 (\sqrt{2}a)^{m-1} \\ &\quad = b\sum_{i=n}^{m-2} b^{i-n}\big[(\sqrt{2}a)^*\big]^i M_1^{\frac{1}{2}} M_1^{\frac{1}{2}} (\sqrt{2}a)^i + b^{m-n-1}\big[(\sqrt{2}a)^*\big]^{m-1} M_1^{\frac{1}{2}} M_1^{\frac{1}{2}} (\sqrt{2}a)^{m-1} \\ &\quad = b\sum_{i=n}^{m-2} b^{i-n}\big(M_1^{\frac{1}{2}}(\sqrt{2}a)^i\big)^*\big(M_1^{\frac{1}{2}}(\sqrt{2}a)^i\big) + b^{m-n-1}\big(M_1^{\frac{1}{2}}(\sqrt{2}a)^{m-1}\big)^*\big(M_1^{\frac{1}{2}}(\sqrt{2}a)^{m-1}\big) \\ &\quad = b\sum_{i=n}^{m-2} b^{i-n}\big|M_1^{\frac{1}{2}}(\sqrt{2}a)^i\big|^2 + b^{m-n-1}\big|M_1^{\frac{1}{2}}(\sqrt{2}a)^{m-1}\big|^2 \\ &\quad \preceq \left\| b\sum_{i=n}^{m-2} b^{i-n}\big|M_1^{\frac{1}{2}}(\sqrt{2}a)^i\big|^2 \right\| 1_{\mathbb{A}} + \big\| b^{m-n-1}\big|M_1^{\frac{1}{2}}(\sqrt{2}a)^{m-1}\big|^2\big\| 1_{\mathbb{A}} \\ &\quad \preceq \|b\| \sum_{i=n}^{m-2} \big\|b^{i-n}\big\| \big\|M_1^{\frac{1}{2}}\big\|^2 \big\|(\sqrt{2}a)^i\big\|^2 1_{\mathbb{A}} + \big\|b^{m-n-1}\big\| \big\|M_1^{\frac{1}{2}}\big\|^2 \big\|(\sqrt{2}a)^{m-1}\big\|^2 1_{\mathbb{A}} \\ &\quad \preceq \|b\|^{1-n} \big\|M_1^{\frac{1}{2}}\big\|^2 \sum_{i=n}^{m-2} \|b\|^i \big\|(\sqrt{2}a)^2\big\|^i 1_{\mathbb{A}} + \|b\|^{-n} \big\|M_1^{\frac{1}{2}}\big\|^2 \|b\|^{m-1} \big\|(\sqrt{2}a)^2\big\|^{m-1} 1_{\mathbb{A}} \\ &\quad = \|b\|^{1-n} \big\|M_1^{\frac{1}{2}}\big\|^2 \sum_{i=n}^{m-2} \big(2\|a\|^2\|b\|\big)^i 1_{\mathbb{A}} + \|b\|^{-n} \big\|M_1^{\frac{1}{2}}\big\|^2 \big(2\|a\|^2\|b\|\big)^{m-1} 1_{\mathbb{A}} \\ &\quad \to 0_{\mathbb{A}} \quad (\text{as } m, n \to \infty), \end{aligned} \tag{2.6}$$

by the condition $2\|a\|^2\|b\| < 1$ and $\|b\| \geq 1$. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. By the completeness of $(X, \mathbb{A}, d)$, there exist $x^*, y^* \in X$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. We now show that $T(x^*, y^*) = x^*$ and $T(y^*, x^*) = y^*$. From Definition 1.1 and (2.1), we get

$$\begin{aligned} 0_{\mathbb{A}} &\preceq d_b\bigl(T(x^*,y^*),x^*\bigr) \preceq b\bigl[d_b\bigl(T(x^*,y^*),x_{n+1}\bigr) + d_b\bigl(x_{n+1},x^*\bigr)\bigr] \\ &= b\bigl[d_b\bigl(T(x^*,y^*),T(x_n,y_n)\bigr) + d_b\bigl(x_{n+1},x^*\bigr)\bigr] \\ &\preceq ba^*d_b\bigl(x^*,x_n\bigr)a + ba^*d_b\bigl(y^*,y_n\bigr)a + bd_b\bigl(x_{n+1},x^*\bigr) \to 0_{\mathbb{A}} \quad (n \to \infty). \end{aligned} \tag{2.7}$$

So, $T(x^*, y^*) = x^*$. Similarly, we have $T(y^*, x^*) = y^*$. Thus, (x^*, y^*) is a coupled fixed point of T.

