Hussain and Taoudi *Fixed Point Theory and Applications* (2016) 2016:65
DOI 10.1186/s13663-016-0555-y

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for multivalued mappings in ordered Banach spaces with application to integral inclusions

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Abstract

This paper provides new common fixed point theorems for pairs of multivalued and single-valued mappings operating between ordered Banach spaces. Our results lead to new existence theorems for a system of integral inclusions.

MSC: 47H10; 47H30

Keywords: fixed point theorems; measure of weak noncompactness; ordered Banach space; integral inclusions

1 Introduction

Over the last decades, a lot of research has been devoted to the study of the existence of common fixed points for pairs of single-valued and multivalued mappings in ordered Banach spaces [1–9]. In a recent paper, Dhage [2] proved some common fixed point theorems for pairs of weakly isotone condensing mappings in an ordered Banach space. Due in part to the immediate application of this interesting result to differential and integral equations, many researchers tried to obtain the conclusion of [2] under weaker assumptions; for example, see [1, 5], and [10]. However, all results so far obtained in this direction need to assume some compactness conditions relative to the strong topology (Monch-type conditions, contractiveness, or condensiveness with respect to a measure of noncompactness). In the present work, we show that some compactness conditions relative to the weak topology are sufficient and reasonably convenient to get the same conclusion as in [2]. More precisely, we combine the advantages of the strong topology (i.e. the involved mappings will be closed with respect to the strong topology) with the advantages of the weak topology (i.e. the maps will satisfy some compactness conditions relative to the weak topology) to draw new conclusions about common fixed point for a pair of multivalued mappings. Our results furnish an efficient tool to develop an existence theory for a system of integral inclusions (see Section 3).

The article is arranged as follows. Firstly, new applicable common fixed point results are presented in Section 2. In Section 3, we prove the existence of continuous solutions to a system of integral inclusions under appropriate assumptions. In particular, we illustrate how the compactness requirements used in the literature for such a class of integral inclusions may be relaxed. In the remainder of this section, we gather some notation and preliminary facts. Let X be a real Banach space, and let P be a subset of X. The set P is called an *order cone* if:

(i) P is closed, nonempty, and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

An order cone permits to define a partial order in X by $x \leq y$ iff $y - x \in P$. The *positive cone* of X is defined by $X^+ = \{x \in X : 0 \leq x\}$. The order cone P is called *normal* if there is $N > 0$ such that, for all $x, y \in X$, we have

$$0 \leq x \leq y \quad \Rightarrow \quad \|x\| \leq N\|y\|.$$

Remark 1.1

(i) Let K be a compact Hausdorff space, and E be an ordered Banach space with normal positive cone. We denote by $C(K, E)$ the Banach space of all continuous E-valued functions on K with the usual maximum norm. $C(K, E)$ is an ordered Banach space with natural ordering whose positive cone is given by

$$C^+(K, E) = \{f \in C(K, E) : f(x) \in E^+, \forall x \in K\}.$$

Since E^+ is normal, so is $C^+(K, E)$.

(ii) Let Ω be a Lebesgue-measurable subset of $\mathbb{R}^n$. Each of the Banach spaces $L_p(\Omega)$, $1 \leq p \leq \infty$, is an ordered Banach space with respect to the natural ordering, whose positive cone is normal [11].

The following definitions are frequently used in the subsequent part of this paper.

Definition 1.1 [10] Let M be a nonempty subset of an ordered Banach space X, and let $S, T\colon M \to 2^M$ be two mappings. We say that S is T-weakly isotone increasing (resp. T-weakly isotone decreasing) if for all $x \in M$, $y \in S(x)$, and $z \in T(y)$, we have $S(x) \leq T(y) \leq S(z)$ (resp. $S(x) \geq T(y) \geq S(z)$). We say that S is T-weakly isotone if it is either T-weakly isotone increasing or T-weakly isotone decreasing.

Definition 1.2 [10] Let M be a nonempty subset of an ordered Banach space X. A mapping $T\colon M \to 2^M$ is said to be monotone-closed (resp. weakly monotone-closed) if for each monotone sequence $\{x_n\}$ in M with $x_n \to x_0$ (resp. $x_n \rightharpoonup x_0$) and for each sequence $\{y_n\}$ with $y_n \in T(x_n)$ and $y_n \to y_0$ (resp. $y_n \rightharpoonup y_0$), we have $y_0 \in T(x_0)$.

Let M be a nonempty subset of an ordered Banach space X, and let $a \in M$. Let $S, T\colon M \to 2^M$ be two mappings. For later use, we introduce the following conditions.

