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Some coupled coincidence point theorems for a mixed monotone operator in a complete metric space endowed with a partial order by using altering distance functions

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Abstract
In this paper, we present some coupled coincidence point results for mixed g-monotone mappings in partially ordered complete metric spaces involving altering distance functions. Moreover, we present an example to illustrate our main result. Our results extend some results in the field.

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1 Introduction and preliminaries

The existence of a fixed point for contractive mappings in partially ordered metric spaces has attracted the attention of many mathematicians (*cf.* [1–11] and the references therein). In [3], Bhaskar and Lakshmikantham introduced the notion of a mixed monotone mapping and proved some coupled fixed point theorems for the mixed monotone mapping. Afterwards, Lakshmikantham and Ciric in [11] introduced the concept of a mixed g-monotone mapping and proved coupled coincidence point results for two mappings F and g, where F has the mixed g-monotone property and the functions F and g commute. It is well known that the concept of commuting has been weakened in various directions. One such notion which is weaker than commuting is the concept of compatibility introduced by Jungck [7]. In [5], Choudhury and Kundu defined the concept of compatibility of F and g. The purpose of this paper is to present some coupled coincidence point theorems for a mixed g-monotone mapping in the context of complete metric spaces endowed with a partial order by using altering distance functions which extend some results of [6]. We also present an example which illustrates the results.

Recall that if $(X, \preceq)$ is a partially ordered set, then f is said to be non-decreasing if for $x, y \in X$, $x \preceq y$, we have $fx \preceq fy$. Similarly, f is said to be non-increasing if for $x, y \in X$, $x \preceq y$, we have $fx \succeq fy$. We also recall the used definitions in the present work.

Definition 1.1 [11] (Mixed g-monotone property) Let $(X, \preceq)$ be a partially ordered set, $g : X \to X$ and $F : X \times X \to X$. We say that the mapping F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y) \tag{1}$$

and

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \quad \Rightarrow \quad F(x, y_1) \succeq F(x, y_2). \tag{2}$$

Definition 1.2 [11] (Coupled coincidence fixed point) Let $(x, y) \in X \times X$, $F : X \times X \to X$ and $g : X \to X$. We say that (x, y) is a coupled coincidence point of F and g if $F(x, y) = gx$ and $F(y, x) = gy$ for $x, y \in X$.

Definition 1.3 [11] Let X be a non-empty set and let $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if, for all $x, y \in X$,

$$g\bigl(F(x, y)\bigr) = F(gx, gy).$$

Definition 1.4 [5] The mappings F and g, where $F : X \times X \to X$ and $g : X \to X$, are said to be compatible if

$$\lim_{n\to\infty} d\bigl(g\bigl(F(x_n, y_n)\bigr), F(gx_n, gy_n)\bigr) = 0$$

and

$$\lim_{n\to\infty} d\bigl(g\bigl(F(y_n, x_n)\bigr), F(gy_n, gx_n)\bigr) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for all $x, y \in X$.

Definition 1.5 (Altering distance function) An altering distance function is a function $\psi : [0, \infty) \to [0, \infty)$ satisfying

1. ψ is continuous and non-decreasing.
2. $\psi(t) = 0$ if and only if $t = 0$.

2 Existence of coupled coincidence points

Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Also, let φ and ϕ be altering distance functions. Now, we are in a position to state our main theorem.

Theorem 2.1 *Let $F : X \times X \to X$ be a mapping having the mixed g-monotone property on X such that*

$$\begin{aligned} &\varphi\bigl(d\bigl(F(x, y), F(u, v)\bigr)\bigr) \\ &\quad \le \varphi\bigl(\max\bigl(d(gx, gu), d(gy, gv)\bigr)\bigr) - \phi\bigl(\max\bigl(d(gx, gu), d(gy, gv)\bigr)\bigr) \end{aligned} \tag{3}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Suppose that $F(X \times X) \subset g(X)$, g is continuous, monotone increasing and suppose also that F and g are compatible mappings. Moreover, suppose either

(a) *F is continuous, or*
(b) *X has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ with $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point.

Proof By using $F(X \times X) \subset g(X)$, we construct sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n) \quad \text{for } n \geq 0. \tag{4}$$

We are going to divide the proof into several steps in order to make it easy to read.