If there exists another coupled fixed point (u, v) of T, then

$$\begin{aligned} 0_{\mathbb{A}} &\preceq d_b\bigl(x^*,u\bigr) = d_b\bigl(T(x^*,y^*),T(u,v)\bigr) \preceq a^*d_b\bigl(x^*,u\bigr)a + a^*d_b\bigl(y^*,v\bigr)a, \\ 0_{\mathbb{A}} &\preceq d_b\bigl(y^*,v\bigr) = d_b\bigl(T(y^*,x^*),T(v,u)\bigr) \preceq a^*d_b\bigl(y^*,v\bigr)a + a^*d_b\bigl(x^*,u\bigr)a, \end{aligned}$$

which implies that

$$0_{\mathbb{A}} \preceq d_b\bigl(x^*,u\bigr) + d_b\bigl(y^*,v\bigr) \preceq (\sqrt{2}a)^*\bigl(d_b\bigl(x^*,u\bigr) + d_b\bigl(y^*,v\bigr)\bigr)(\sqrt{2}a).$$

Thus, we have

$$\begin{aligned} 0 &\leq \bigl\|d_b\bigl(x^*,u\bigr) + d_b\bigl(y^*,v\bigr)\bigr\| \leq \|\sqrt{2}a\|^2\bigl\|d_b\bigl(x^*,u\bigr) + d_b\bigl(y^*,v\bigr)\bigr\| \\ &< \frac{1}{\|b\|}\bigl\|d_b\bigl(x^*,u\bigr) + d_b\bigl(y^*,v\bigr)\bigr\| \leq \bigl\|d_b\bigl(x^*,u\bigr) + d_b\bigl(y^*,v\bigr)\bigr\|, \end{aligned}$$

which is a contradiction. Thus, $(u, v) = (x^*, y^*)$, that is, the coupled fixed point is unique. Finally, we will prove that T has a unique fixed point. Since

$$\begin{aligned} 0_{\mathbb{A}} &\preceq d_b\bigl(x^*,y^*\bigr) = d_b\bigl(T(x^*,y^*),T(y^*,x^*)\bigr) \\ &\preceq a^*d_b\bigl(x^*,y^*\bigr)a + a^*d_b\bigl(y^*,x^*\bigr)a = (\sqrt{2}a)^*d_b\bigl(x^*,y^*\bigr)(\sqrt{2}a), \end{aligned}$$

we have

$$\bigl\|d_b\bigl(x^*,y^*\bigr)\bigr\| \leq 2\|a\|^2\bigl\|d_b\bigl(x^*,y^*\bigr)\bigr\|.$$

It follows from the condition $2\|a\|^2 < \frac{1}{\|b\|} \leq 1$ that $\|d_b(x^*, y^*)\| = 0$. Hence, $x^* = y^*$. The proof is completed. □

Remark 2.1 Taking $b = 1_{\mathbb{A}}$, Theorem 2.1 of [13] becomes a special case of Theorem 2.1.

Theorem 2.2 *Let $(X, \mathbb{A}, d_b)$ be a complete C^*-valued b-metric space. Assume that the mapping $T : X \times X \to X$ satisfies the following condition*:

$$d_b\bigl(T(x,y),T(u,v)\bigr) \preceq a_1d_b\bigl(T(x,y),u\bigr) + a_2d_b\bigl(T(u,v),x\bigr), \quad \forall x,y,u,v \in X, \tag{2.8}$$

where $a_1, a_2 \in \mathbb{A}'_+$ with $\|a_1 + a_2\| \|b\|^2 < 1$. Then T has a unique coupled fixed point in X. Moreover, T has a unique fixed point in X.

Proof From $a_1, a_2 \in \mathbb{A}'_+$ and Lemma 1.2, we see that $a_1 d_b(T(x,y),u) + a_2 d_b(T(u,v),x)$ is a positive element. Choose $x_0, y_0 \in X$. Set $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$ for $n = 0, 1, \ldots$. Applying (2.8), we have

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b\bigl(T(x_{n-1}, y_{n-1}), T(x_n, y_n)\bigr) \\ &\preceq a_1 d_b\bigl(T(x_{n-1}, y_{n-1}), x_n\bigr) + a_2 d_b\bigl(T(x_n, y_n), x_{n-1}\bigr) \\ &= a_2 d_b(x_{n+1}, x_{n-1}) \preceq a_2 b\bigl[d_b(x_{n+1}, x_n) + d_b(x_n, x_{n-1})\bigr] \\ &\preceq a_2 b^2 d_b(x_{n+1}, x_n) + a_2 b d_b(x_n, x_{n-1}), \end{aligned}$$

which implies that

$$\bigl(1_{\mathbb{A}} - a_2 b^2\bigr) d_b(x_n, x_{n+1}) \preceq a_2 b d_b(x_n, x_{n-1}). \tag{2.9}$$

