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A new class of $\mathcal{S}$-contractions in complete metric spaces and $\mathcal{G}_{\mathcal{P}}$-contractions in ordered metric spaces

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Abstract

The basic purpose of this article is to define new so-called $\mathcal{S}$-contractions and discuss the presence of common best proximity point theorems for such contractions in the setting of Cauchy metric spaces. We also calculate some common optimal approximate solutions of some fixed point equations when there does not exist any common fixed point. We also introduce the notions of $\mathcal{G}_{\mathcal{P}}$-functions and $\mathcal{G}_{\mathcal{P}}$-contractions with the help of $\mathcal{P}$-functions and prove the existence of a unique best proximity point in partially order metric spaces. We give some examples that justify the validity of our results. These results extend and unify many existing results in the literature.

MSC: 58C30; 47H10

Keywords: common best proximity point; common fixed point; best proximity point; fixed point; $\mathcal{S}$-contraction; $\mathcal{G}_{\mathcal{P}}$-function; $\mathcal{G}_{\mathcal{P}}$-contraction; proximally commuting mappings; *P*-property

1 Introduction

When discussing fixed points of various mappings satisfying certain conditions, we see that these maps have many applications and are important tools in various research activities. The Banach contraction principle [1] helps many mathematicians and researchers working in mathematics and mathematical sciences. Many important results of [1, 2], and [3] have become the source of motivation for many researchers and mathematicians that do research in the metric fixed point theory and best proximity point theory. When some self-mapping in a metric space, topological vector space, or any other appropriate space has no fixed points, then we are interested in the existence and uniqueness of some point that minimizes the distance between the origin and its corresponding image known as best proximity point. Best proximity point theorems for several types using different contraction maps are considered in [2, 4–10], and [11]. Best proximity point theorems establish a generalization of fixed points by considering self-mappings. Let sets $A, B \neq \phi$ of (X, d) with mappings $S : A \to B$ and $T : A \to B$ be such that the equations $Tx = x$ and $Sx = x$ have no common fixed point of the mappings S and T. In such a situation, when there does not exist any type of common solution, it is essential to find an element that is in close distance to Sx and Tx, and such an optimal approximate solution is known as the common best proximity point of the given non-self-mappings. If x is an element that gives the global minimum value for these two mappings S and T, then we write $d(x, Sx) = d(x, Tx) = d(A, B)$.

Now, the main aim of this paper is to furnish new $\mathcal{S}$-contractions and to derive a common best proximity point theorem in the framework of metric spaces for the pair of non-self-mappings and to derive $\mathcal{G}_{\mathcal{P}}$-contractions and functions to find optimal approximate solutions of certain contractive maps. We present some theorems and examples in favor of our work.

2 Preliminaries

In this section, we consider subsets $A, B \neq \phi$ of a metric space X with metric d and collect some definitions and mathematical symbols, which will be used in this paper.

Definition 2.1 [8] Let X be a metric space, and A and B two nonempty subsets of X. Define

$$\begin{aligned} &d(A,B) = \inf\bigl\{d(a,b) : a \in A, b \in B\bigr\}, \\ &A_0 = \bigl\{a \in A : \text{ there exists } b \in B \text{ such that } d(a,b) = d(A,B)\bigr\}, \\ &B_0 = \bigl\{b \in B : \text{ there exists } a \in A \text{ such that } d(a,b) = d(A,B)\bigr\}. \end{aligned}$$

Definition 2.2 [8] Given non-self-mappings $S : A \to B$ and $T : A \to B$, an element x^* is called a common best proximity point of the mappings if the following condition is satisfied:

$$d\bigl(x^*, Sx^*\bigr) = d\bigl(x^*, Tx^*\bigr) = d(A,B).$$

We know that best proximity point x for mapping T from A to B is defined as $d(x, Tx) = d(A, B)$ and a common best proximity point is an element at which both functions S and T attain their global minimum since $d(x, Sx) \geq d(A, B)$ and $d(x, Tx) \geq d(A, B)$ for all x.

Definition 2.3 [8] Mappings $S : A \to B$ and $T : A \to B$ are said to commute proximally if the following condition is satisfied:

$$\bigl[d(u, Sx) = d(v, Tx) = d(A,B)\bigr] \quad \Rightarrow \quad Sv = Tu$$

for all x, u, and v in A.

