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Fixed point of Suzuki-Zamfirescu hybrid contractions in partial metric spaces via partial Hausdorff metric

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Abstract
Coincidence point theorems for hybrid pairs of single-valued and multi-valued mappings on an arbitrary non-empty set with values in a partial metric space using a partial Hausdorff metric have been proved. As an application of our main result, the existence and uniqueness of common and bounded solutions of functional equations arising in dynamic programming are discussed.
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1 Introduction and preliminaries

Fixed point theory plays a fundamental role in solving functional equations [1] arising in several areas of mathematics and other related disciplines as well. The Banach contraction principle is a key principle that made a remarkable progress towards the development of metric fixed point theory. Markin [2] and Nadler [3] proved a multi-valued version of the Banach contraction principle employing the notion of a Hausdorff metric. Afterwards, a number of generalizations (see [4–9]) were obtained using different contractive conditions. The study of hybrid type contractive conditions involving single-valued and multi-valued mappings is a valuable addition to the metric fixed point theory and its applications (for details, see [10–14]). Among several generalizations of the Banach contraction principle, Suzuki's work [15, Theorem 2.1] led to a number of results (for details, see [13, 16–21]).

On the other hand, Matthews [22] introduced the concept of a partial metric space as a part of the study of denotational semantics of dataflow networks. He obtained a modified version of the Banach contraction principle, more suitable in this context (see also [23, 24]). Since then, results obtained in the framework of partial metric spaces have been used to constitute a suitable framework to model the problems related to the theory of computation (see [22, 25–28]). Recently, Aydi *et al.* [29] initiated the concept of a partial Hausdorff metric and obtained an analogue of Nadler's fixed point theorem [3] in partial metric spaces.

The aim of this paper is to obtain some coincidence point theorems for a hybrid pair of single-valued and multi-valued mappings on an arbitrary non-empty set with values in a partial metric space. Our results extend, unify and generalize several known results in the existing literature (see [13, 15, 21, 30]). As an application, we obtain the existence and uniqueness of a common and bounded solution for Suzuki-Zamfirescu class of functional equations under contractive conditions weaker than those given in [1, 31–34].

Throughout this work, a mapping $\omega : [0,1) \to (\frac{1}{2}, 1]$ is defined by

$$\omega(r) = \frac{1}{1+r} \quad \text{for all } r \in [0,1). \tag{1.1}$$

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all non-negative real numbers and the set of all positive integers, respectively. Consistent with [22, 29, 35, 36], the following definitions and results will be needed in the sequel.

Definition 1.1 [22] Let X be any non-empty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(P1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;
(P2) $p(x,x) \leq p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space. If $p(x,y) = 0$, then (P1) and (P2) imply that $x = y$. But the converse does not hold in general. A classical example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p : X \times X \to \mathbb{R}^+$ is defined as $p(x,y) = \max\{x,y\}$ (see also [37]).

Example 1.2 [22] If $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$, then

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$$

defines a partial metric p on X.

For more interesting examples, we refer to [23, 27, 28, 35, 38, 39]. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open balls (p-balls) $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in a partial metric space (X,p) is called convergent to a point $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$ (for details, see [22]). If p is a partial metric on X, then the mapping $p^S : X \times X \to \mathbb{R}^+$ given by $p^S(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X. Furthermore, a sequence $\{x_n\}$ converges in a metric space (X,p^S) to a point $x \in X$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{1.2}$$

Definition 1.3 [22] Let (X,p) be a partial metric space, then

(a) A sequence $\{x_n\}$ in X is called Cauchy if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(b) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Lemma A [22, 35] *Let (X,p) be a partial metric space, then*

(c) *A sequence $\{x_n\}$ in X is Cauchy in (X,p) if and only if it is Cauchy in (X,p^S).*
(d) *A partial metric space (X,p) is complete if and only if (X,p^S) is complete.*

Consistent with [29], let $CB^p(X)$ be the family of all non-empty, closed and bounded subsets of the partial metric space (X,p), induced by the partial metric p. Note that closedness is taken from (X,τ_p) (τ_p is the topology induced by p) and boundedness is given as follows: A is a bounded subset in (X,p) if there exists an $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(a,a) + M$. For $A,B \in CB^p(X)$ and $x \in X$, define $\delta_p : CB^p(X) \times CB^p(X) \to [0,\infty)$ and

$$p(x,A) = \inf\{p(x,a) : a \in A\},$$

$$\delta_p(A,B) = \sup\{p(a,B) : a \in A\},$$

$$\delta_p(B,A) = \sup\{p(b,A) : b \in B\},$$

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}.$$

It can be verified that $p(x,A) = 0$ implies $p^S(x,A) = 0$, where $p^S(x,A) = \inf\{p^S(x,a) : a \in A\}$.