Moreover, we obtain

$$\begin{aligned} d_b(x_{n+1}, x_n) &= d_b\bigl(T(x_n, y_n), T(x_{n-1}, y_{n-1})\bigr) \\ &\preceq a_1 d_b\bigl(T(x_n, y_n), x_{n-1}\bigr) + a_2 d_b\bigl(T(x_{n-1}, y_{n-1}), x_n\bigr) \\ &= a_1 d_b(x_{n+1}, x_{n-1}) \preceq a_1 b^2 d_b(x_{n+1}, x_n) + a_1 b d_b(x_n, x_{n-1}), \end{aligned}$$

which yields

$$\bigl(1_{\mathbb{A}} - a_1 b^2\bigr) d_b(x_n, x_{n+1}) \preceq a_1 b d_b(x_n, x_{n-1}). \tag{2.10}$$

From (2.9) and (2.10), we get

$$\left(1_{\mathbb{A}} - \frac{(a_1 + a_2)b^2}{2}\right) d_b(x_n, x_{n+1}) \preceq \frac{(a_1 + a_2)b}{2} d_b(x_n, x_{n-1}). \tag{2.11}$$

Since $a_1, a_2, b \in \mathbb{A}'_+$, we have $\frac{(a_1+a_2)b}{2} \in \mathbb{A}'_+$ and $\frac{(a_1+a_2)b^2}{2} \in \mathbb{A}'_+$. Moreover, from the condition $\|(a_1 + a_2)\| \|b\|^2 < 1$, we get

$$\left\| \frac{(a_1 + a_2)b}{2} \right\| \leq \frac{1}{2} \bigl\|(a_1 + a_2)\bigr\| \|b\| \leq \frac{1}{2} \bigl\|(a_1 + a_2)\bigr\| \|b\|^2 < \frac{1}{2}$$

and

$$\left\| \frac{(a_1 + a_2)b^2}{2} \right\| \leq \frac{1}{2} \bigl\|(a_1 + a_2)\bigr\| \|b\|^2 < \frac{1}{2},$$

which implies that $(1_{\mathbb{A}} - \frac{(a_1+a_2)b}{2})^{-1} \in \mathbb{A}'_+$ and $(1_{\mathbb{A}} - \frac{(a_1+a_2)b^2}{2})^{-1} \in \mathbb{A}'_+$ with

$$\left\| \left(1_{\mathbb{A}} - \frac{(a_1 + a_2)b^2}{2}\right)^{-1} \frac{(a_1 + a_2)b^2}{2} \right\| < 1 \tag{2.12}$$

by Lemma 1.1(2). Thus, we have by (2.11)

$$d_b(x_{n+1},x_n)\preceq hd_b(x_n,x_{n-1}),$$

where

$$h=\left(1_{\mathbb{A}}-\frac{(a_1+a_2)b^2}{2}\right)^{-1}\frac{(a_1+a_2)b}{2} \tag{2.13}$$

with $\|h\|\preceq\|hb\|<1$ by (2.12). Inductively, for all $n\in\mathbb{N}$, we have

$$d_b(x_{n+1},x_n)\preceq h^n d_b(x_1,x_0)=h^n m_0, \tag{2.14}$$

where $m_0=d_b(x_1,x_0)$. Let $n,m\in\mathbb{N}$ with $m>n$, by using Definition 1.1 and (2.12)-(2.14), we have

$$\begin{aligned}
d_b(x_n,x_m) &\preceq b\bigl[d_b(x_n,x_{n+1})+d_b(x_{n+1},x_m)\bigr]\\
&\preceq bd_b(x_n,x_{n+1})+b^2\bigl[d_b(x_{n+1},x_{n+2})+d_b(x_{n+2},x_m)\bigr]\\
&\preceq bd_b(x_n,x_{n+1})+b^2d_b(x_{n+1},x_{n+2})+\cdots\\
&\quad +b^{m-n-1}\bigl[d_b(x_{m-2},x_{m-1})+d_b(x_{m-1},x_m)\bigr]\\
&\preceq bh^n m_0+b^2h^{n+1}m_0+\cdots+b^{m-n-1}h^{m-2}m_0+b^{m-n-1}h^{m-1}m_0.\\
&=\sum_{i=1}^{m-n-1}b^ih^{n+i-1}m_0+b^{m-n-1}h^{m-1}m_0\\
&=\sum_{i=1}^{m-n-1}\bigl|m_0^{\frac{1}{2}}h^{\frac{n+i-1}{2}}b^{\frac{i}{2}}\bigr|^2+\bigl|m_0^{\frac{1}{2}}h^{\frac{m-1}{2}}b^{\frac{m-n-1}{2}}\bigr|^2\\
&\preceq \|m_0\|\sum_{i=1}^{m-n-1}\|h\|^{n-1}\|hb\|^i1_{\mathbb{A}}+\|m_0\|\|h\|^n\|hb\|^{m-n-1}1_{\mathbb{A}}\\
&\preceq \frac{\|m_0\|\|h\|^{n-1}\|hb\|}{1-\|hb\|}1_{\mathbb{A}}+\|M_0\|\|h\|^n\|hb\|^{m-n-1}1_{\mathbb{A}}\\
&\to 0_{\mathbb{A}}\quad(\text{as } m,n\to\infty).
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Similarly, we can prove that $\{y_n\}$ is also a Cauchy sequence in X. Since $(X,\mathbb{A},d_b)$ is complete, we see that $\{x_n\}$ and $\{y_n\}$ converge to some $u\in X$ and $v\in X$, respectively. In the following, we will show that $T(u,v)=u$ and $T(v,u)=v$. By (2.8), we get