Definition 2.4 [8] A mapping $T : A \to B$ is said to dominate a mapping $S : A \to B$ proximally if there exists a nonnegative number $\beta < 1$ such that

$$d(u_1, Sx_1) = d(u_2, Sx_2) = d(A,B)$$

and

$$d(v_1, Tx_1) = d(v_2, Tx_2) = d(A,B)$$

imply that $d(u_1, u_2) \leq \beta d(v_1, v_2)$ for all $x_1, x_2, u_1, u_2, v_1, v_2 \in A$.

Definition 2.5 [7] Mappings $S : A \to B$ and $T : A \to B$ are said to be swapped proximally if the following condition satisfied:

$$\bigl[d(y,u) = d(y,v) = d(A,B) \text{ and } Su = Tv\bigr] \quad \Rightarrow \quad Sv = Tu,$$

for all $x, u \in A$, $y \in B$. It is clear that any two self-mappings on a same set can be swapped proximally.

Theorem 2.1 [1] *Let (X,d) be a complete metric space. Then every contraction mapping has a unique fixed point. It is known as the Banach contraction principle.*

Definition 2.6 [12] Let $(X,\preceq,d)$ be a partially ordered metric space. A function $\mathcal{P} : X \times X \to \mathbb{R}$ is called a $\mathcal{P}$-function w.r.t. $\preceq$ in X if the following conditions are satisfied:

1. $\mathcal{P}(x,y) \geq 0$ for every comparable $x, y \in A$;
2. for any sequences $\{x_n\}$, $\{y_n\}$ in X such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} \mathcal{P}(x_n,y_n) = \mathcal{P}(x,y)$;
3. for any sequences $\{x_n\}$, $\{y_n\}$ in X such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} \mathcal{P}(x_n,y_n) = 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.
4. for any sequences $\{x_n\}$, $\{y_n\}$ in X such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} d(x_n,y_n)$ exists, then $\lim_{n\to\infty} \mathcal{P}(x_n,y_n)$ also exists.

Definition 2.7 [12] Let $(X,\preceq,d)$ be a partially ordered metric space. A mapping $f : X \to X$ is called a $\mathcal{P}$-contraction w.r.t. $\preceq$ if there exists a $\mathcal{P}$-function $\mathcal{P} : X \times X \to \mathbb{R}$ w.r.t. $\preceq$ in X such that

$$d(fx,fy) \leq d(x,y) - \mathcal{P}(x,y)$$

for all $x, y \in X$.

3 $\mathcal{S}$-Functions and $\mathcal{S}$-contractions

Definition 3.1 Let (X,d) be a complete metric space. Then a function $\mathcal{S} : A \to B$, where A and B are subsets of (X,d), is called an $\mathcal{S}$-function in X if it satisfies the following hypotheses:

1. if there exists another mapping $F : A \to B$ in (X,d), then $d(Fx,Fy) < d(\mathcal{S}x,\mathcal{S}y)$ with $F(A_0) \subseteq \mathcal{S}(A_0)$;
2. for any $A, B \subseteq (X,d)$, if A_0 and B_0 are nonempty, then $\mathcal{S}(A_0) \subseteq B_0$;
3. for any sequence $\{x_m\}$ in A, if $\lim_{m\to\infty} x_m = x \in A$, then $\lim_{m\to\infty} \mathcal{S}x_m = \mathcal{S}x \in B$, where $A \subseteq X$ and $m \in \mathbb{N}$.

Definition 3.2 Let (X,d) be a Cauchy (complete) metric space. A mapping $T : A \to B$ with $T(A_0) \subseteq B_0$ is called an $\mathcal{S}$-contraction in (X,d) if there is an $\mathcal{S}$-function in (X,d) such that

$$d(Tx,Ty) \leq \beta\bigl(d(x,y)\bigr)d(\mathcal{S}x,\mathcal{S}y)$$

for $x, y \in A$ and $\beta \in \mathcal{F}$.

We denote by $\mathcal{F}$ the collection of all mappings $\beta : [0,\infty) \to [0,1)$ such that $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$. The following theorem is based on the existence of a unique common best proximity point for non-self-maps and also furnishes fixed point results in Cauchy metric spaces.