($\mathcal{A}1$) If $\{x_n\}$ is a monotone sequence of M such that

$$\{x_n\} \subset \{a\} \cup S(\{x_n\}) \cup T(\{x_n\}),$$

then $\{x_n\}$ has a weakly convergent subsequence.

($\mathcal{A}2$) If $\{x_n\}$ is a monotone sequence of M such that

$$\{x_n\} \subset \{a\} \cup S(T(\{x_n\})),$$

then $\{x_n\}$ has a weakly convergent subsequence.

Remark 1.2 Note that hypothesis ($\mathcal{A}1$) holds for every pair (S,T) satisfying *condition* D_M or *weak condition* D_M (see [10]).

In what follows, by ψ we denote a measure of weak noncompactness (MWNC) on the Banach space X. We refer the reader to [12] for the axiomatic definition of such a measure. One of the most frequently exploited measures of weak noncompactness was defined by De Blasi [13] as follows:

$$w(M) = \inf\{r > 0 : \text{there exists } W \text{ weakly compact such that } M \subseteq W + B_r\},$$

for each bounded subset M of X; here, B_r stands for the closed ball of X centered at origin with radius r.

Let M be a closed convex subset of X, and let $T, S\colon M \subseteq X \to 2^X$ be two mappings. The pair (S,T) is said to be *weakly condensing* (resp. *weakly countably condensing*) if $T(M)$ and $S(T(M))$ are bounded and there is an MWNC ψ on X such that for every bounded (resp. countable bounded) subset A of M such that $\psi(A) > 0$ and $\psi(T(A)) > 0$, we have $\psi(S(T(A))) < \psi(A)$. The pair (S,T) is said to be *weakly monotone-condensing* if $T(M)$ and $S(T(M))$ are bounded and there is an MWNC ψ on X such that for every bounded monotone sequence $\{x_n\}$ such that $\psi(\{x_n\}) > 0$ and $\psi(T(\{x_n\})) > 0$, we have $\psi(S(T(\{x_n\}))) < \psi(\{x_n\})$.

Remark 1.3 It is worth mentioning that if (S,T) is weakly monotone-condensing, then (S,T) satisfies ($\mathcal{A}2$).

2 Common fixed point theorems

Before proceeding with the main results, we give a useful technical lemma, which we will employ in the sequel. This result is really interesting and may have several applications. It says that every monotone sequence that has a weakly convergent subsequent is strongly convergent. The proof can be adapted from [11], Theorem 2.2(c). However, we provide here a different proof.

Lemma 2.1 *Let X be an ordered real Banach space with a normal order cone. Suppose that $\{x_n\}$ is a monotone sequence that has a subsequence $\{x_{n_k}\}$ converging weakly to x_∞. Then $\{x_n\}$ converges strongly to x_∞. Moreover, if $\{x_n\}$ is an increasing sequence, then $x_n \le x_\infty$ $(n = 1,2,3,\ldots)$; if $\{x_n\}$ is a decreasing sequence, then $x_\infty \le x_n$ $(n = 1,2,3,\ldots)$.*

Proof Suppose that $\{x_n\}_n$ is increasing. Let $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$ that converges weakly to x_∞, and let

$$\mathcal{F} = \overline{\operatorname{conv}}\{x_{n_k} : k \ge 1\}$$

be the closed convex hull of $\{x_{n_k} : k \ge 1\}$. Since the norm-closure of $\mathcal{F}$ coincides with the weak closure, it follows that $x_\infty \in \mathcal{F}$. Hence, for each $\epsilon > 0$, there exist

$$y = \alpha_1 x_{n_1} + \cdots + \alpha_p x_{n_p} \in \mathcal{F}, \quad \alpha_1 \ge 0,\ldots,\alpha_n \ge 0, \alpha_1 + \cdots + \alpha_p = 1, \tag{2.1}$$

such that $\|y - x_\infty\| < \frac{\epsilon}{N}$. Now (2.1) and $x_{n_k} \le x_\infty$ imply $0 \le x_\infty - x_{n_k} \le x_\infty - y$ for all $k \ge p$. Keeping in mind that the cone is normal with constant N, we infer that $\|x_{n_k} - x_\infty\| \le N\|y - x_\infty\| < \epsilon$ for all $k \ge p$. As a result, $x_{n_k} \to x_\infty$, and so

$$0 \le x_\infty - x_m \le x_\infty - x_{n_k}$$

for $m \ge n_k$. Using once again the fact that the cone is normal with constant N, we get $\|x_m - x_\infty\| \le N\|x_{n_k} - x_\infty\|$. Consequently, $\{x_n\}$ converges strongly to x_∞, as desired. The case where $\{x_n\}$ is decreasing is similar. This completes the proof. □

Now we are in a position to state the main result of this section.