$$\begin{aligned}
d_b\bigl(T(u,v),u\bigr) &\preceq b\bigl[d_b\bigl(x_{n+1},T(u,v)\bigr)+d_b(x_{n+1},u)\bigr]\\
&\preceq b\bigl[d_b\bigl(T(x_n,y_n),T(u,v)\bigr)+d_b(x_{n+1},u)\bigr]\\
&\preceq ba_1d_b\bigl(T(x_n,y_n),u\bigr)+ba_2d_b\bigl(T(u,v),x_n\bigr)+bd_b(x_{n+1},u)\\
&\preceq ba_1d_b(x_{n+1},u)+ba_2d_b\bigl(T(u,v),x_n\bigr)+bd_b(x_{n+1},u).
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\|d_b\bigl(T(u,v),u\bigr)\right\| \\
&\quad\le \|ba_1\|\left\|d_b(x_{n+1},u)\right\| + \|ba_2\|\left\|d_b\bigl(T(u,v),x_n\bigr)\right\| + \|b\|\left\|d_b(x_{n+1},u)\right\| \\
&\quad\to \|ba_2\|\left\|d_b\bigl(T(u,v),u\bigr)\right\|, \quad n\to\infty.
\end{aligned} \tag{2.15}$$

Since $0_{\mathbb{A}} \preceq ba_2 \preceq (a_1+a_2)b$, we have $\|a_2b\| \le \|(a_1+a_2)b\| < 1$ by Lemma 1.1(6). This and (2.15) imply that $\|d_b(T(u,v),u)\| = 0$. Hence $T(u,v) = u$. Similarly, we obtain $T(v,u) = v$. Thus (u,v) is a coupled fixed point of T.

Now if (u^*,v^*) is another coupled fixed point of T, then

$$\begin{aligned}
0_{\mathbb{A}} \preceq d_b\bigl(u,u^*\bigr) &= d_b\bigl(T(u,v),T\bigl(u^*,v^*\bigr)\bigr) \\
&\preceq a_1 d_b\bigl(T(u,v),u^*\bigr) + a_2 d_b\bigl(T\bigl(u^*,v^*\bigr),u\bigr) \\
&\preceq a_1 d_b\bigl(u,u^*\bigr) + a_2 d_b\bigl(u^*,u\bigr) = (a_1+a_2)d_b\bigl(u,u^*\bigr),
\end{aligned}$$

so, we get

$$0 \le \left\|d_b\bigl(u,u^*\bigr)\right\| \le \|a_1+a_2\|\left\|d_b\bigl(u,u^*\bigr)\right\| < \frac{1}{\|b\|^2}\left\|d_b\bigl(u,u^*\bigr)\right\| \le \left\|d_b(u,v)\right\|,$$

which implies that $\|d_b(u,u^*)\| = 0$, then we have $u = u^*$. Similarly, we can get $v = v^*$. Hence, the coupled fixed point is unique. Moreover, we will prove the uniqueness of fixed points of T. By (2.8), we have

$$\begin{aligned}
d_b(u,v) &= d_b\bigl(T(u,v),T(v,u)\bigr) \\
&\preceq a_1 d_b\bigl(T(u,v),v\bigr) + a_2 d_b\bigl(T(v,u),u\bigr) = (a_1+a_2)d_b(u,v),
\end{aligned}$$

then

$$\left\|d_b\bigl(u,u^*\bigr)\right\| \le \|a_1+a_2\|\left\|d_b\bigl(u,u^*\bigr)\right\| < \frac{1}{\|b\|^2}\left\|d_b\bigl(u,u^*\bigr)\right\| \le \left\|d_b(u,v)\right\|,$$

which yields $u = v$. This completes the proof. □

Remark 2.2 Taking $b = 1_{\mathbb{A}}$, Theorem 2.3 of [13] becomes a special case of Theorem 2.2.