Example 3.1 Consider $\mathbb{R}^2$ with Euclidean metric. Let $A = \{0\} \times [0,\infty)$ and $B = \{1\} \times [0,\infty)$. Let us define $\mathcal{S} : A \to B$ and $T : A \to B$ as

$$T(0,y) = \left(1, \frac{y}{3}\right)$$

and

$$\mathcal{S}(0,y) = \left(1, \frac{y}{2}\right).$$

Let us take $A_0 = A$ and $B_0 = B$, where

$$\beta\bigl(d\bigl((x_1,y_1),(x_2,y_2)\bigr)\bigr) = \frac{1}{2} \quad \text{for } x_1, x_2 \in \{0,1\}$$

and 0 otherwise. Then

$$\begin{aligned} d(Tx,Ty) &= d\bigl(T(0,y_1),T(0,y_2)\bigr) \\ &= d\left(\left(1,\frac{y_1}{3}\right),\left(1,\frac{y_2}{3}\right)\right) \\ &\leq \left|\frac{y_1-y_2}{3}\right| \\ &\leq \left|\frac{y_1-y_2}{2}\right| \\ &= \frac{1}{2}|y_1-y_2| \\ &= \beta\bigl(d\bigl((x_1,y_1),(x_2,y_2)\bigr)\bigr)d\bigl(\mathcal{S}(0,y_1),\mathcal{S}(0,y_2)\bigr). \end{aligned}$$

Here since $x_1, x_2 \in \{0,1\}$, it follows that $0 \leq \frac{1}{2} = \beta(d((x_1,y_1),(x_2,y_2))) < 1$. Therefore, $d(T(0,y_1),T(0,y_2)) \leq \beta(d((x_1,y_1),(x_2,y_2)))d(\mathcal{S}(0,y_1),\mathcal{S}(0,y_2))$. The given mapping T is $\mathcal{S}$-contraction.

Theorem 3.1 *Let A and B be closed $(A,B \neq \phi)$ be subsets of a complete metric space (X,d), and let $T : A \to B$ be a continuous $\mathcal{S}$-contraction such that for any $x_0 \in A_0$ such that $\mathcal{S}$ and T commute proximally, there exists a unique common best proximity point in A such that*

$$d(x,Tx) = d(A,B)$$

and

$$d(x,\mathcal{S}x) = d(A,B).$$

Proof Let us take an element $x_0 \in A_0$. Since T is an $\mathcal{S}$-contraction, so that $T(A_0) \subseteq \mathcal{S}(A_0)$, we get an element $x_1 \in A_0$ such that $Tx_0 = \mathcal{S}x_1$. Again, since $T(A_0) \subseteq \mathcal{S}(A_0)$, there exists an element x_2 in A_0 such that $Tx_1 = \mathcal{S}x_2$. Continuing in this manner inductively, we obtain a sequence $\{x_n\}$ in A_0 with

$$Tx_{n-1} = \mathcal{S}x_n$$

for all positive integers n by using the fact $T(A_0) \subseteq \mathcal{S}(A_0)$.

Since $T(A_0) \subseteq B_0$, there occurs a point u_n in A_0 such that

$$d(Tx_n, u_n) = d(A, B)$$

for any nonnegative integer n.

Since $Tx_{n-1} = \mathcal{S}x_n$, it follows for any x_m and x_n that

$$\begin{aligned} d(Tx_m, Tx_n) &\leq \beta d(\mathcal{S}x_m, \mathcal{S}x_n) \\ &= \beta d(Tx_{m-1}, Tx_{n-1}). \end{aligned}$$

This shows that $\{Tx_n\}$ is a Cauchy sequence and thus converges to some element y in B. Similarly, the sequence $\{\mathcal{S}x_n\}$ also converges to $y \in B$. Since the sets A and B are closed, this means that if we take any point from these sets, then it will converge in the same set. For any $u_k \in A$, we have a sequence $\{u_k\}$ in A, and it converges to some $u \in A$ because A is closed.

