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Discussion on some coupled fixed point theorems

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Abstract
In this paper, we show that, unexpectedly, most of the coupled fixed point theorems (on ordered metric spaces) are in fact immediate consequences of well-known fixed point theorems in the literature.

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1 Introduction

In recent years, there has been recent interest in establishing fixed point theorems on ordered metric spaces with a contractivity condition which holds for all points that are related by partial ordering.

In [1], Ran and Reurings established the following fixed point theorem that extends the Banach contraction principle to the setting of ordered metric spaces.

Theorem 1.1 (Ran and Reurings [1]) *Let $(X,\preceq)$ be an ordered set endowed with a metric d and $T:X\to X$ be a given mapping. Suppose that the following conditions hold*:

(i) *(X,d) is complete*;
(ii) *T is continuous nondecreasing* (*with respect to $\preceq$*);
(iii) *there exists $x_0\in X$ such that $x_0\preceq Tx_0$*;
(iv) *there exists a constant $k\in(0,1)$ such that for all $x,y\in X$ with $x\succeq y$*,

$$d(Tx,Ty)\le kd(x,y).$$

Then T has a fixed point. Moreover, if for all $(x,y)\in X\times X$ there exists a $z\in X$ such that $x\preceq z$ and $y\preceq z$, we obtain uniqueness of the fixed point.

Nieto and López [2] extended the above result for a mapping T not necessarily continuous by assuming an additional hypothesis on $(X,\preceq,d)$.

Theorem 1.2 (Nieto and López [2]) *Let $(X,\preceq)$ be an ordered set endowed with a metric d and $T:X\to X$ be a given mapping. Suppose that the following conditions hold*:

(i) *(X,d) is complete*;
(ii) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x\in X$, then $x_n\preceq x$ for all n*;
(iii) *T is nondecreasing*;

(iv) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;
(v) *there exists a constant* $k \in (0,1)$ *such that for all* $x, y \in X$ *with* $x \succeq y$,

$$d(Tx, Ty) \leq kd(x, y).$$

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X \times X$ *there exists a* $z \in X$ *such that* $x \preceq z$ *and* $y \preceq z$, *we obtain uniqueness of the fixed point.*

Theorems 1.1 and 1.2 are extended and generalized by many authors. Before presenting some of theses results, we need to introduce some functional sets.

Denote by Φ the set of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Φ_1) φ is continuous nondecreasing;
(Φ_2) $\varphi^{-1}(\{0\}) = \{0\}$.

Denote by $\mathcal{S}$ the set of functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \longrightarrow 1 \quad \text{implies} \quad t_n \longrightarrow 0.$$

Denote by Θ the set of functions $\theta : [0,\infty)^2 \to [0,1)$ which satisfy the condition:

$$\theta(s_n, t_n) \longrightarrow 1 \quad \text{implies} \quad t_n, s_n \longrightarrow 0.$$

Denote by Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ_1) $\psi(t) < t$ for all $t > 0$;
(Ψ_2) $\lim_{r\to t^+} \psi(r) < t$.

In [3], Harjani and Sadarangani established the following results.

Theorem 1.3 (Harjani and Sadarangani [3]) *Let* $(X, \preceq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:
(i) (X, d) *is complete*;
(ii) T *is continuous nondecreasing*;
(iii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;
(iv) *there exist* $\varphi, \psi \in \Phi$ *such that for all* $x, y \in X$ *with* $x \succeq y$,

$$\psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big).$$

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X \times X$ *there exists a* $z \in X$ *such that* $x \preceq z$ *and* $y \preceq z$, *we obtain uniqueness of the fixed point.*

Theorem 1.4 (Harjani and Sadarangani [3]) *Let* $(X, \preceq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:
(i) (X, d) *is complete*;
(ii) *if a nondecreasing sequence* $\{x_n\}$ *in* X *converges to some point* $x \in X$, *then* $x_n \preceq x$ *for all* n;
(iii) T *is nondecreasing*;
(iv) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;

(v) *there exist* $\varphi, \psi \in \Phi$ *such that for all* $x, y \in X$ *with* $x \succeq y$,

$$\psi\big(d(Tx, Ty)\big) \le \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big).$$

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X \times X$ *there exists a* $z \in X$ *such that* $x \preceq z$ *and* $y \preceq z$, *we obtain uniqueness of the fixed point.*

In [4], Amini-Harandi and Emami established the following results.