Lemma B [35] *Let (X,p) be a partial metric space and A be a non-empty subset of X, then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$.*

Proposition 1.4 [29] *Let (X,p) be a partial metric space. For any $A,B,C \in CB^p(X)$, we have the following*:

(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$;
(ii) $\delta_p(A,A) \leq \delta_p(A,B)$;
(iii) $\delta_p(A,B) = 0$ *implies* $A \subseteq B$;
(iv) $\delta_p(A,B) \leq \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Proposition 1.5 [29] *Let (X,p) be a partial metric space. For any $A,B,C \in CB^p(X)$, we have the following*:

(h1) $H_p(A,A) \leq H_p(A,B)$;
(h2) $H_p(A,B) = H_p(B,A)$;
(h3) $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c,c)$;
(h4) $H_p(A,B) = 0$ *implies that* $A = B$.

The mapping $H_p : CB^p(X) \times CB^p(X) \to [0,\infty)$ is called a partial Hausdorff metric induced by a partial metric p. Every Hausdorff metric is a partial Hausdorff metric, but the converse is not true (see Example 2.6 in [29]).

Lemma C [29] *Let (X,p) be a partial metric space and $A,B \in CB^p(X)$ and $h > 1$, then for any $a \in A$, there exists a $b \in B$ such that $p(a,b) \leq hH_p(A,B)$.*

Theorem 1.6 [29] *Let (X,p) be a partial metric space. If $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have $H_p(Tx,Ty) \leq kp(x,y)$, where $k \in (0,1)$. Then T has a fixed point.*

Definition 1.7 Let (X,p) be a partial metric space and $f : X \longrightarrow X$ and $T : X \longrightarrow CB^p(X)$. A point $x \in X$ is said to be (i) a *fixed point of* f if $x = f(x)$, (ii) a *fixed point of* T if $x \in T(x)$, (iii) a *coincidence point of a pair* (f,T) if $fx \in Tx$, (iv) a *common fixed point of the pair* (f,T) if $x = fx \in Tx$.

We denote the set of all fixed points of f, the set of all coincidence points of the pair (f,T) and the set of all common fixed points of the pair (f,T) by $F(f)$, $C(f,T)$ and $F(f,T)$, respectively. Motivated by the work of [4, 13], we give the following definitions in partial metric spaces.

Definition 1.8 Let (X,p) be a partial metric space and $f : X \longrightarrow X$ and $T : X \longrightarrow CB^p(X)$. The pair (f,T) is called (i) *commuting* if $Tfx = fTx$ for all $x \in X$, (ii) *weakly compatible* if the pair (f,T) commutes at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f,T)$, (iii) IT-commuting [11] at $x \in X$ if $fTx \subseteq Tfx$.

Definition 1.9 Let (X,p) be a partial metric space and Y be any non-empty set. Let $f : Y \longrightarrow X$ and $T : Y \longrightarrow CB^p(X)$ be single-valued and multi-valued mappings, respectively. Suppose that $x_0 \in Y$, then the set

$$O(f,T;x_0) = \{y_n : y_{n+1} = fx_{n+1} \in Tx_n \text{ for } n = 0,1,2,\ldots\} \tag{1.3}$$

is called an orbit for the pair (f,T) at x_0. A partial metric space X is called (f,T)-orbitally complete if and only if every Cauchy sequence in the orbit for (f,T) at x_0 converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_m,x_n)$.

Singh and Mishra [13] introduced Suzuki-Zamfirescu type hybrid contractive condition in complete metric spaces. In the context of partial metric spaces, the condition is given as follows.

Definition 1.10 Let (X,p) be a partial metric space, $f : Y \longrightarrow X$ and $T : Y \longrightarrow CB^p(X)$ be single-valued and multi-valued mappings, respectively. The hybrid pair (f,T) is said to satisfy *Suzuki-Zamfirescu hybrid contraction condition* if there exists $r \in [0,1)$ such that $\omega(r)p(fx,Tx) \le p(fx,fy)$ implies that

$$H_p(Tx,Ty) \le rM_{p,f}(x,y) \tag{1.4}$$

for all $x,y \in Y$ and

$$M_{p,f}(x,y) = \max\left\{p(fx,fy), \frac{p(fx,Tx)+p(fy,Ty)}{2}, \frac{p(fx,Ty)+p(fy,Tx)}{2}\right\}. \tag{1.5}$$

Lemma D *Let (X,p) be a partial metric space, $f : Y \longrightarrow X$ and $T : Y \longrightarrow CB^p(X)$ be single-valued and multi-valued mappings, respectively. Then the partial metric space (X,p) is (f,T)-orbitally complete if and only if (X,p^S) is (f,T)-orbitally complete.*