Since $T(A_0) \subseteq B_0$, there exists a point u_n in A_0 such that

$$d(Tx_n, u_n) = d(A, B)$$

for any $n \in \mathbb{Z}^+ \cup \{0\}$. So, for any $x_n \in A_0$, it follows that

$$d(\mathcal{S}x_n, u_{n-1}) = d(Tx_{n-1}, u_{n-1}) = d(A, B)$$

for any nonnegative integer n. Since $\mathcal{S}$ and T commute proximally, we obtain

$$\mathcal{S}u_n = Tu_{n-1}$$

for any nonnegative integer n. Therefore, $\mathcal{S}$ and T are continuous mappings, so that $Tu = \lim_{n\to\infty} Tu_{n-1}$ and $\mathcal{S}u = \lim_{n\to\infty} \mathcal{S}u_n$; hence, $\mathcal{S}u$ and Tu are identical mappings.

Since $T(A_0) \subseteq B_0$, there is an element $x \in A$ such that

$$d(x, Tu) = d(A, B)$$

and

$$d(x, \mathcal{S}u) = d(A, B).$$

Again, since $\mathcal{S}$ and T both commute proximally, $\mathcal{S}x = Tx$. Thus, we get

$$d(Tu, Tx) \leq \beta d(\mathcal{S}u, \mathcal{S}x) = \beta d(Tu, Tx),$$

which shows that $Tu = Tx$ and hence $\mathcal{S}u = \mathcal{S}x$. So, we have

$$d(x, \mathcal{S}x) = d(x, \mathcal{S}u) = d(A, B),$$
$$d(x, Tx) = d(x, Tu) = d(A, B).$$

Thus, x is a common best proximity point of the mappings T and $\mathcal{S}$.

Now, we have to prove the uniqueness of common optimal approximate solution. Let there exists another common best proximity point x^* of mappings $\mathcal{S}$ and T. Then

$$d(x^*, Tx^*) = d(A, B),$$
$$d(x^*, \mathcal{S}x^*) = d(A, B).$$

As we know, $\mathcal{S}$ and T commute proximally; therefore, $\mathcal{S}x = Tx$ and $\mathcal{S}x^* = Tx^*$. We may write

$$d(Tx, Tx^*) \leq \beta d(\mathcal{S}x, \mathcal{S}x^*) = \beta d(Tx, Tx^*),$$

which shows that $Tx = Tx^*$. Hence, to conclude we have

$$d(x, x^*) \leq d(x, Tx) + d(Tx, Tx^*) + d(Tx^*, x^*) = 2d(A, B),$$

which implies that $\frac{1}{2}d(x, x^*) \leq d(A, B)$. If $d(A, B) = 0$, then the uniqueness is proved. If $d(A, B) > 0$, then it is a contradiction because we know that $d(A, B)$ itself is a minimum distance. This completes the proof. □

Our main result asserts that if we take $\beta(t) = k \in [0, 1)$ and take self-mappings in $A = B = X$ in Theorem 3.1, by the definition of $\mathcal{S}$-functions and $\mathcal{S}$-contractions we get the following fixed point result of [7] and [8].

Corollary 3.1 *Let (X, d) be a complete metric space. Define an $\mathcal{S}$-function as a self-mapping $S : X \to X$ and an $\mathcal{S}$-contraction as a self-mapping $T : X \to X$ that obeys the following conditions*:

1. *There is a nonnegative real number $k < 1$ such that*

   $$d(Tx_n, Tx_{n+1}) \leq kd(Sx_n, Sx_{n+1})$$

   for any x_n and x_{n+1} in A.
2. *S and T commute and are continuous.*
3. *$T(X) \subseteq S(X)$.*

Then the mappings S and T have a unique common fixed point.

Further, if we take $\beta(t) = k \in [0, 1)$ and add two extra conditions in Theorem 3.1, then we get the main result of [7]:

Corollary 3.2 *Let A and B be nonempty closed subsets of a metric space (X,d), which is a Cauchy space such that A_0 and B_0 are nonempty. Let $\mathcal{S}$-functions $S : A \to B$ and $T : A \to B$ satisfy the following conditions*:

1. *There is a nonnegative real number $\beta < 1$ such that*

$$d(Tx_n, Tx_{n+1}) \leq \beta d(Sx_n, Sx_{n+1})$$

   for any x_n and x_{n+1} in A.
2. *S and T commute proximally, swapped proximally and continuous.*
3. *$T(A_0) \subseteq B_0$, $S(A_0)$.*
4. *A is approximatively compact with respect to B.*

Then we have a common best proximity point x in A. In addition, if x^* is another common best proximity point of S and T, then

$$d(x, x^*) \leq 2d(A,B).$$

Proof By adding hypotheses (4) and the condition that S and T swapped proximally in our Theorem 3.1 we obtain the above well-known result. □