Theorem 1.5 (Amini-Harandi and Emami [4]) *Let* $(X, \preceq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:

(i) (X, d) *is complete*;
(ii) T *is continuous nondecreasing*;
(iii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;
(iv) *there exists* $\beta \in \mathcal{S}$ *such that for all* $x, y \in X$ *with* $x \succeq y$,

$$d(Tx, Ty) \le \beta\big(d(x,y)\big)d(x,y).$$

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X \times X$ *there exists a* $z \in X$ *such that* $x \preceq z$ *and* $y \preceq z$, *we obtain uniqueness of the fixed point.*

Theorem 1.6 (Amini-Harandi and Emami [4]) *Let* $(X, \preceq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:

(i) (X, d) *is complete*;
(ii) *if a nondecreasing sequence* $\{x_n\}$ *in* X *converges to some point* $x \in X$, *then* $x_n \preceq x$ *for all* n;
(iii) T *is nondecreasing*;
(iv) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;
(v) *there exists* $\beta \in \mathcal{S}$ *such that for all* $x, y \in X$ *with* $x \succeq y$,

$$d(Tx, Ty) \le \beta\big(d(x,y)\big)d(x,y).$$

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X \times X$ *there exists a* $z \in X$ *such that* $x \preceq z$ *and* $y \preceq z$, *we obtain uniqueness of the fixed point.*

Remark 1.1 Jachymski [5] established that Theorem 1.5 (resp. Theorem 1.6) follows from Theorem 1.3 (resp. Theorem 1.4).

Remark 1.2 Theorems 1.3 and 1.4 hold if $\varphi : [0,\infty) \to [0,\infty)$ satisfies only the following conditions: φ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}$ (see, for example, [6]).

The following results are special cases of Theorem 2.2 in [7].

Theorem 1.7 (Ćirić *et al.* [7]) *Let* $(X, \preceq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:

(i) (X, d) *is complete*;
(ii) T *is continuous nondecreasing*;
(iii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;

(iv) *there exists a continuous function* $\varphi : [0,\infty) \to [0,\infty)$ *with* $\varphi(t) < t$ *for all* $t > 0$ *such that for all* $x, y \in X$ *with* $x \succeq y$,

$$d(Tx, Ty) \leq \varphi\big(d(x,y)\big).$$

Then T *has a fixed point.*

Theorem 1.8 (Ćirić *et al.* [7]) *Let* $(X, \preceq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:

(i) (X,d) *is complete*;
(ii) *if a nondecreasing sequence* $\{x_n\}$ *in* X *converges to some point* $x \in X$, *then* $x_n \preceq x$ *for all* n;
(iii) T *is nondecreasing*;
(iv) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$;
(v) *there exists a continuous function* $\varphi : [0,\infty) \to [0,\infty)$ *with* $\varphi(t) < t$ *for all* $t > 0$ *such that for all* $x, y \in X$ *with* $x \succeq y$,

$$d(Tx, Ty) \leq \varphi\big(d(x,y)\big).$$

Then T *has a fixed point.*

Remark 1.3 Theorems 1.7 and 1.8 hold if we suppose that $\varphi \in \Psi$ (see, for example, [8]).

Let X be a nonempty set and $F : X \times X \to X$ be a given mapping. We say that $(x,y) \in X \times X$ is a coupled fixed point of F if

$$x = F(x,y) \quad \text{and} \quad y = F(y,x).$$

In [9], Bhaskar and Lakshmikantham established some coupled fixed point theorems on ordered metric spaces and applied the obtained results to the study of existence and uniqueness of solutions to a class of periodic boundary value problems. The obtained results in [9] have been extended and generalized by many authors (see, for example, [8, 10–23]).

In this paper, we will prove that most of the coupled fixed point theorems are in fact immediate consequences of well-known fixed point theorems in the literature.

2 Main results

Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and $F : X \times X \to X$ be a given mapping. We endow the product set $X \times X$ with the partial order:

$$(x,y), (u,v) \in X \times X, \quad (x,y) \preceq_2 (u,v) \quad \Longleftrightarrow \quad x \preceq u, \qquad y \succeq v.$$

Definition 2.1 F is said to have the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad \Longrightarrow \quad F(x_1, y) \preceq F(x_2, y);$$

$$y_1, y_2 \in X, \quad y_1 \succeq y_2 \quad \Longrightarrow \quad F(x, y_1) \preceq F(x, y_2).$$

Let $Y = X \times X$. It is easy to show that the mappings $\eta, \delta : Y \times Y \to [0,\infty)$ defined by

$$\eta\big((x,y),(u,v)\big) = d(x,u) + d(y,v);$$
$$\delta\big((x,y),(u,v)\big) = \max\big\{d(x,u), d(y,v)\big\}$$

for all $(x,y),(u,v) \in Y$, are metrics on Y.