Theorem 2.3 *Let $(X,\mathbb{A},d_b)$ be a complete C^*-valued b-metric space. Assume that the mapping $T: X\times X \to X$ satisfies the following condition*:

$$d_b\bigl(T(x,y),T(u,v)\bigr) \preceq a_1 d_b\bigl(T(x,y),x\bigr) + a_2 d_b\bigl(T(u,v),u\bigr), \quad \forall x,y,u,v\in X, \tag{2.16}$$

where $a_1,a_2 \in \mathbb{A}'_+$ with $(\|a_1\|+\|a_2\|)\|b\| < 1$. Then T has a unique coupled fixed point in X. Moreover, T has a unique fixed point.

Proof Since $a_1,a_2\in\mathbb{A}'_+$, we see that $a_1 d_b(T(x,y),x) + a_2 d_b(T(u,v),u)$ is a positive element. Similar to the proof of Theorem 2.2, we construct $\{x_n\}$ and $\{y_n\}$ such that $x_{n+1} = T(x_n,y_n)$

and $y_{n+1} = T(y_n, x_n)$. By (2.16), we obtain

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b\bigl(T(x_{n-1}, y_{n-1}), T(x_n, y_n)\bigr) \\ &\preceq a_1 d_b\bigl(T(x_{n-1}, y_{n-1}), x_{n-1}\bigr) + a_2 d_b\bigl(T(x_n, y_n), x_n\bigr) \\ &= a_1 d_b(x_n, x_{n-1}) + a_2 d_b(x_{n+1}, x_n), \end{aligned}$$

which implies that

$$(1_{\mathbb{A}} - a_2) d_b(x_n, x_{n+1}) \preceq a_1 d_b(x_n, x_{n-1}).$$

Since $a_1, a_2 \in \mathbb{A}'_+$ with $\|a_1\| + \|a_2\| < \frac{1}{\|b\|} \leq 1$, we have $1_{\mathbb{A}} - a_2$ is invertible and $(1_{\mathbb{A}} - a_2)^{-1} a_1 \in \mathbb{A}'_+$. Hence

$$d_b(x_n, x_{n+1}) \preceq (1_{\mathbb{A}} - a_2)^{-1} a_1 d_b(x_n, x_{n-1}).$$

Inductively, for all $n \in \mathbb{N}$, we have

$$d_b(x_n, x_{n+1}) \preceq k^n m_0, \tag{2.17}$$

where $k = (1_{\mathbb{A}} - a_2)^{-1} a_1$ and $m_0 = d_b(x_1, x_0)$. Since $\|a_1\|\|b\| + \|a_2\| \leq (\|a_1\| + \|a_2\|)\|b\| < 1$, we have

$$\|bk\| = \bigl\|(1_{\mathbb{A}} - a_2)^{-1} a_1 b\bigr\| \leq \bigl\|(1_{\mathbb{A}} - a_2)^{-1}\bigr\| \|a_1\|\|b\| = \sum_{i=0}^{\infty} \|a_2\|^i \|a_1\|\|b\| = \frac{\|a_1\|\|b\|}{1 - \|a_2\|} < 1.$$

And $\|k\| \leq \|bk\| < 1$ by Lemma 1.1(6).

Let $n, m \in \mathbb{N}$ with $m > n$, by using Definition 1.1, (2.16), and (2.17), we have

$$\begin{aligned} d_b(x_n, x_m) &\preceq b\bigl[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_m)\bigr] \\ &\preceq b d_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + b^{m-n-1}\bigl[d_b(x_{m-2}, x_{m-1}) + d_b(x_{m-1}, x_m)\bigr] \\ &\preceq bk^n M_0 + b^2 k^{n+1} M_0 + \cdots + b^{m-n-1} k^{m-2} M_0 + b^{m-n-1} k^{m-1} M_0 \\ &= \sum_{i=1}^{m-n-1} b^i k^{n+i-1} M_0 + b^{m-n-1} k^{m-1} M_0 \\ &= \sum_{i=1}^{m-n-1} \bigl|M_0^{\frac{1}{2}} k^{\frac{n+i-1}{2}} b^{\frac{i}{2}}\bigr|^2 + \bigl|M_0^{\frac{1}{2}} k^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}\bigr|^2 \\ &\preceq \sum_{i=1}^{m-n-1} \bigl\|M_0^{\frac{1}{2}} k^{\frac{n+i-1}{2}} b^{\frac{i}{2}}\bigr\|^2 1_{\mathbb{A}} + \bigl\|M_0^{\frac{1}{2}} k^{\frac{m-1}{2}} b^{\frac{m-n-1}{2}}\bigr\|^2 1_{\mathbb{A}} \\ &\preceq \|M_0\| \sum_{i=1}^{m-n-1} \bigl\|(bk)^{\frac{i}{2}}\bigr\|^2 \bigl\|k^{\frac{n-1}{2}}\bigr\|^2 1_{\mathbb{A}} + \|M_0\| \bigl\|(bk)^{\frac{m-n-1}{2}}\bigr\|^2 \bigl\|k^{\frac{n}{2}}\bigr\|^2 1_{\mathbb{A}} \\ &= \|M_0\| \|k\|^{n-1} \sum_{i=1}^{m-n-1} \|bk\|^i 1_{\mathbb{A}} + \|M_0\| \|bk\|^{m-n-1} \|k\|^n 1_{\mathbb{A}} \end{aligned}$$