Proof Suppose that (X,p^S) is (f,T)-orbitally complete and x_0 is an arbitrary element of X. If $\{y_n\}$ is a Cauchy sequence in $O(f,T;x_0)$ in (X,p), then it is also Cauchy in (X,p^S). Therefore, by (1.2) we deduce that there exists y in X such that

$$p(y,y) = \lim_{n\to\infty} p(y,y_n) = \lim_{n,m\to\infty} p(y_n,y_m),$$

and $\{y_n\}$ converges to y in (X,p). Conversely, let (X,p) be (f,T)-orbitally complete. If $\{y_n\}$ is a Cauchy sequence in $O(f,T;x_0)$ in (X,p^S), then it is also a Cauchy sequence in (X,p). Therefore,

$$p(y,y) = \lim_{n\to\infty} p(y,y_n) = \lim_{n,m\to\infty} p(y_n,y_m).$$

For given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\left|p(y,y_n) - p(y,y)\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|p(y,y_n) - p(y_n,y_m)\right| < \frac{\varepsilon}{2}$$

for all $m,n > n_\varepsilon$. Consequently, we have

$$\begin{aligned} p^S(y,y_n) &= 2p(y,y_n) - p(y,y) - p(y_n,y_m) \\ &\leq \left|p(y,y_n) - p(y,y) + p(y,y_n) - p(y_n,y_m)\right| \\ &\leq \left|p(y,y_n) - p(y,y)\right| + \left|p(y,y_n) - p(y_n,y_m)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

whenever $m,n > n_\varepsilon$. The result follows. □

2 Coincidence points of a hybrid pair of mappings

In the following theorem, the existence of coincidence points of a hybrid pair of single-valued and multi-valued mappings that satisfy Suzuki-Zamfirescu hybrid contraction condition in partial metric spaces is established.

Theorem 2.1 *Let (X,p) be a partial metric space and Y be any non-empty set. Assume that a pair of mappings $f : Y \longrightarrow X$ and $T : Y \longrightarrow CB^p(X)$ satisfies Suzuki-Zamfirescu hybrid contraction condition with $T(Y) \subset f(Y)$. If there exists $u_0 \in Y$ such that $f(Y)$ is (f,T)-orbitally complete at u_0, then $C(f,T) \neq \phi$. If $Y = X$ and (f,T) is IT-commuting at coincidence points of (f,T), then $F(f,T) \neq \phi$ provided that fz is a fixed point of f for some $z \in C(f,T)$.*

Proof Let $h = 1/\sqrt{r}$ and $u_0 \in Y$ be such that $y_0 = fu_0$. By the given assumption, we have $Tu_0 \subseteq f(Y)$. So, there exists a point $u_1 \in Y$ such that $y_1 = fu_1 \in Tu_0$. As $h > 1$, so by Lemma C, there exists a point $y_2 \in Tu_1$ such that

$$p(fu_1,y_2) \leq hH_p(Tu_0,Tu_1).$$

Using the fact that $Tu_1 \subseteq f(Y)$, we obtain a point $u_2 \in Y$ such that $y_2 = fu_2 \in Tu_1$. Therefore,

$$p(fu_1,fu_2) \leq hH_p(Tu_0,Tu_1).$$

Since

$$\omega(r)p(fu_0, Tu_0) \leq \omega(r)p(fu_0, fu_1) \leq p(fu_0, fu_1),$$

we have

$$\begin{aligned} p(fu_1, fu_2) &\leq hH_p(Tu_0, Tu_1) \\ &\leq hr\max\left\{p(fu_0, fu_1), \frac{p(fu_0, Tu_0) + p(fu_1, Tu_1)}{2}, \frac{p(fu_0, Tu_1) + p(fu_1, Tu_0)}{2}\right\} \\ &\leq \frac{1}{\sqrt{r}}r\max\left\{p(y_0, y_1), \frac{p(y_0, y_1) + p(y_1, y_2)}{2}, \frac{p(y_0, y_2) + p(y_1, y_1)}{2}\right\} \\ &\leq \sqrt{r}\max\left\{p(y_0, y_1), \frac{p(y_0, y_1) + p(y_1, y_2)}{2}\right\}. \end{aligned}$$

If

$$\max\left\{p(y_0, y_1), \frac{p(y_0, y_1) + p(y_1, y_2)}{2}\right\} = p(y_0, y_1),$$

then

$$p(y_1, y_2) \leq hH_p(Tu_0, Tu_1) \leq \sqrt{r}p(y_0, y_1).$$

If

$$\max\left\{p(y_0, y_1), \frac{p(y_0, y_1) + p(y_1, y_2)}{2}\right\} = \frac{p(y_0, y_1) + p(y_1, y_2)}{2},$$

then we obtain

$$p(y_1, y_2) \leq \frac{\sqrt{r}}{2 - \sqrt{r}}p(y_0, y_1) \leq \sqrt{r}p(y_0, y_1).$$

As $fu_2 \in Tu_1$, we choose $y_3 \in Tu_2$ such that $p(fu_2, y_3) \leq hH(Tu_1, Tu_2)$. Using the fact that $Tu_2 \subseteq f(Y)$, we obtain a point $u_3 \in Y$ such that $y_3 = fu_3 \in Tu_2$ and

$$p(fu_2, fu_3) \leq hH_p(Tu_1, Tu_2).$$

Since

$$\omega(r)p(fu_1, Tu_1) \leq \omega(r)p(fu_1, fu_2) \leq p(fu_1, fu_2),$$

so we have

$$\begin{aligned} p(fu_2, fu_3) &\leq hH_p(Tu_1, Tu_2) \\ &\leq hr\max\left\{p(fu_1, fu_2), \frac{p(fu_1, Tu_1) + p(fu_2, Tu_2)}{2}, \frac{p(fu_1, Tu_2) + p(fu_2, Tu_1)}{2}\right\} \end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{\sqrt{r}} r \max\left\{p(y_1,y_2), \frac{p(y_1,y_2)+p(y_2,y_3)}{2}, \frac{p(y_1,y_3)+p(y_2,y_2)}{2}\right\}\\ &\leq \sqrt{r}\max\left\{p(y_1,y_2), \frac{p(y_1,y_2)+p(y_2,y_3)}{2}\right\}.\end{aligned}$$