Example 3.2 Let us take $\mathbb{R}^2$ with Euclidean metric. Let $A = \{(x,y) : x \leq 1\}$ and $B = \{(x,y) : x \geq 2\}$. Let us define $\mathcal{S} : A \to B$ and $T : A \to B$ as

$$T(x,y) = \left(-4x + 6, \frac{y}{5}\right)$$

and

$$\mathcal{S}(x,y) = \left(-5x + 7, \frac{y}{4}\right).$$

We get $d(A,B) = 1$, $A_0 = \{(1,y) : y \in [0,\infty)\}$ and $B_0 = \{(2,y) : y \in [0,\infty)\}$, where

$$\beta\big(d\big((x_1,y_1),(x_2,y_2)\big)\big) = \frac{4}{5} \quad \text{for } x_1, x_2 \leq 1$$

and 0 otherwise.

Hence, given that non-self-mappings commute proximally, we also have

$$\begin{aligned} d(Tx, Ty) &= d\big(T(x_1,y_1), T(x_2,y_2)\big) \\ &= d\left(\left(-4x_1 + 6, \frac{y_1}{5}\right), \left(-4x_2 + 6, \frac{y_2}{5}\right)\right) \\ &\leq \big|(-4x_1 + 4x_2)\big| + \left|\left(\frac{y_1 - y_2}{5}\right)\right| \\ &\leq \frac{4}{5}\left(\big|(-5x_1 + 5x_2 + 7 - 7)\big| + \left|\left(\frac{y_1}{4} - \frac{y_2}{4}\right)\right|\right) \\ &= \frac{4}{5} d\left(\left(-5x_1 + 7, \frac{y_1}{4}\right), \left(-5x_2 + 7, \frac{y_2}{4}\right)\right) \\ &= \beta\big(d\big((x_1,y_1),(x_2,y_2)\big)\big) d\big(\mathcal{S}(x_1,y_1), \mathcal{S}(x_2,y_2)\big), \end{aligned}$$

where since $x_1, x_2 \leq 1$, $0 \leq \frac{4}{5} = \beta(d((x_1, y_1), (x_2, y_2))) < 1$. Therefore, $d(T(x_1, y_1), T(x_2, y_2)) \leq \beta(d((x_1, y_1), (x_2, y_2)))d(\mathcal{S}(x_1, y_1), \mathcal{S}(x_2, y_2))$. The given mapping T is an $\mathcal{S}$-contraction. Furthermore, all other hypotheses of Theorem 3.1 are also satisfied, so that there exists a unique common best proximity point for the non-self-mappings $\mathcal{S}$ and T, which is $(1, 0)$.

Remark 3.1 Replacing condition (1) in Corollary 3.2 with the condition that T dominates S proximally, we get the main result of [8].

4 $\mathcal{G}_{\mathcal{P}}$-Functions and $\mathcal{G}_{\mathcal{P}}$-contractions

Motivated by [12], we define here the generalized $\mathcal{P}$-functions and contractions.

Definition 4.1 Let $(X, \preceq, d)$ be a partially ordered metric space. A mapping $\mathcal{A} : A \times A \to \mathbb{R}$, where $A \subseteq X$, is said to be a generalized $\mathcal{P}$-function ($\mathcal{G}_{\mathcal{P}}$-function) w.r.t. $\preceq$ in X if it satisfies the following conditions:

1. $\mathcal{A}(x, y) \geq 0$ for every comparable $x, y \in A$;
2. for any sequences $\{x_n\}$, $\{y_n\}$ in A such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} \mathcal{A}(x_n, y_n) = \mathcal{A}(x, y)$;
3. for any sequences $\{x_n\}$, $\{y_n\}$ in A such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} \mathcal{A}(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.
4. for any sequences $\{x_n\}$, $\{y_n\}$ in A such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} d(x_n, y_n)$ exists, then $\lim_{n\to\infty} \mathcal{A}(x_n, y_n)$ also exists.