$$\begin{aligned}
&= \|M_0\|\|k\|^{n-1}\frac{\|bk\| - \|bk\|^{m-n}}{1-\|bk\|}1_{\mathbb{A}} + \|M_0\|\|bk\|^{m-n-1}\|k\|^n 1_{\mathbb{A}} \\
&\preceq \frac{\|M_0\|\|bk\|}{1-\|bk\|}\|k\|^{n-1}1_{\mathbb{A}} + \|M_0\|\|bk\|^{m-n-1}\|k\|^n 1_{\mathbb{A}} \\
&\to 0_{\mathbb{A}} \quad (\text{as } m,n\to\infty).
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. Similarly, we can prove that $\{y_n\}$ is also a Cauchy sequence. Since $(X,\mathbb{A},d_b)$ is complete, there are $u,v\in X$ such that $x_n\to u$ and $y_n\to v$ as $n\to\infty$. In the following, we will show that $T(u,v)=u$ and $T(v,u)=v$. From (2.16), we get

$$\begin{aligned}
d_b\bigl(T(u,v),u\bigr) &\preceq b\bigl[d_b\bigl(x_{n+1},T(u,v)\bigr)+d_b(x_{n+1},u)\bigr] \\
&\preceq b\bigl[d_b\bigl(T(x_n,y_n),T(u,v)\bigr)+d_b(x_{n+1},u)\bigr] \\
&\preceq b\bigl[a_1 d_b\bigl(T(x_n,y_n),x_n\bigr)+a_2 d_b\bigl(T(u,v),u\bigr)+d_b(x_{n+1},u)\bigr] \\
&= ba_1 d_b(x_{n+1},x_n)+ba_2 d_b\bigl(T(u,v),u\bigr)+bd_b(x_{n+1},u),
\end{aligned}$$

which implies that

$$d_b\bigl(T(u,v),u\bigr)\preceq (1_{\mathbb{A}}-a_2b)^{-1}ba_1 d_b(x_{n+1},x_n)+(1_{\mathbb{A}}-a_2b)^{-1}d_b(x_{n+1},u).$$

Thus $d_b(T(u,v),u)=0_{\mathbb{A}}$. Equivalently, $T(u,v)=u$. Similarly, we can obtain $T(v,u)=v$.

Now if (u^*,v^*) is another coupled fixed point of T, then

$$\begin{aligned}
0_{\mathbb{A}} &\preceq d_b\bigl(u,u^*\bigr)=d_b\bigl(T(u,v),T\bigl(u^*,v^*\bigr)\bigr) \\
&\preceq a_1 d_b\bigl(T(u,v),u\bigr)+a_2 d_b\bigl(T\bigl(u^*,v^*\bigr),u^*\bigr)=a_1 d_b(u,u)+a_2 d_b\bigl(u^*,u^*\bigr)=0_{\mathbb{A}},
\end{aligned}$$

so, we get $d_b(u,u^*)=0_{\mathbb{A}}$, which yields $u^*=u$. Similarly, we have $v^*=v$. Thus, (u,v) is the unique coupled fixed point of T. Finally, we will show the uniqueness of fixed points of T. By (2.16), we have

$$\begin{aligned}
d_b(u,v) &= d_b\bigl(T(u,v),T(v,u)\bigr) \\
&\preceq a_1 d_b\bigl(T(u,v),u\bigr)+a_2 d_b\bigl(T(v,u),v\bigr)=a_1 d_b(u,u)+a_2 d_b(v,v)=0_{\mathbb{A}},
\end{aligned}$$

which implies that $u=v$. □

Remark 2.3 Taking $b=1_{\mathbb{A}}$ [13], Theorem 2.2 becomes a special case of Theorem 2.3.

3 Application

As an application of coupled fixed point theorems on complete C^*-algebra-valued b-metric spaces, we prove here the existence and uniqueness of a solution for a Fredholm nonlinear integral equation.