Following the arguments similar to those given above, we obtain

$$p(y_2,y_3) \leq \sqrt{r}p(y_1,y_2),$$

which further implies that

$$p(y_2,y_3) \leq (\sqrt{r})^2 p(y_0,y_1).$$

Continuing this process, we obtain a sequence $\{y_n\} \subset Y$ such that for any integer $n \geq 0$, $y_{n+1} = fu_{n+1} \in Tu_n$ and

$$p(y_n,y_{n+1}) \leq (\sqrt{r})^n p(y_0,y_1)$$

for every $n \in \mathbb{N}$. This shows that $\lim_{n\to\infty} p(y_n,y_{n+1}) = 0$. Since

$$p(y_n,y_n) + p(y_{n+1},y_{n+1}) \leq 2p(y_n,y_{n+1}),$$

so we obtain

$$\lim_{n\to\infty} p(y_n,y_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} p(y_{n+1},y_{n+1}) = 0.$$

Now, for $m > n \geq 1$, we have

$$\begin{aligned}p^S(y_n,y_{n+m}) &= 2p(y_n,y_{n+m}) - p(y_n,y_n) - p(y_{n+m},y_{n+m})\\ &\leq 2p(y_n,y_{n+1}) + 2p(y_{n+1},y_{n+2}) + \cdots + 2p(y_{n+m-1},y_{n+m})\\ &\leq 2\big((\sqrt{r})^n + (\sqrt{r})^{n+1} + \cdots + (\sqrt{r})^{n+m-1}\big)p(y_0,y_1).\end{aligned}$$

It follows that $\{y_n\}$ is a Cauchy sequence in $(f(Y),p^S)$. By Lemma A, we have $\{y_n\}$ is a Cauchy sequence in $(f(Y),p)$. Since $(f(Y),p)$ is (f,T)-orbitally complete at u_0, so again by Lemma D, $(f(Y),p^S)$ is (f,T)-orbitally complete at u_0. Hence, there exists an element $u \in f(Y)$ such that $\lim_{n\to\infty} p^S(y_n,y) = 0$. This implies that

$$\lim_{n\to\infty} p(y_n,u) = \lim_{n\to\infty} p(y_n,y_n) = p(u,u) = 0. \tag{2.1}$$

Let $z \in f^{-1}u$, then $z \in Y$ and $u = fz$. Now,

$$\begin{aligned}&p(fz,Tx) \leq p(fz,fu_{n+1}) + p(fu_{n+1},Tx) - p(fu_{n+1},fu_{n+1}) \quad \text{and}\\ &p(fu_{n+1},Tx) \leq p(fu_{n+1},fu_n) + p(fu_n,fz) + p(fz,Tx) - p(fu_n,fu_n) - p(fz,fz)\end{aligned}$$

give

$$\lim_{n\to\infty} p(fu_{n+1},Tx) = p(fz,Tx).$$

Similarly, we can show that

$$\lim_{n\to\infty} p(fu_n, Tx) = p(fz, Tx).$$

Now, we will claim that

$$p(fz, Tx) \le rp(fz, fx) \quad \text{for any } fx \in f(Y) - \{fz\}. \tag{2.2}$$

If $x = z$ or $fx = fz$, then $p(fx, Tx) = 0$. This gives $p^S(fx, Tx) = 0$, which implies that $fx \in Tx$ and we are done. Now from (2.1), there exists a positive integer n_0 such that for all $n \ge n_0$,

$$p(fz, fu_{n+1}) \le \frac{1}{3}p(fz, fx) \quad \text{and} \quad p(fz, fu_n) \le \frac{1}{3}p(fz, fx).$$

So, for any $n \ge n_0$, we have

$$\begin{aligned}
\omega(r)p(fu_n, Tu_n) &\le p(fu_n, Tu_n) \le p(fu_n, fu_{n+1}) \\
&\le p(fu_n, fz) + p(fz, fu_{n+1}) - p(fz, fz) \le \frac{2}{3}p(fz, fx) \\
&\le p(fz, fx) - \frac{1}{3}p(fz, fx) \le p(fz, fx) - p(fz, fu_n) \\
&\le p(fu_n, fx) - p(fu_n, fu_n) \le p(fu_n, fx).
\end{aligned}$$