Now, define the mapping $T : Y \to Y$ by

$$T(x,y) = \big(F(x,y), F(y,x)\big) \quad \text{for all } (x,y) \in Y.$$

It is easy to show the following.

Lemma 2.1 *The following properties hold*:

(a) *(X,d) is complete if and only if (Y,η) and (Y,δ) are complete*;

(b) *F has the mixed monotone property if and only if T is monotone nondecreasing with respect to $\preceq_2$*;

(c) *$(x,y) \in X \times X$ is a coupled fixed point of F if and only if (x,y) is a fixed point of T.*

2.1 Bhaskar and Lakshmikantham's coupled fixed point results

We present the obtained results in [9] in the following theorem.

Theorem 2.1 (see Bhaskar and Lakshmikantham [9]) *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d. Let $F : X \times X \to X$ be a given mapping. Suppose that the following conditions hold*:

(i) *(X,d) is complete*;

(ii) *F has the mixed monotone property*;

(iii) *F is continuous or X has the following properties*:

  (X_1) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \preceq x$ for all n,*

  (X_2) *if a decreasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \succeq y$ for all n*;

(iv) *there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$*;

(v) *there exists a constant $k \in (0,1)$ such that for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,*

$$d\big(F(x,y), F(u,v)\big) \le \frac{k}{2}\big[d(x,u) + d(y,v)\big].$$

Then F has a coupled fixed point $(x^*,y^*) \in X \times X$. Moreover, if for all $(x,y),(u,v) \in X \times X$ there exists $(z_1,z_2) \in X \times X$ such that $(x,y) \preceq_2 (z_1,z_2)$ and $(u,v) \preceq_2 (z_1,z_2)$, we have uniqueness of the coupled fixed point and $x^* = y^*$.

We will prove the following result.

Theorem 2.2 *Theorem* 2.1 *follows from Theorems* 1.1 *and* 1.2.

Proof From (v), for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$, we have

$$d\big(F(x,y),F(u,v)\big) \leq \frac{k}{2}\big[d(x,u)+d(y,v)\big]$$

and

$$d\big(F(v,u),F(y,x)\big) \leq \frac{k}{2}\big[d(x,u)+d(y,v)\big].$$

This implies that for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,

$$d\big(F(x,y),F(u,v)\big)+d\big(F(v,u),F(y,x)\big) \leq k\big[d(x,u)+d(y,v)\big],$$

that is,

$$\eta\big(T(x,y),T(u,v)\big) \leq k\eta\big((x,y),(u,v)\big)$$

for all $(x,y),(u,v) \in Y$ with $(x,y) \succeq_2 (u,v)$. From Lemma 2.1, since (X,d) is complete, (Y,η) is also complete. Since F has the mixed monotone property, T is a nondecreasing mapping with respect to $\preceq_2$. From (iv), we have $(x_0,y_0) \preceq_2 T(x_0,y_0)$. Now, if F is continuous, then T is continuous. In this case, applying Theorem 1.1, we get that T has a fixed point, which implies from Lemma 2.1 that F has a coupled fixed point. If conditions (X_1) and (X_2) are satisfied, then Y satisfies the following property: if a nondecreasing (with respect to $\preceq_2$) sequence $\{u_n\}$ in Y converges to some point $u \in Y$, then $u_n \preceq_2 u$ for all n. Applying Theorem 1.2, we get that T has a fixed point, which implies that F has a coupled fixed point. If, in addition, we suppose that for all $(x,y),(u,v) \in X \times X$ there exists $(z_1,z_2) \in X \times X$ such that $(x,y) \preceq_2 (z_1,z_2)$ and $(u,v) \preceq_2 (z_1,z_2)$, from the last part of Theorems 1.1 and 1.2, we obtain the uniqueness of the fixed point of T, which implies the uniqueness of the coupled fixed point of F. Now, let $(x^*,y^*) \in X \times X$ be a unique coupled fixed point of F. Since (y^*,x^*) is also a coupled fixed point of F, we get $x^* = y^*$. □

2.2 Harjani, López and Sadarangani's coupled fixed point results

We present the results obtained in [16] in the following theorem.