Step 1. We will show that $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$ for $n \geq 0$.

We use the mathematical induction to show that. From the assumption of the theorem, it follows that $gx_0 \preceq F(x_0, y_0) = gx_1$ and $gy_0 \succeq F(y_0, x_0) = gy_1$, so our claim is satisfied for $n = 0$. Now, suppose that our claim holds for some fixed $n > 0$. Since $gx_{n-1} \preceq gx_n$, $gy_n \preceq gy_{n-1}$ and F has the mixed g-monotone property, then we get

$$gx_{n+1} = F(x_n, y_n) \succeq F(x_{n-1}, y_n) \succeq F(x_{n-1}, y_{n-1}) = gx_n$$

and

$$gy_{n+1} = F(y_n, x_n) \preceq F(y_{n-1}, x_n) \preceq F(y_{n-1}, x_{n-1}) = gy_n.$$

Thus the claim holds for $n + 1$ and by the mathematical induction our claim is proved.

Step 2. We will show that $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0$.

In fact, using (3), $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, we get

$$\begin{aligned}\varphi\bigl(d(gx_{n+1}, gx_n)\bigr) &= \varphi\bigl(d\bigl(F(x_n, y_n), F(x_{n-1}, y_{n-1})\bigr)\bigr) \\ &\leq \varphi\bigl(\max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr)\bigr) \\ &\quad - \phi\bigl(\max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr)\bigr).\end{aligned} \tag{5}$$

Since ϕ is non-negative, we have

$$\varphi\bigl(d(gx_{n+1}, gx_n)\bigr) \leq \varphi\bigl(\max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr)\bigr),$$

and since φ is non-decreasing, we have

$$d(gx_{n+1}, gx_n) \leq \max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr). \tag{6}$$

In the same way, we get the following:

$$\begin{aligned}\varphi\bigl(d(gy_{n+1}, gy_n)\bigr) &= \varphi\bigl(d\bigl(F(y_n, x_n), F(y_{n-1}, x_{n-1})\bigr)\bigr) \\ &= \varphi\bigl(d\bigl(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\bigr)\bigr) \\ &\leq \varphi\bigl(\max\bigl(d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\bigr)\bigr)\end{aligned}$$

$$\begin{aligned}&- \phi\bigl(\max\bigl(d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\bigr)\bigr)\\ &\le \varphi\bigl(\max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr)\bigr),\end{aligned} \tag{7}$$

and hence

$$d(gy_{n+1}, gy_n) \le \max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr). \tag{8}$$

Using (6) and (8), we have

$$\max\bigl(d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\bigr) \le \max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr).$$

From the last inequality, we notice that the sequence $(\max(d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)))$ is non-negative decreasing. This implies that there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \max\bigl(d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\bigr) = r. \tag{9}$$

It is easily seen that if $\varphi : [0,\infty) \to [0,\infty)$ is non-decreasing, we have $\varphi(\max(a, b)) = \max(\varphi(a), \varphi(b))$ for $a, b \in [0,\infty)$ for $a, b \in [0,\infty)$. Using this, (5) and (7), we obtain

$$\begin{aligned}\max(\varphi\bigl(d(gx_{n+1}, gx_n)\bigr), \varphi\bigl(d(gy_{n+1}, gy_n)\bigr) &= \varphi\bigl(\max\bigl(d(gx_{n+1}, gx_n)d(gy_{n+1}, gy_n)\bigr)\bigr)\\ &\le \varphi\bigl(\max\bigl(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\bigr)\bigr)\\ &\quad - \phi\bigl(\max\bigl(d(gx_n, gx_{n-1})d(gy_n, gy_{n-1})\bigr)\bigr).\end{aligned} \tag{10}$$

Letting $n \to \infty$ in the last inequality and using (6), we have

$$\varphi(r) \le \varphi(r) - \phi(r) \le \varphi(r),$$

and this implies $\phi(r) = 0$. Thus, using the fact that ϕ is an altering distance function, we have $r = 0$. Therefore,

$$\lim_{n\to\infty} \max\bigl(d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\bigr) = 0. \tag{11}$$

Hence, $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0$ and this completes the proof of our claim.