Hence, for any $n \ge n_0$, we obtain

$$\omega(r)p(fu_n, Tu_n) \le p(fu_n, fx).$$

This implies

$$\begin{aligned}
p(fu_{n+1}, Tx) &\le H_p(Tu_n, Tx) \\
&\le r\max\left\{p(fu_n, fx), \frac{p(fu_n, Tu_n) + p(fx, Tx)}{2}, \frac{p(fu_n, Tx) + p(fx, Tu_n)}{2}\right\} \\
&\le r\max\left\{p(y_n, fx), \frac{p(y_n, y_{n+1}) + p(fx, Tx)}{2}, \frac{p(y_n, Tx) + p(fx, y_{n+1})}{2}\right\} \\
&\le r\max\left\{p(y_n, u) + p(u, fx) - p(u, u), \frac{p(y_n, y_{n+1}) + p(fx, Tx)}{2},\right. \\
&\qquad \left.\frac{p(y_n, u) + p(u, Tx) - p(u, u) + p(fx, u) + p(u, y_{n+1}) - p(u, u)}{2}\right\}.
\end{aligned}$$

On taking limit as n tends to ∞, we obtain

$$\begin{aligned}
p(fz, Tx) &\le r\max\left\{p(u, fx), \frac{p(fx, Tx)}{2}, \frac{p(u, Tx) + p(fx, u)}{2}\right\} \\
&= r\max\left\{p(fz, fx), \frac{p(fx, Tx)}{2}, \frac{p(fz, Tx) + p(fx,, fz)}{2}\right\} \\
&\le r\max\left\{p(fz, fx), \frac{p(fz, Tx) + p(fx,, fz)}{2}\right\}.
\end{aligned}$$

If

$$\max\left\{p(fz,fx),\frac{p(fz,Tx)+p(fx,,fz)}{2}\right\}=p(fz,fx),$$

then we are done. If

$$\max\left\{p(fz,fx),\frac{p(fz,Tx)+p(fx,,fz)}{2}\right\}=\frac{p(fz,Tx)+p(fx,,fz)}{2},$$

then we obtain

$$p(fz,Tx)\leq\frac{r}{2-r}p(fx,fz)\leq rp(fx,fz)$$

and hence (2.2) holds. Next, we show that

$$H_p(Tz,Tx)\leq r\max\left\{p(fz,fx),\frac{p(fx,Tx)+p(fz,Tz)}{2},\frac{p(fx,Tz)+p(fz,Tx)}{2}\right\} \tag{2.3}$$

for any $x\in Y$. If $x=z$, then $fx=fz$, and the claim follows from (2.2). Suppose that $x\neq z$, then $fx\neq fz$. As f is a non-constant single-valued mapping, we have

$$\begin{aligned}p(fx,Tx)&\leq p(fx,fz)+p(fz,Tx)-p(fz,fz)\\&\leq p(fx,fz)+rp(fx,fz)\leq(1+r)p(fx,fz).\end{aligned}$$

This implies

$$\omega(r)p(fx,Tx)\leq p(fx,fz).$$

Therefore,

$$H_p(Tz,Tx)\leq r\max\left\{p(fz,fx),\frac{p(fx,Tx)+p(fz,Tz)}{2},\frac{p(fx,Tz)+p(fz,Tx)}{2}\right\}.$$

Hence, (2.3) holds for any $x\in Y$. Note that

$$\begin{aligned}p(Tz,fu_{n+2})&\leq H_p(Tz,Tu_{n+1})\\&\leq r\max\left\{p(fz,fu_{n+1}),\frac{p(fu_{n+1},Tu_{n+1})+p(fz,Tz)}{2},\right.\\&\qquad\left.\frac{p(fu_{n+1},Tz)+p(fz,Tu_{n+1})}{2}\right\}\\&\leq r\max\left\{p(fz,y_{n+2}),\frac{p(y_{n+2},y_{n+2})+p(fz,Tz)}{2},\right.\\&\qquad\left.\frac{p(y_{n+2},fz)+p(fz,Tz)-p(fz,fz)+p(fz,y_{n+2})}{2}\right\}.\end{aligned}$$

On taking limit as $n\to\infty$, we obtain

$$p(fz,Tz)\leq\frac{r}{2}p(fz,Tz).$$

We obtain $p(fz, Tz) = 0$, which further implies that $p^S(fz, Tz) \leq 2p(fz, Tz) = 0$. Hence, $fz \in Tz$. Further if $Y = X$ and $ffz = fz$, then due to IT-commutativity of the pair (f, T), we have $fz = ffz \in fTz \subseteq Tfz$. This shows that fz is a common fixed point of the pair (f, T). $\square$

Corollary A *Let (X, p) be a partial metric space and Y be any non-empty set. Assume that here exists $r \in [0, 1)$ such that the mappings $f : Y \longrightarrow X$ and $T : Y \longrightarrow CB^p(X)$ satisfy*

$$\omega(r)p(fx, Tx) \leq p(fx, fy) \quad \Rightarrow \quad H_p(Tx, Ty) \leq rp(fx, fy)$$

for all $x, y \in Y$, with $T(Y) \subset f(Y)$. If there exists $u_0 \in Y$ such that $f(Y)$ is (f, T)-orbitally complete at u_0, then $C(f, T) \neq \phi$. If $Y = X$ and (f, T) is IT-commuting at coincidence points of the pair (f, T), then $F(f, T) \neq \phi$ provided that fz is a fixed point of f for some $z \in C(f, T)$.