Theorem 2.3 (see Harjani *et al.* [16]) *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d. Let $F: X \times X \to X$ be a given mapping. Suppose that the following conditions hold*:

- (i) *(X,d) is complete*;
- (ii) *F has the mixed monotone property*;
- (iii) *F is continuous or X has the following properties*:
  - (X_1) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \preceq x$ for all n,*
  - (X_2) *if a decreasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \succeq y$ for all n;*
- (iv) *there exist $x_0,y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$;*

(v) *there exist* $\psi, \varphi \in \Phi$ *such that for all* $(x,y),(u,v) \in X \times X$ *with* $x \succeq u$ *and* $y \preceq v$,

$$\psi\big(d\big(F(x,y),F(u,v)\big)\big) \le \psi\big(\max\big\{d(x,u),d(y,v)\big\}\big) - \varphi\big(\max\big\{d(x,u),d(y,v)\big\}\big).$$

Then F *has a coupled fixed point* $(x^*,y^*) \in X \times X$. *Moreover, if for all* $(x,y),(u,v) \in X \times X$ *there exists* $(z_1,z_2) \in X \times X$ *such that* $(x,y) \preceq_2 (z_1,z_2)$ *and* $(u,v) \preceq_2 (z_1,z_2)$, *we have uniqueness of the coupled fixed point and* $x^* = y^*$.

We will prove the following result.

Theorem 2.4 *Theorem* 2.3 *follows from Theorems* 1.3 *and* 1.4.

Proof From (v), for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$, we have

$$\psi\big(d\big(F(x,y),F(u,v)\big)\big) \le \psi\big(\max\big\{d(x,u),d(y,v)\big\}\big) - \varphi\big(\max\big\{d(x,u),d(y,v)\big\}\big)$$

and

$$\psi\big(d\big(F(v,u),F(y,x)\big)\big) \le \psi\big(\max\big\{d(x,u),d(y,v)\big\}\big) - \varphi\big(\max\big\{d(x,u),d(y,v)\big\}\big).$$

This implies (since ψ is nondecreasing) that for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,

$$\begin{aligned}&\psi\big(\max\big\{d\big(F(x,y),F(u,v)\big),d\big(F(v,u),F(y,x)\big)\big\}\big)\\&\quad\le \psi\big(\max\big\{d(x,u),d(y,v)\big\}\big) - \varphi\big(\max\big\{d(x,u),d(y,v)\big\}\big),\end{aligned}$$

that is,

$$\psi\big(\delta\big(T(x,y),T(u,v)\big)\big) \le \psi\big(\delta\big((x,y),(u,v)\big)\big) - \varphi\big(\delta\big((x,y),(u,v)\big)\big)$$

for all $(x,y),(u,v) \in Y$ with $(x,y) \succeq_2 (u,v)$. Thus we proved that the mapping T satisfies the condition (iv) (resp. (v)) of Theorem 1.3 (resp. Theorem 1.4). The rest of the proof is similar to the above proof. □

2.3 Lakshmikantham and Ćirić's coupled fixed point results

In [8], putting $g = i_X$ (the identity mapping), we obtain the following result.

Theorem 2.5 (see Lakshmikantham and Ćirić's [8]) *Let* $(X,\preceq)$ *be a partially ordered set endowed with a metric* d. *Let* $F : X \times X \to X$ *be a given mapping. Suppose that the following conditions hold*:

(i) (X,d) *is complete*;
(ii) F *has the mixed monotone property*;
(iii) F *is continuous or* X *has the following properties*:

(X_1) *if a nondecreasing sequence* $\{x_n\}$ *in* X *converges to some point* $x \in X$, *then* $x_n \preceq x$ *for all* n,

(X_2) *if a decreasing sequence* $\{y_n\}$ *in* X *converges to some point* $y \in X$, *then* $y_n \succeq y$ *for all* n;

(iv) *there exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0)$ *and* $y_0 \succeq F(y_0, x_0)$;

(v) *there exists* $\varphi \in \Psi$ *such that for all* $(x,y),(u,v) \in X \times X$ *with* $x \succeq u$ *and* $y \preceq v$,

$$d\big(F(x,y), F(u,v)\big) \le \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right).$$

Then F *has a coupled fixed point.*

We will prove the following result.