Step 3. We will prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Suppose that one of the sequences $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. This implies that $\lim_{n,m\to\infty} d(gx_n, gx_m) \not\to 0$ or $\lim_{n,m\to\infty} d(gy_n, gy_m) \not\to 0$, and hence

$$\lim_{n,m\to\infty} \max\bigl(d(gx_n, gx_m), d(gy_n, gy_m)\bigr) \not\to 0.$$

This means that there exists $\epsilon > 0$, for which we can find subsequences $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$ with $n(k) > m(k) > k$, such that

$$\max\bigl(d(gx_{m(k)}, gx_{n(k)}), d(gy_{m(k)}, gy_{n(k)})\bigr) \ge \epsilon. \tag{12}$$

Further, we can choose $n(k)$ corresponding to $m(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (12). Then

$$\max\bigl(d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})\bigr) < \epsilon. \tag{13}$$

Using (3), $gx_{n(k)-1} \succeq gx_{m(k)-1}$ and $gy_{n(k)-1} \preceq gy_{m(k)-1}$, we get

$$\begin{aligned}\varphi\bigl(d(gx_{n(k)}, gx_{m(k)})\bigr) &= \varphi\bigl(d\bigl(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1})\bigr)\bigr)\\ &\le \varphi\bigl(\max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr)\bigr)\\ &\quad - \phi\bigl(\max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr)\bigr),\end{aligned} \tag{14}$$

and also we get

$$\begin{aligned}\varphi\bigl(d(gy_{n(k)}, gy_{m(k)})\bigr) &= \varphi\bigl(d\bigl(F(y_{n(k)-1}, x_{n(k)-1}), F(y_{m(k)-1}, x_{m(k)-1})\bigr)\bigr)\\ &= \varphi\bigl(d\bigl(F(y_{m(k)-1}, x_{m(k)-1}), F(y_{n(k)-1}, x_{n(k)-1})\bigr)\bigr)\\ &\le \varphi\bigl(\max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr)\bigr)\\ &\quad - \phi\bigl(\max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr)\bigr).\end{aligned} \tag{15}$$

Combining (14) and (15), we obtain

$$\begin{aligned}&\max\bigl(\varphi\bigl(d(gx_{n(k)}, gx_{m(k)})\bigr), \varphi\bigl(d(gy_{n(k)}, gy_{m(k)})\bigr)\bigr)\\ &\quad\le \varphi\bigl(\max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr)\bigr)\\ &\qquad - \phi\bigl(\max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr)\bigr).\end{aligned} \tag{16}$$

Using the triangular inequality and (13), we get

$$\begin{aligned}d(gx_{n(k)}, gx_{m(k)}) &\le d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)})\\ &< d(gx_{n(k)}, gx_{n(k)-1}) + \epsilon\end{aligned} \tag{17}$$

and

$$\begin{aligned}d(gy_{n(k)}, gy_{m(k)}) &\le d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)})\\ &< d(gy_{n(k)}, gy_{n(k)-1}) + \epsilon.\end{aligned} \tag{18}$$

Using (12), (17) and (18), we have

$$\begin{aligned}\epsilon &\le \max\bigl(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\bigr)\\ &\le \max\bigl(d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\bigr) + \epsilon.\end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (11), we have

$$\lim_{k\to\infty} \max\bigl(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\bigr) = \epsilon. \tag{19}$$

Similarly, using the triangular inequality and (13), we have

$$\begin{aligned} d(gx_{n(k)-1}, gx_{m(k)-1}) &\leq d(gx_{n(k)-1}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)-1}) \\ &< \epsilon + d(gx_{m(k)}, gx_{m(k)-1}) \end{aligned} \tag{20}$$

and

$$\begin{aligned} d(gy_{n(k)-1}, gy_{m(k)-1}) &\leq d(gy_{n(k)-1}, gy_{m(k)}) + d(gy_{m(k)}, gy_{m(k)-1}) \\ &< \epsilon + d(gy_{m(k)}, gy_{m(k)-1}). \end{aligned} \tag{21}$$

Combining (20) and (21), we obtain

$$\begin{aligned} &\max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr) \\ &\quad < \max\bigl(d(gx_{m(k)}, gx_{m(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\bigr) + \epsilon. \end{aligned} \tag{22}$$