Example 2.2 Let $X = \{0, 1, 2\}$ and $Y = \{0, 1, 2, 3\}$. Define a mapping $p : X \times X \to \mathbb{R}^+$ as follows:

$$p(0,0) = p(1,1) = 0, \qquad p(0,1) = p(1,0) = \frac{1}{4}, \qquad p(2,2) = \frac{1}{3},$$
$$p(0,2) = p(2,0) = \frac{2}{5}, \qquad p(1,2) = p(2,1) = \frac{13}{20}.$$

Then p is a partial metric on X. Let $\omega(r)$ be as given in Theorem 2.1 and the mappings $T : Y \longrightarrow CB^p(X)$ and $f : Y \to X$ be given as

$$Tx = \begin{cases} \{0\} & \text{when } x \neq 2, \\ \{0,1\} & \text{when } x = 2, \end{cases} \quad \text{and} \quad fx = \begin{cases} 0, & \text{if } x \in \{0,1\}, \\ 2, & \text{if } x = 2, \\ 1, & \text{if } x = 3. \end{cases}$$

Note that

$$p(f0, T0) = p(f1, T1) = p(f0, f1) = p(f1, f1) = 0,$$
$$p(f3, f2) = \frac{13}{20}, \qquad p(f2, f2) = \frac{1}{3},$$
$$p(f1, f2) = p(f0, f2) = p(f2, T2) = \frac{2}{5},$$
$$p(f3, T3) = p(f3, f0) = p(f3, f1) = \frac{1}{4}.$$

If we take $r \geq \frac{3}{5}$ and $\omega(r) \leq \frac{5}{8}$, then for all $x, y \in Y$,

$$\omega(r)p(fx, Tx) \leq p(fx, fy)$$

holds. If we consider $r = \frac{5}{6}$, then $\omega(r) = \frac{1}{6}$. Then, for $x, y \in \{0, 1, 3\}$, we have $H_p(Tx, Ty) = 0$, hence $H_p(Tx, Ty) \leq rp(fx, fy)$ is satisfied trivially. Now consider

$$H_p(T0, T2) = p(0,1) = \frac{1}{4} \leq \frac{1}{3} = rp(f0, f2),$$
$$H_p(T1, T2) = p(0,1) = \frac{1}{4} \leq \frac{1}{3} = rp(f1, f2),$$

$$H_p(T2, T3) = p(0,1) = \frac{1}{4} < \frac{13}{24} = rp(f2, f3),$$
$$H_p(T2, T2) = p(0,1) = \frac{1}{4} < \frac{5}{18} = rp(f2, f2).$$

Hence, for all $x, y \in Y$,

$$\omega(r)p(fx, Tx) \le p(fx, fy)$$

implies

$$H_p(Tx, Ty) \le rp(fx, fy).$$

Let $u_0 = 1$, $y_0 = f(1) = 0$. As $T(0) \subseteq f(Y)$, there exists a point $u_1 = 1$ in Y such that $y_1 = f(1) = 0 \in T(1)$ and $T(0) = \{0\} \subseteq f(Y)$, we obtain a point $u_2 = 1$ in Y such that $y_2 = 0 = f(1) \in T(1)$. Continuing this way, we construct an orbit $\{y_0 = y_1 = y_2 = \cdots = 0\}$ for (f, T) at $u_0 = 1$. Also, $f(Y)$ is (f, T)-orbitally complete at $u_0 = 0$. So, all the conditions of Corollary A are satisfied. Moreover, $C(f, T) = \{0, 1\}$.

On the other hand, the metric p^S induced by the partial metric p is given by

$$p^S(0,0) = p^S(1,1) = p^S(2,2) = 0,$$
$$p^S(0,1) = p^S(1,0) = \frac{1}{2}, \qquad p^S(1,2) = p^S(2,1) = \frac{29}{30},$$
$$p^S(0,2) = p^S(2,0) = \frac{7}{15}.$$

Now, we show that Corollary A is not applicable (in the case of a metric induced by a partial metric p) in this case. Since

$$\omega(r)p^S(f1, T1) = \omega(r)p^S(0,0) = 0 \le p^S(fx, fy)$$

is satisfied for any $r \in [0,1)$, x and y in X, so it must imply $H_{p^S}(T1, T2) \le rp(f1, f2)$. But

$$H_{p^S}(T1, T2) = H_{p^S}(\{0\}, \{0,1\}) = \frac{1}{2}$$

and

$$p^S(f1, f2) = p^S(0,2) = \frac{7}{15} < \frac{1}{2}.$$

Hence, for any $r \in [0,1)$,

$$H_{p^S}(T1, T2) \nleq rp(f1, f2).$$

Corollary B *Let (X, p) be a partial metric space, Y be any non-empty set and $f, T : Y \longrightarrow X$ be such that $T(Y) \subset f(Y)$. Suppose that there exists $u_0 \in Y$ such that $f(Y)$ is (f, T)-orbitally complete at u_0. Assume further that there exists an $r \in [0,1)$ such that*

$$\omega(r)p(fx, Tx) \le p(fx, fy)$$

implies that

$$p(Tx,Ty) \le r\max\left\{p(fx,fy), \frac{p(fx,Tx)+p(fy,Ty)}{2}, \frac{p(fx,Ty)+p(fy,Tx)}{2}\right\}$$

for all $x, y \in Y$. Then $C(f,T) \neq \phi$. Further, if $Y = X$ and the pair (f,T) is commuting at x where $x \in C(f,T)$, then $F(f,T)$ is a singleton.