Let E be a Lebesgue-measurable set with $m(E)<\infty$ and $X=L^\infty(E)$ denote the class of essentially bounded measurable functions on E.

Consider the Hilbert space $L^2(E)$. Let the set of all bounded linear operators on $L^2(E)$ be denoted by $B(L^2(E))$. Obviously, $B(L^2(E))$ is a C^*-algebra with usual operator norm.

Let $K_1, K_2 : E \times E \to \mathbb{R}$, assume that there exist two continuous functions $f, g : E \times E \to \mathbb{R}$ and a constant $\alpha \in (0, \frac{1}{4})$ such that for all $x, y \in X$ and $u, v \in E$, we have

$$\bigl|K_1\bigl(u,v,x(v)\bigr) - K_1\bigl(u,v,y(v)\bigr)\bigr| \le \alpha\bigl|f(u,v)\bigr|\bigl|x(v) - y(v)\bigr|, \tag{3.1}$$

$$\bigl|K_2\bigl(u,v,x(v)\bigr) - K_2\bigl(u,v,y(v)\bigr)\bigr| \le \alpha\bigl|g(u,v)\bigr|\bigl|x(v) - y(v)\bigr|. \tag{3.2}$$

Example 3.1 Consider the integral equation

$$x(t) = \int_E \bigl(K_1\bigl(t,s,x(s)\bigr) + K_2\bigl(t,s,x(s)\bigr)\bigr)\, ds, \quad t \in E. \tag{3.3}$$

Assume that (3.1) and (3.2) hold. Moreover, if

$$\sup_{u\in E} \int_E \bigl|f(u,v)\bigr|\, dv \le 1, \qquad \sup_{v\in E} \int_E \bigl|g(u,v)\bigr|\, dv \le 1, \tag{3.4}$$

then the integral equation (3.3) has a unique solution in $L^\infty(E)$.

Proof Define $d_b : X \times X \to B(L^2(E))$ as follows:

$$d_b(f,g) = \pi_{(f-g)^2},$$

where $\pi_h : L^2(E) \to L^2(E)$ is the product operator given by

$$\pi_h(u) = h \cdot u \quad \text{for } u \in L^2(E).$$

Working in the same lines as in [5], Example 3.2, we easily see that $(X, B(L^2(E)), d_b)$ is a complete C^*-valued b-metric space with $b = 2 \cdot 1_{B(L^2(E))}$.

Let $T : X \times X \to X$ be

$$T(x,y)(t) = \int_E \bigl(K_1\bigl(t,s,x(s)\bigr) + K_2\bigl(t,s,y(s)\bigr)\bigr)\, ds, \quad t \in E.$$

Then by (3.1), (3.2), and (3.4), we obtain

$$\begin{aligned}
\bigl\|d_b\bigl(T(x,y),T(u,v)\bigr)\bigr\| &= \sup_{\|p\|=1} \langle \pi_{|T(x,y)-T(u,v)|}p, p\rangle \quad \text{for every } p \in L^2(E) \\
&= \sup_{\|p\|=1} \int_E \bigl|T(x,y) - T(u,v)\bigr|^2 p(t)\overline{p(t)}\, dt \\
&\le 2 \sup_{\|p\|=1} \int_E \biggl[\int_E \bigl|K_1\bigl(t,s,x(s)\bigr) - K_1\bigl(t,s,u(s)\bigr)\bigr|\, ds\biggr]^2 \bigl|p(t)\bigr|^2\, dt \\
&\quad + 2 \sup_{\|p\|=1} \int_E \biggl[\int_E \bigl|K_2\bigl(t,s,y(s)\bigr) - K_2\bigl(t,s,v(s)\bigr)\bigr|\, ds\biggr]^2 \bigl|p(t)\bigr|^2\, dt \\
&\le 2 \sup_{\|p\|=1} \int_E \alpha^2 \biggl[\int_E \bigl|f(t,s)\bigr|\bigl|x(s) - u(s)\bigr|\, ds\biggr]^2 \bigl|p(t)\bigr|^2\, dt \\
&\quad + 2 \sup_{\|p\|=1} \int_E \alpha^2 \biggl[\int_E \bigl|g(t,s)\bigr|\bigl|y(s) - v(s)\bigr|\, ds\biggr]^2 \bigl|p(t)\bigr|^2\, dt
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\alpha^2 \sup_{\|p\|=1} \int_E \left[\int_E \left|f(t,s)\right| ds\right]^2 \left|p(t)\right|^2 dt \cdot \left\|(x-u)^2\right\|_\infty \\
&\quad + 2\alpha^2 \sup_{\|p\|=1} \int_E \left[\int_E \left|g(t,s)\right| ds\right]^2 \left|p(t)\right|^2 dt \cdot \left\|(y-v)^2\right\|_\infty \\
&\leq 2\alpha^2 \sup_{t\in E} \left[\int_E \left|f(t,s)\right| ds\right]^2 \cdot \sup_{\|p\|=1} \int_E \left|p(t)\right|^2 dt \cdot \left\|(x-u)^2\right\|_\infty \\
&\quad + 2\alpha^2 \sup_{t\in E} \left[\int_E \left|g(t,s)\right| ds\right]^2 \cdot \sup_{\|p\|=1} \int_E \left|p(t)\right|^2 dt \cdot \left\|(y-v)^2\right\|_\infty \\
&\leq 2\alpha^2 \left(\left\|(x-u)^2\right\|_\infty + \left\|(y-v)^2\right\|_\infty\right) \\
&= 2\alpha^2 d_b(x,u) + 2\alpha^2 d_b(y,v).
\end{aligned}
$$