Theorem 2.1 *Let X be an ordered Banach space with a normal order cone. Let M be a nonempty closed convex subset of X, and $S, T: M \to 2^M$ be two monotone-closed* (*or weakly monotone-closed*) *mappings satisfying*:

(i) *The pair (S,T) verifies $(\mathcal{A}1)$ or $(\mathcal{A}2)$*;
(ii) *S is T-weakly isotone.*

Then T and S have a common fixed point.

Proof Let $x \in M$ be fixed. Consider the sequence $\{x_n\}$ defined by

$$x_0 = x, \qquad x_{2n+1} \in Sx_{2n}, \qquad x_{2n+2} \in Tx_{2n+1}, \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

Suppose first that S is T-weakly isotone increasing on M. Notice that $x_1 \in Sx_0$ and $x_2 \in Tx_1$. Since $Sx_0 \le Ty \le Sz$ for all $y \in Sx_0$ and $z \in Ty$, it follows that $Sx_0 \le Tx_1 \le Sx_2$. In particular, $x_1 \le x_2 \le x_3$. Similar arguments yield

$$x_1 \le x_2 \le \cdots \le x_n \le \cdots. \tag{2.3}$$

Now, it is a simple matter to check that

$$\{x_2, x_4, \ldots\} \subseteq \{x_2\} \cup S\bigl(T\bigl(\{x_2, x_4, \ldots\}\bigr)\bigr) \tag{2.4}$$

and

$$\{x_1, x_2, \ldots\} \subseteq \{x_1\} \cup S\bigl(\{x_1, x_2, \ldots\}\bigr) \cup T\bigl(\{x_1, x_2, \ldots\}\bigr). \tag{2.5}$$

From our assumptions we know that $\{x_n\}$ has a weakly convergent subsequence. Referring to Lemma 2.1, we see that $\{x_n\}$ is strongly convergent. Let x^* be its strong limit. Now since $x_{2n+1} \in Sx_{2n}$ and S is monotone-closed (or weakly monotone-closed), we have $x^* \in Sx^*$. Similar arguments yield $x^* \in Tx^*$. To complete the proof, we consider the case where S is T-weakly isotone decreasing on M. In this case, the sequence $\{x_n\}$ is monotone decreasing. Following the previous reasoning, we show that $\{x_n\}$ converges strongly to a common fixed point of S and T. □

Remark 2.1

(i) If X is a reflexive Banach space and M is bounded, then in Theorem 2.1, we assume only (ii) (since the closed unit ball is weakly compact).
(ii) If S and T are only weakly monotone closed, then we can draw the same conclusion.

From Theorem 2.1 we can derive several important corollaries. We present a number of selected corollaries that extend and encompass several well-known results in the literature.

Corollary 2.1 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to 2^M$ be two monotone-closed mappings. In addition, suppose that S is a T-weakly isotone mapping and the pair (S, T) satisfies $(\mathcal{A}1)$. Then S and T have a common fixed point.*

Remark 2.2 Corollary 2.1 extends [10], Theorem 4.1 and [14], Theorem 3.1.

As a consequence of Theorem 2.1, we obtain the following sharpening of [14], Theorem 3.1 and [14], Corollary 3.1.

Corollary 2.2 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to 2^M$ be two closed (with closed graph) weakly isotone mappings satisfying condition D_M. Then S and T have a common fixed point.*

Corollary 2.3 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to 2^M$ be two closed (with closed graph), weakly countably condensing weakly isotone mappings. Then S and T have a common fixed point.*

Corollary 2.4 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to 2^M$ be monotone-closed and such that the pair (S, T) is weakly monotone-condensing. If S is a T-weakly isotone mapping, then S and T have a common fixed point.*

Remark 2.3 Corollary 2.4 extends [10], Theorem 4.2. In [10] the pair (S, T) is assumed to be monotone-condensing.

Another consequence of Theorem 2.1 is the following.

Corollary 2.5 *Let X be an ordered Banach space with a normal order cone, and ψ a measure of weak noncompactness on X. Let M be a nonempty closed convex subset of X, and $S, T: M \to 2^M$ be two monotone-closed mappings satisfying:*

(i) *S is $1 - \psi$-monotone-contractive, that is, $S(M)$ is bounded, and for every bounded monotone sequence $\{x_n\}$ of M, we have $\psi(S(\{x_n\})) \leq \psi(\{x_n\})$;*
(ii) *T is ψ-monotone-condensing, that is, $T(M)$ is bounded, and for every bounded monotone sequence $\{x_n\}$ of M with $\psi(\{x_n\}) > 0$, we have $\psi(T(\{x_n\})) < \psi(\{x_n\})$;*
(iii) *S is T-weakly isotone.*

Then T and S have a common fixed point.