Definition 4.2 Let $(X, \preceq, d)$ be a partially ordered metric space. A mapping $f : A \to B$ is called a $\mathcal{G}_{\mathcal{P}}$-contraction w.r.t. $\preceq$ if there is a $\mathcal{G}_{\mathcal{P}}$-function $\mathcal{A} : A \times A \to \mathbb{R}$, where $A \subseteq X$, w.r.t. $\preceq$ in X such that

$$d(fx, fy) \leq d(x, y) - \mathcal{A}(x, y)$$

for any $x, y \in A$.

Theorem 4.1 *Let $A, B \neq \phi$ be closed subsets of a Cauchy partially ordered metric space $(X, \preceq, d)$ such that A_0 is nonempty. Define a map $f : A \to B$ satisfying the following conditions:*

1. *f is a continuous generalized $\mathcal{P}$-contraction w.r.t. $\preceq$ with $f(A_0) \subseteq B_0$;*
2. *the pair (A, B) has the P-property.*

Then there exists a unique x^* in A such that $d(x^*, fx^*) = d(A, B)$.

Proof Since for any $x_n \in A$, $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and A_0 is nonempty, so if we take $x_0 \in A_0$, then since $f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$d(x_1, fx_0) = d(A, B). \tag{1}$$

Again, since $f(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, fx_1) = d(A, B). \tag{2}$$

Repeating this technique, we have a sequence $\{x_n\}$ in A_0 satisfying $d(x_{n+1}, fx_n) = d(A, B)$ for any $n \in \mathbb{N}$. Since the pair (A, B) has the P-property, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n) - \mathcal{A}(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n) \end{aligned} \tag{3}$$

for all $n \in \mathbb{N}$. Therefore, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence.

Suppose that there exists $n_0 \in \mathbb{N}$ such that $0 = d(x_{n_0}, x_{n_0+1}) = d(fx_{n_0-1}, fx_{n_0})$ and, consequently,

$$fx_{n_0-1} = fx_{n_0}.$$

Therefore, we obtain

$$d(A, B) = d(x_{n_0}, fx_{n_0-1}) = d(x_{n_0}, fx_{n_0}).$$

Note that $x_0 \in A_0$, $x_1 \in B_0$, and $x_0 = x_1$, so that $A \cap B$ is nonempty, and then $d(A, B) = 0$. Thus, in this case, there exists a best proximity point, that is, there exists a unique $x^* \in A$ such that $d(x^*, fx^*) = d(A, B)$.

In the contrary case, let $d(x_n, x_{n+1}) > 0$ for any $n \in \mathbb{N}$. Since $\{d(x_n, x_{n+1})\}$ is a bounded sequence of real numbers, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Thus, there is $s \geq 0$ such that $\lim_{n\to\infty} \mathcal{A}(x_n, x_{n+1}) = s$. We have to prove that $r = 0$. Let $r \neq 0$ and $r > 0$. Then by the generalized $\mathcal{P}$-contractivity of f we have

$$r \leq r - s.$$

Thus, $s = 0$, so we get $r = 0$, a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$, $\{x_{n_k}\}$ of $\{x_n\}$ such that, for any positive integers $n_k > m_k \geq k$,

$$r_k := d(x_{m_k}, x_{n_k}) \geq \epsilon$$

and $d(x_{m_k}, x_{n_k-1}) < \epsilon$ for any $k \in \{1, 2, 3, \ldots\}$.

For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. Then, we have

$$\begin{aligned} \epsilon \leq r_k &= d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \epsilon + \gamma_{n_k-1}. \end{aligned} \tag{4}$$

Taking the limit as $k \to \infty$, we get

$$\begin{aligned}
\epsilon &\leq \lim_{k\to\infty} r_k \\
&< \epsilon + \lim_{k\to\infty} \gamma_{n_k-1} \\
&\Rightarrow \quad \epsilon \leq \lim_{k\to\infty} r_k < \epsilon + 0,
\end{aligned}$$

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{5}$$

Notice also that

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{n_k}, x_{m_k}) + d(x_{n_k1}, x_{n_k-1}).$$