Set $a = \sqrt{2}\alpha 1_{B(L^2(E))}$, then $a \in B(L^2(E))$ and $\|a\| = \sqrt{2}\alpha < \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}\|b\|}$. Hence, all the conditions of Theorem 2.1 hold. Applying Theorem 2.1, we see that the integral equation (3.3) has a unique solution in $L^\infty(E)$. $\square$

Competing interests
The author declares that he has no competing interests.

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References

1. Bakhtin, IA: The contraction principle in quasimetric spaces. In: Functional Analysis, vol. 30, pp. 26-37 (1989)
2. Alghamdi, MA, Hussain, N, Salimi, P: Fixed point and coupled fixed point theorems on *b*-metric-like spaces. J. Inequal. Appl. **2013**, 402 (2013)
3. Shah, MH, Hussain, N: Nonlinear contractions in partially ordered quasi *b*-metric spaces. Commun. Korean Math. Soc. **27**(1), 117-128 (2012)
4. Zhu, CX, Chen, CF, Zhang, X: Some results in quasi-*b*-metric-like spaces. J. Inequal. Appl. **2014**, 437 (2014)
5. Ma, Z, Jiang, L: C^*-Algebra-valued *b*-metric spaces and related fixed point theorems. Fixed Point Theory Appl. **2015**, 222 (2015)
6. Kamran, T, Postolache, M, Ghiura, A, Batul, S, Ali, R: The Banach contraction principle in C^*-algebra-valued *b*-metric spaces with application. Fixed Point Theory Appl. **2016**, 10 (2016)
7. Guo, D, Lakshmikantham, V: Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. **11**, 623-632 (1987)
8. Samet, B: Coupled fixed point theorems for s generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. **72**, 4508-4517 (2010)
9. Aydi, H, Postolache, M, Shatanawi, W: Coupled fixed point results for (ψ,ϕ)-weakly contractive mappings in ordered *G*-metric spaces. Comput. Math. Appl. **63**, 298-309 (2012)
10. Berinde, V: Coupled fixed point theorems for ϕ-contractive mixed monotone mappings in partially ordered metric spaces. Nonlinear Anal. **75**, 3218-3228 (2012)
11. Asgari, MS, Mousavi, B: Solving a class of nonlinear matrix equations via the coupled fixed point theorem. Appl. Math. Comput. **259**, 364-373 (2015)
12. Luong, NV, Thuan, NX: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74**, 983-992 (2011)
13. Cao, T: Some coupled fixed point theorems in C^*-algebra-valued metric spaces (2016) arXiv:1601.07168v1
14. Huang, H, Radenovic, S: Common fixed point theorems of generalized Lipschitz mappings in cone *b*-metric spaces over Banach algebras and applications. J. Nonlinear Sci. Appl. **8**(5), 787-799 (2015)
15. Latif, A, Kadelburg, Z, Parvaneh, V, Roshan, JR: Some fixed point theorems for G-rational Geraghty contractive mappings in ordered generalized *b*-metric spaces. J. Nonlinear Sci. Appl. **8**(6), 1212-1227 (2015)
16. Yamaod, O, Sintunavarat, W, Cho, YJ: Common fixed point theorems for generalized cyclic contraction pairs in *b*-metric spaces with applications. Fixed Point Theory Appl. **2015**, 164 (2015)
17. Kadelburg, Z, Radenovic, S: Pata-type common fixed point results in *b*-metric and *b*-rectangular metric spaces. J. Nonlinear Sci. Appl. **8**(6), 944-954 (2015)
18. Murphy, GJ: C^*-Algebras and Operator Theory. Academic Press, London (1990)
19. Douglas, RG: Banach Algebra Techniques in Operator Theory. Springer, Berlin (1998)