Proof Thanks to Theorem 2.1, it suffices to show that the pair (S, T) is weakly monotone-condensing. To see this, let $\{x_n\}$ a bounded monotone sequence of M with $\psi(\{x_n\}) > 0$ and $\psi(\{Tx_n\}) > 0$. Then

$$\psi(S(T(\{x_n\}))) \leq \psi(T(\{x_n\})) < \psi(\{x_n\}).$$

This achieves the proof. □

Remark 2.4 Corollary 2.5 extends [1], Theorem 1.1 and [2], Theorem 2.1.

Whereas our interest lies in ensuring the existence of common fixed points for multivalued mappings in ordered Banach spaces, we will not rule out the possibility that the involved operators are single-valued. Consequently, the results reported here also hold for single-valued mappings with slight modifications. Specifically, the single-valued version of Theorem 2.1 reduces to the following:

Theorem 2.2 *Let X be an ordered Banach space with a normal order cone. Let M be a nonempty closed convex subset of X, and $S, T: M \to M$ be two monotone-continuous* (*or weakly monotone-continuous*) *mappings satisfying*:

(i) *The pair (S, T) satisfies $(\mathcal{A}1)$ or $(\mathcal{A}2)$;*

(ii) *S is T-weakly isotone.*

Then T and S have a common fixed point.

Recall that $T: M \to M$ is said to be monotone-continuous (resp. weakly monotone continuous) if for any monotone sequence (x_n) that converges strongly (resp. weakly) to x the sequence (Tx_n) converges strongly (resp. weakly) to Tx.

Similarly to the multivalued case, we can get the following corollaries and related results.

Corollary 2.6 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to M$ be two monotone-continuous mappings. Suppose in addition that S is a T-weakly isotone mapping and the pair (S, T) satisfies $(\mathcal{A}1)$. Then S and T have a common fixed point.*

Remark 2.5 Corollary 2.6 extends [10], Theorem 3.1 and [14], Theorem 2.1.

Corollary 2.7 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to M$ be two continuous weakly isotone mappings satisfying condition D_M. Then S and T have a common fixed point.*

As an immediate consequence of Theorem 2.2, we can derive the following sharpening of [14], Theorem 2.1.

Corollary 2.8 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to M$ be two continuous weakly countably condensing weakly isotone mappings. Then S and T have a common fixed point.*

Corollary 2.9 *Let M be a closed subset of an ordered Banach space X with normal order cone, and let $S, T: M \to M$ be monotone-continuous such that the pair (S, T) is weakly monotone-condensing. If S is a T-weakly isotone mapping, then S and T have a common fixed point.*

Remark 2.6 Corollary 2.9 extends [10], Theorem 3.4. In [10] the pair (S, T) is assumed to be monotone-condensing.

Another consequence of Theorem 2.1 is the following result.

Corollary 2.10 *Let X be an ordered Banach space with a normal order cone, and ψ a measure of weak noncompactness on X. Let M be a nonempty closed convex subset of X, and $S, T\colon M \to M$ be two monotone-continuous mappings satisfying*:

(i) *S is $1-\psi$-monotone-contractive,*

(ii) *T is ψ-monotone-condensing,*

(iii) *S is T-weakly isotone.*

Then T and S have a common fixed point.

3 Application

Consider the nonlinear integral inclusions

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s)F\big(s,x(s)\big)\,ds \tag{3.1}$$

and

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s)G\big(s,x(s)\big)\,ds \tag{3.2}$$

for $t \in [0,1]$, where $\sigma: [0,1] \to [0,1]$, $q\colon \to E$, $k\colon [0,1] \times [0,1] \to \mathbb{R}$ are continuous, and $F, G\colon [0,1] \times E \to C(E)$; Here E is a Banach space with norm $\|\cdot\|_E$. By a common solution for the integral inclusions (3.1) and (3.2) we mean a continuous function $x\colon [0,1] \to E$ such that

$$x(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v_1(s)\,ds$$

and

$$x(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v_2(s)\,ds$$

for some $v_1, v_2 \in B([0,1],E)$ satisfying $v_1(t) \in F(t,x(t))$ and $v_2(t) \in G(t,x(t))$, $t \in [0,1]$, where $B([0,1],E)$ is the space of all E-valued Bochner-integrable functions on $[0,1]$. In 2007, Turkoglu and Altun [15] proved an existence theorem of common solutions for the integral inclusions (3.1) and (3.2) by using a common fixed point theorem of Dhage et al. [14]. In this section, we provide sufficient conditions ensuring the existence of common solutions for (3.1) and (3.2) within the same framework and under weaker hypotheses compared with those of [15].