Proof It follows from Theorem 2.1, that $C(f,T) \neq \phi$. If $u \in C(f,T)$, then $fu = Tu$. Further, if $Y = X$ and (f,T) is commuting at u, then $ffu = fTu = Tfu$. Now,

$$\omega(r)p(fu,Tfu) \le p(fu,Tfu) = p(fu,ffu)$$

implies that

$$\begin{aligned} p(fu,ffu) &= p(Tu,Tfu) \\ &\le rM_{p,f}(u,fu) \\ &\le r\max\left\{p(fu,ffu), \frac{p(ffu,Tfu)+p(fu,Tu)}{2}, \frac{p(ffu,Tu)+p(fu,Tfu)}{2}\right\} \\ &\le r\max\left\{p(fu,ffu), \frac{p(ffu,ffu)+p(fu,Tu)}{2}, \frac{p(ffu,fu)+p(fu,ffu)}{2}\right\} \\ &\le r\max\left\{p(fu,ffu), \frac{p(ffu,fu)+p(fu,ffu)}{2}, \frac{p(ffu,fu)+p(fu,ffu)}{2}\right\} \\ &\le rp(fu,ffu). \end{aligned}$$

As $r < 1$, we obtain $p(fu,ffu) = 0$, which further implies that $p^S(fu,ffu) \le 2p(fu,ffu) = 0$. Hence, fu is a common fixed point of f and T.

For uniqueness, assume there exist $z_1 \neq z_2$, such that $z_1 = fz_1 = Tz_1$ and $z_2 = fz_2 = Tz_2$. Then

$$\psi(r)p(fz_1,Tz_1) \le p(fz_1,Tz_1) = p(fz_1,fz_1) \le p(fz_1,fz_2),$$

which implies

$$\begin{aligned} p(z_1,z_2) &= p(Tz_1,Tz_2) \\ &\le r\max\left\{p(fz_1,fz_2), \frac{p(fz_1,Tz_1)+p(fz_2,Tz_2)}{2}, \frac{p(fz_2,Tz_1)+p(fz_1,Tz_2)}{2}\right\} \\ &\le r\max\{p(z_1,z_2), p(z_1,z_2), p(z_1,z_2)\} \\ &\le rp(z_1,z_2). \end{aligned}$$

We obtain $p(z_1,z_2) = 0$, which further implies that $p^S(z_1,z_2) \le 2p(z_1,z_2) = 0$. Hence, $z_1 = z_2$. □

3 An application

In this section, we assume that U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$. Suppose that

$$\tau : W \times D \longrightarrow W,$$

$$g, g', h, h' : W \times D \longrightarrow \mathbb{R},$$

$$G, F : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}.$$

Considering W and D as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p(x) = \sup_{y \in D}\bigl\{h(x,y) + G\bigl(x,y,p\bigl(\tau(x,y)\bigr)\bigr)\bigr\}, \quad \text{for } x \in W, \tag{3.1}$$

$$q(x) = \sup_{y \in D}\bigl\{h'(x,y) + F\bigl(x,y,q\bigl(\tau(x,y)\bigr)\bigr)\bigr\}, \quad \text{for } x \in W. \tag{3.2}$$

Then equations (3.1) and (3.2) can be reformulated as

$$p(x) = \sup_{y \in D}\bigl\{g(x,y) + G\bigl(x,y,p\bigl(\tau(x,y)\bigr)\bigr)\bigr\} - b, \quad \text{for } x \in W, \tag{3.3}$$

$$q(x) = \sup_{y \in D}\bigl\{g'(x,y) + F\bigl(x,y,q\bigl(\tau(x,y)\bigr)\bigr)\bigr\} - b, \quad \text{for } x \in W. \tag{3.4}$$

For more on the multistage process involving such functional equations, we refer to [23, 31–34]. Now, we study the existence and uniqueness of a common and bounded solution of the functional equations (3.3)-(3.4) arising in dynamic programming in the setup of partial metric spaces.

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space endowed with the metric d defined as $d(h,k) = \sup_{x \in W} |hx - kx|$. Now, consider

$$p_B(h,k) = d(h,k) + b = \sup_{x \in W} |hx - kx| + b, \tag{3.5}$$

where $h, k \in B(W)$, $b > 0$ and p_B is a partial metric on $B(W)$. Let $\omega(r)$ be defined as in Section 1. Suppose that the following conditions hold:

(C1): G, F, g, and g' are bounded.