Theorem 2.6 *Theorem* 2.5 *follows from Theorems* 1.7 *and* 1.8.

Proof From (v), for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$, we have

$$d\big(F(x,y), F(u,v)\big) \le \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right)$$

and

$$d\big(F(v,u), F(y,x)\big) \le \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right).$$

This implies that for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,

$$\frac{d(F(x,y), F(u,v)) + d(F(v,u), F(y,x))}{2} \le \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right),$$

that is,

$$\eta'\big(T(x,y), T(u,v)\big) \le \varphi\big(\eta'\big((x,u),(y,v)\big)\big)$$

for all $(x,y),(u,v) \in Y$ with $(x,y) \succeq_2 (u,v)$. Here, $\eta' : Y \times Y \to [0,\infty)$ is the metric on Y given by

$$\eta'\big((x,y),(u,v)\big) = \frac{\eta((x,y),(u,v))}{2} \quad \text{for all } (x,y),(u,v) \in Y.$$

Thus we proved that the mapping T satisfies the condition (iv) (resp. (v)) of Theorem 1.7 (resp. Theorem 1.8). Then T has a fixed point, which implies that F has a coupled fixed point. □

2.4 Luong and Thuan's coupled fixed point results

Luong and Thuan [18] presented a coupled fixed point result involving an ICS mapping.

Definition 2.2 Let (X,d) be a metric space. A mapping $S : X \to X$ is said to be ICS if S is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Sx_n\}$ is convergent, then $\{x_n\}$ is also convergent.

We have the following result.

Lemma 2.2 *Let (X,d) be a metric space and $S : X \to X$ be an ICS mapping. Then the mapping $d_S : X \times X \to [0,\infty)$ defined by*

$$d_S(x,y) = d(Sx,Sy) \quad \textit{for all } x,y \in X,$$

is a metric on X. Moreover, if (X,d) is complete, then (X,d_S) is also complete.

Proof Let us prove that d_S is a metric on X. Let $x,y \in X$ such that $d_S(x,y) = 0$. This implies that $Sx = Sy$. Since S is injective, we obtain that $x = y$. Other properties of the metric can be easily checked. Now, suppose that (X,d) is complete and let $\{x_n\}$ be a Cauchy sequence in the metric space (X,d_S). This implies that $\{Sx_n\}$ is Cauchy in (X,d). Since (X,d) is complete, $\{Sx_n\}$ is convergent in (X,d) to some point $y \in X$. Since S is an ICS mapping, $\{x_n\}$ is also convergent in (X,d) to some point $x \in X$. On the other hand, the continuity of S implies the convergence of $\{Sx_n\}$ in (X,d) to Sx. By the uniqueness of the limit in (X,d), we get that $y = Sx$, which implies that $d_S(x_n,x) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus $\{x_n\}$ is a convergent sequence in (X,d_S). This proves that (X,d_S) is complete. □

The obtained result in [18] is the following.

Theorem 2.7 (see Luong and Thuan [18]) *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d. Let $S : X \to X$ be an ICS mapping. Let $F : X \times X \to X$ be a given mapping. Suppose that the following conditions hold*:

(i) *(X,d) is complete*;
(ii) *F has the mixed monotone property*;
(iii) *F is continuous or X has the following properties*:

  (X_1) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \preceq x$ for all n,*
  (X_2) *if a decreasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \succeq y$ for all n*;

(iv) *there exist $x_0,y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$*;
(v) *there exists $\psi \in \Psi$ such that for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,*

$$d\big(SF(x,y),SF(u,v)\big) \le \frac{1}{2}\psi\big(d(Sx,Su) + d(Sy,Sv)\big).$$

Then F has a coupled fixed point.

We will prove the following result.

Theorem 2.8 *Theorem* 2.7 *follows from Theorems* 1.7 *and* 1.8.

Proof The condition (v) implies that for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,

$$d_S\big(F(x,y),F(u,v)\big) + d_S\big(F(v,u),F(y,x)\big) \le \psi\big(d_S(x,u) + d_S(y,v)\big),$$

that is,

$$\eta_S\big(T(x,y),T(u,v)\big) \le \psi\big(\eta_S\big((x,y),(u,v)\big)\big),$$

for all $(x,y),(u,v) \in Y$ with $(x,y) \succeq_2 (u,v)$, where η_S is the metric (see Lemma 2.2) on Y defined by

$$\eta_S\big((x,y),(u,v)\big) = d_S(x,u) + d_S(y,v) \quad \text{for all } (x,y),(u,v) \in Y.$$

Thus we proved that the mapping T satisfies the condition (iv) (resp. (v)) of Theorem 1.7 (resp. Theorem 1.8). Then T has a fixed point, which implies that F has a coupled fixed point. □

2.5 Berind's coupled fixed point results

The following result was established in [11].