From now on, we will simply follow the notation and terminology of [15], and the integral inclusions (3.1)-(3.2) will be discussed under the same assumptions as in [15], except for the compactness assumption ($\mathcal{H}2$), which will be replaced with the weaker assumption:

($\mathcal{H}'2$) For any countable and bounded set A of E, $w(F([0,1]\times A)) \le \lambda_F w(A)$ and $w(G([0,1]\times A)) \le \lambda_G w(A)$ for some reals $\lambda_F, \lambda_G > 0$, where w is the De Blasi measure of weak noncompactness.

Theorem 3.1 *Under the assumptions above, the integral inclusions* (3.1) *and* (3.2) *have a common solution in $C([0,1],E)$, provided that $\lambda_F\lambda_G M^2 < 1$.*

Proof Let $X = C([0,1],E)$ and consider the order interval $[a,b]$, which is well defined in view of $(H5)$. To allow the abstract formulation of our problem, we define two multivalued mappings $S,T\colon [a,b] \to 2^X$ as in [15] by

$$Sx = \left\{u : u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(s)\,ds, v \in S_F^1(x)\right\},$$

$$Tx = \left\{u : u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(s)\,ds, v \in S_G^1(x)\right\}.$$

Our strategy is to apply Corollary 2.4 to find a common fixed point for the multivalued mappings S and T that is, in turn, a common solution to the integral inclusions (3.1) and (3.2). First, notice that a similar reasoning as in [15] yields that S and T are weakly isotone and have closed graphs. Now we show that the pair (S,T) is weakly countably condensing. The main idea of the proof is similar, in spirit, to that of [15]. Let us for now choose an arbitrary $A \subset [a,b]$ countable. Then invoking [16], Theorem 1.1, the mean value theorem, and the properties of the De Blasi measure of weak noncompactness [13], we get, for $t \in [0,1]$,

$$\begin{aligned} w\bigl(T\bigl(A(t)\bigr)\bigr) &\le w\left(\bigcup\left\{q(t) + \int_0^{\sigma(t)} k(t,s)G\bigl(s,x(s)\bigr)\,ds : x \in A\right\}\right) \\ &\le w\left(\bigcup\left\{\int_0^{\sigma(t)} k(t,s)G\bigl(s,x(s)\bigr)\,ds : x \in A\right\}\right) \\ &\le w\left(\int_0^{\sigma(t)} k(t,s)G\bigl(s,A(s)\bigr)\right) \\ &\le w\bigl(\sigma(t)\overline{\operatorname{conv}}\bigl(\bigl\{k(t,s)G\bigl(s,A(s)\bigr)\bigr\}\bigr)\bigr) \\ &\le Mw(G\bigl([0,1] \times A\bigl([0,1]\bigr)\bigr) \\ &\le M\lambda_F w\bigl(A\bigl([0,1]\bigr)\bigr) \\ &\le M\lambda_F w(A). \end{aligned}$$

Now, following a standard argument used in [15], we can prove that $T(A)$ is uniformly bounded and equicontinuous, and therefore

$$w\bigl(T(A)\bigr) \le M\lambda_G w(A). \tag{3.3}$$

Similarly, we have

$$w\bigl(S(A)\bigr) \le M\lambda_F w(A). \tag{3.4}$$

Combining (3.3) and (3.4), we arrive at

$$w\bigl(S\bigl(T(A)\bigr)\bigr) \le M^2\lambda_F\lambda_G w(A), \tag{3.5}$$

where $M^2\lambda_F\lambda_G < 1$. This shows that the pair (S,T) is weakly countably condensing. The result follows from Corollary 2.4. □

Remark 3.1

1. The results of [15] are established under much stronger hypotheses on the multivalued mappings F and G, made necessary by the fact that some compactness conditions are imposed (Condition ($\mathcal{H}2$)). In our considerations, only some conditions expressed in terms of the De Blasi measure of weak noncompactness are required (Condition ($\mathcal{H}'2$)). We also emphasize that it is straightforward to guarantee the existence of solutions for the integral inclusions (3.1) and (3.2) if the De Blasi measure of noncompactness is replaced by any axiomatic measure of weak noncompactness satisfying conditions of Ambrosetti type [16], Theorem 1.1.
2. If E is reflexive and F and G are bounded (i.e. map bounded sets into bounded sets), then condition ($\mathcal{H}'2$) is automatically satisfied.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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Received: 8 October 2015 Accepted: 20 May 2016 Published online: 31 May 2016

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