(C2): For $x \in W$, $h \in B(W)$ and $b > 0$, define

$$Kh(x) = \sup_{y \in D}\bigl\{g(x,y) + G\bigl(x,y,h\bigl(\tau(x,y)\bigr)\bigr)\bigr\} - b, \tag{3.6}$$

$$Jh(x) = \sup_{y \in D}\bigl\{g'(x,y) + F\bigl(x,y,h\bigl(\tau(x,y)\bigr)\bigr)\bigr\} - b. \tag{3.7}$$

Moreover, assume that there exists $r \in [0,1)$ such that for every $(x,y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\omega(r)p_B\bigl(Kh(t), Jh(t)\bigr) \le p_B\bigl(Jh(t), Jk(t)\bigr) \tag{3.8}$$

implies

$$\big|G\big(x,y,h(t)\big)-G\big(x,y,k(t)\big)\big|\le rM_{p_B,J}\big(h(t),k(t)\big), \tag{3.9}$$

where

$$\begin{aligned}M_{p_B,J}\big(h(t),k(t)\big) = \max\bigg\{&p_B\big(Jh(t),Jk(t)\big),\frac{p_B(Jk(t),Kk(t))+p_B(Jh(t),Kh(t))}{2},\\ &\frac{p_B(Jh(t),Kk(t))+p_B(Jk(t),Kh(t))}{2}\bigg\}.\end{aligned}$$

(C3): For any $h\in B(W)$, there exists $k\in B(W)$ such that for $x\in W$,

$$Kh(x)=Jk(x).$$

(C4): There exists $h\in B(W)$ such that

$$Kh(x)=Jh(x)\quad\text{implies that}\quad JKh(x)=KJh(x).$$

Theorem 3.1 *Assume that the conditions* (C1)-(C4) *are satisfied. If* $J(B(W))$ *is a closed convex subspace of* $B(W)$, *then the functional equations* (3.3) *and* (3.4) *have a unique, common and bounded solution.*

Proof Note that $(B(W),p_B)$ is a complete partial metric space. By (C1), J, K are self-maps of $B(W)$. The condition (C3) implies that $K(B(W))\subseteq J(B(W))$. It follows from (C4) that J and K commute at their coincidence points. Let λ be an arbitrary positive number and $h_1,h_2\in B(W)$. Choose $x\in W$ and $y_1,y_2\in D$ such that

$$Kh_j<g(x,y_j)+G(x,y_j,h_j(x_j)) - b+\lambda, \tag{3.10}$$

where $x_j=\tau(x,y_j)$, $j=1,2$. Further, from (3.5) and (3.6), we have

$$Kh_1\ge g(x,y_2)+G\big(x,y_2,h_1(x_2)\big), \tag{3.11}$$

$$Kh_2\ge g(x,y_1)+G\big(x,y_1,h_2(x_1)\big). \tag{3.12}$$

Therefore, (3.8) in (C2) becomes

$$\omega(r)p_B\big(Kh_1(x),Jh_1(x)\big)\le p_B\big(Jh_1(x)-Jh_2(x)\big). \tag{3.13}$$

Then (3.13) together with (3.10) and (3.12) implies

$$\begin{aligned}Kh_1(x)-Kh_2(x) &< G\big(x,y_1,h_1(x_1)\big)-G\big(x,y_1,h_2(x_2)\big)-b+\lambda\\ &\le \big|G\big(x,y_1,h_1(x_1)\big)-G\big(x,y_1,h_2(x_2)\big)\big|-b+\lambda\\ &\le rM_{p_B,J}\big(h(t),k(t)\big)-b+\lambda.\end{aligned} \tag{3.14}$$

Now, (3.10), (3.11) and (3.13) imply

$$\begin{aligned} Kh_2(x) - Kh_1(x) &\le G\bigl(x, y_1, h_2(x_2)\bigr) - G\bigl(x, y_1, h_1(x_1)\bigr) - b \\ &\le \bigl|G\bigl(x, y_1, h_1(x_1)\bigr) - G\bigl(x, y_1, h_2(x_2)\bigr)\bigr| - b \\ &\le rM_{p_B,J}\bigl(h(t), k(t)\bigr) - b. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we have

$$\bigl|Kh_1(x) - Kh_2(x)\bigr| + b \le rM_{p_B,J}\bigl(h(t), k(t)\bigr). \tag{3.16}$$

As the above inequality is true for any $x \in W$ and $\lambda > 0$ is taken arbitrarily, so from (3.13) we obtain

$$\omega(r)p_B(Kh_1, Jh_2) \le p_B(Jh_1, Ih_2) \tag{3.17}$$

implies

$$p_B(Kh_1, Kh_2) \le rM_{p_B,J}\bigl(h(t), k(t)\bigr). \tag{3.18}$$

Therefore, by Corollary B, the pair (K, J) has a common fixed point h^*, that is, $h^*(x)$ is a unique, bounded and common solution of (3.3) and (3.4). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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