By taking limits as $n \to \infty$ we get $\lim_{n\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon$, which implies that $\lim_{n\to\infty} \mathcal{A}(x_{m_k-1}, x_{n_k-1})$ also exists. Now, by the generalized $\mathcal{P}$-contractivity we have $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) - \mathcal{A}(x_{m_k-1}, x_{n_k-1})$. After taking limits, we get

$$0 \leq \lim_{k\to\infty} \mathcal{A}(x_{m_k-1}, x_{n_k-1}),$$

which implies that $\lim_{k\to\infty} \mathcal{A}(x_{m_k-1}, x_{n_k-1}) = 0$. Thus,

$$\lim_{n\to\infty} d(x_{m_k-1}, x_{n_k-1}) = 0.$$

Hence $\epsilon = 0$, which is a contradiction. So, $\{x_n\}$ is a Cauchy sequence in A, and A is a closed subset of X. There is $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. Since f is continuous, we have

$$\begin{aligned}
&fx_n \to fx^* \\
&\quad \Rightarrow \quad d(x_{n+1}, fx_n) \to d\bigl(x^*, fx^*\bigr).
\end{aligned}$$

Note that $\{d(x_{n+1}, fx_n)\}$ is a constant sequence that has the value $d(A, B)$, so that

$$d\bigl(x^*, fx^*\bigr) = d(A, B),$$

that is, x^* is a unique best proximity point of f. □

Remark 4.1 By taking $A = B = X$ in the last theorem we obtain the result of [12].

Example 4.1 Consider $X = \mathbb{R}^2$. Let $A = \{1\} \times [0, \infty)$ and $B = \{0\} \times [0, \infty)$, and take $A_0 = A$ and $B_0 = B$. Here, $d(A, B) = 3$ and $A, B \neq \phi$ are closed subsets of X,

We define $f : A \to B$ as

$$f(1, x) = \left(0, \frac{x^3}{9}\right),$$

where $(0, x) \in A$ and $\frac{x^3}{9} \in [0, \infty)$.

Let d and $\mathcal{A}: A \times A \to \mathbb{R}$ be defined as

$$\mathcal{A}\big((x_1, x_2), (y_1, y_2)\big) = \begin{cases} 1 & \text{if } (x_1, x_2) = (y_1, y_2), \\ 2\max\{2x_1, x_2 + y_2\} & \text{elsewhere,} \end{cases}$$

and

$$d\big((x_1, x_2), (y_1, y_2)\big) = \begin{cases} 3 & \text{if } (x_1, x_2) = (y_1, y_2), \\ 3\max\{2(x_1 + y_1), x_2 + y_2\} & \text{elsewhere,} \end{cases}$$

where $\sqsubseteq$ is an order in X, and $\mathcal{A}$ is a generalized $\mathcal{P}$-function. Clearly, X is a partially ordered metric space, and f is continuous w.r.t. $\sqsubseteq$ Now, if we take $x, y \in A$ such that $x = y$, then the given mapping f is a $\mathcal{G}_{\mathcal{P}}$-contraction, and thus for $x \neq y$, we can check that for $(1, x_2), (1, y_2) \in A$, we have

$$\begin{aligned} d(fx, fy) &= d\big(f(1, x_2), f(1, y_2)\big) \\ &= d\left(\left(0, \frac{x_2^3}{9}\right), \left(0, \frac{y_2^3}{9}\right)\right) \\ &= 3\max\left\{0, \frac{x_2^3 + y_2^3}{9}\right\} \\ &\leq \frac{x_2^3 + y_2^3}{3} \\ &\leq \frac{x_2}{3} + \frac{y_2}{3} \\ &\leq x_2 + y_2 \\ &\leq \max\{2x_1, x_2 + y_2\} \\ &= 3\max\{2x_1, x_2 + y_2\} - 2\max\{2x_1, x_2 + y_2\} \\ &\leq 3\max\big\{2(x_1 + y_1), x_2 + y_2\big\} - 2\max\{2x_1, x_2 + y_2\} \\ &= d(x, y) - \mathcal{A}(x, y). \end{aligned}$$

Thus, the given mapping f is a $\mathcal{G}_{\mathcal{P}}$-contraction. Also, the P-property is satisfied here, and so Theorem 4.1 is verified. Hence, there is a unique best proximity point for the given mapping f, and it is $(1, 0)$.

5 Conclusions

In this article, the authors introduced the new notions of $\mathcal{S}$-contractions and $\mathcal{G}_{\mathcal{P}}$-contractions. These contractions and the results in this paper introduced new techniques for finding optimal approximate and global optimal approximate solutions in ordered metric spaces.

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All authors declared that they have no competing interests.

Authors' contributions
All the authors contributed equally, and they also read and finalize manuscript.

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