Theorem 2.9 (see Berinde [11]) *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d. Let $F : X \times X \to X$ be a given mapping. Suppose that the following conditions hold*:

(i) *(X,d) is complete*;
(ii) *F has the mixed monotone property*;
(iii) *F is continuous or X has the following properties*:

  (X_1) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \preceq x$ for all n*,
  (X_2) *if a decreasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \succeq y$ for all n*;

(iv) *there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$*;
(v) *there exists a constant $k \in (0,1)$ such that for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$*,

$$d\big(F(x,y),F(u,v)\big) + d\big(F(y,x),F(v,u)\big) \le k\big[d(x,u) + d(y,v)\big].$$

Then F has a coupled fixed point $(x^*,y^*) \in X \times X$. Moreover, if for all $(x,y),(u,v) \in X \times X$ there exists $(z_1,z_2) \in X \times X$ such that $(x,y) \preceq_2 (z_1,z_2)$ and $(u,v) \preceq_2 (z_1,z_2)$, we have uniqueness of the coupled fixed point and $x^* = y^*$.

We have the following result.

Theorem 2.10 *Theorem* 2.9 *follows from Theorems* 1.1 *and* 1.2.

Proof From the condition (v), the mapping T satisfies

$$\eta\big(T(x,y),T(u,v)\big) \le k\eta\big((x,y),(u,v)\big)$$

for all $(x,y),(u,v) \in Y$ with $(x,y) \succeq_2 (u,v)$. Thus we proved that the mapping T satisfies the condition (iv) (resp. (v)) of Theorem 1.1 (resp. Theorem 1.2). Then T has a fixed point, which implies that F has a coupled fixed point. The rest of the proof is similar to the above proofs. □

2.6 Rasouli and Bahrampour's coupled fixed point results

Theorem 2.11 (see Rasouli and Bahrampour [20]) *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d. Let $F: X\times X\to X$ be a given mapping. Suppose that the following conditions hold*:

(i) *(X,d) is complete*;

(ii) *F has the mixed monotone property*;

(iii) *F is continuous or X has the following properties*:

- (X_1) *if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x\in X$, then $x_n\preceq x$ for all n*,
- (X_2) *if a decreasing sequence $\{y_n\}$ in X converges to some point $y\in X$, then $y_n\succeq y$ for all n*;

(iv) *there exist $x_0,y_0\in X$ such that $x_0\preceq F(x_0,y_0)$ and $y_0\succeq F(y_0,x_0)$*;

(v) *there exists $\beta\in\mathcal{S}$ such that for all $(x,y),(u,v)\in X\times X$ with $x\succeq u$ and $y\preceq v$*,

$$d\big(F(x,y),F(u,v)\big)\le\beta\big(\max\big\{d(x,u),d(y,v)\big\}\big)\max\big\{d(x,u),d(y,v)\big\}.$$

Then F has a coupled fixed point $(x^*,y^*)\in X\times X$. Moreover, if for all $(x,y),(u,v)\in X\times X$ there exists $(z_1,z_2)\in X\times X$ such that $(x,y)\preceq_2(z_1,z_2)$ and $(u,v)\preceq_2(z_1,z_2)$, we have uniqueness of the coupled fixed point and $x^*=y^*$.

We have the following result.

Theorem 2.12 *Theorem* 2.11 *follows from Theorems* 1.5 *and* 1.6.

Proof From the condition (v), the mapping T satisfies

$$\delta\big(T(x,y),T(u,v)\big)\le\beta\big(\delta\big((x,y),(u,v)\big)\big)\delta\big((x,y),(u,v)\big)$$

for all $(x,y),(u,v)\in Y$ with $(x,y)\succeq_2(u,v)$. Thus we proved that the mapping T satisfies the condition (iv) (resp. (v)) of Theorem 1.5 (resp. Theorem 1.6). Then T has a fixed point, which implies that F has a coupled fixed point. The rest of the proof is similar to the above proofs. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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