Using the triangular inequality, we have

$$\begin{aligned} d(gx_{n(k)}, gx_{m(k)}) \leq{}& d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)-1}) \\ &+ d(gx_{m(k)-1}, gx_{m(k)}) \end{aligned}$$

and

$$\begin{aligned} d(gy_{n(k)}, gy_{m(k)}) \leq{}& d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1}) \\ &+ d(gy_{m(k)-1}, gy_{m(k)}). \end{aligned}$$

Using the two last inequalities and (12), we have

$$\begin{aligned} \epsilon &\leq \max\bigl(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\bigr) \\ &\leq \max\bigl(d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\bigr) \\ &\quad + \max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr) \\ &\quad + \max\bigl(d(gx_{m(l)-1}, gx_{m(k)}), d(gy_{m(k)-1}, gy_{m(k)})\bigr). \end{aligned} \tag{23}$$

Using (22) and (23), we get

$$\begin{aligned} \epsilon &\leq \max\bigl(d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\bigr) \\ &\leq \max\bigl(d(gx_{m(k)-1}, gx_{m(k)}), d(gy_{m(k)-1}, gy_{mk)})\bigr) \\ &\leq \max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr) \\ &< \max\bigl(d(gx_{m(k)}, gx_{m(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\bigr) + \epsilon. \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (11), we obtain

$$\lim_{k\to\infty} \max\bigl(d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\bigr) = \epsilon. \tag{24}$$

Finally, letting $k \to \infty$ in (15) and using (18), (23) and the continuity of φ and ϕ, we have

$$\varphi(\epsilon) \le \varphi(\epsilon) - \phi(\epsilon) \le \varphi(\epsilon)$$

and, consequently, $\phi(\epsilon) = 0$. Since ϕ is an altering distance function, we get $\epsilon = 0$, and this is a contradiction. This proves our claim.

Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y. \tag{25}$$

Since F and g are compatible mappings, we have

$$\lim_{n\to\infty} d\big(g\big(F(x_n, y_n)\big), F(gx_n, gy_n)\big) = 0 \tag{26}$$

and

$$\lim_{n\to\infty} d\big(g\big(F(y_n, x_n)\big), F(gy_n, gx_n)\big) = 0. \tag{27}$$

We now show that $gx = F(x, y)$ and $gy = F(y, x)$. Suppose that assumption (a) holds. For all $n \ge 0$, we have

$$d\big(gx, F(gx_n, gy_n)\big) \le d\big(gx, g\big(F(x_n, y_n)\big)\big) + d\big(g\big(F(x_n, y_n)\big), F(gx_n, gy_n)\big).$$

Taking the limit as $n \to \infty$, using (3), (25), (26) and the fact that F and g are continuous, we have $d(gx, F(x, y)) = 0$. Similarly, using (3), (25), (27) and the fact that F and g are continuous, we have $d(gy, F(y, x)) = 0$. Hence, we get

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x).$$

Finally, suppose that (b) holds. In fact, since $\{gx_n\}$ is non-decreasing and $gx_n \to x$ and $\{gy_n\}$ is non-increasing and $gy_n \to y$, by our assumption, $gx_n \preceq x$ and $gy_n \succeq y$ for every $n \in N$.

Applying (3), we have

$$\begin{aligned} \varphi\big(d\big(F(x, y), F(x_n, y_n)\big)\big) &\le \varphi\big(\max\big(d(gx, gx_n), d(gy, gy_n)\big)\big) - \phi(\max\big(d(gx, gx_n), d(gy, gy_n)\big) \\ &\le \varphi\big(\max\big(d(gx, gx_n), d(gy, gy_n)\big)\big), \end{aligned}$$

and as φ is non-decreasing, we obtain

$$d\big(F(x, y), F(x_n, y_n)\big) \le \max\big(d(gx, gx_n), d(gy, gy_n)\big). \tag{28}$$

Using the triangular inequality and (28), we get

$$\begin{aligned} d\big(gx, F(x, y)\big) &\le \lim_{n\to\infty} d(gx, ggx_{n+1}) + d\big(ggx_{n+1}, F(x, y)\big) \\ &= \lim_{n\to\infty} d(gx, ggx_{n+1}) + d\big(F(x, y), gF(x_n, y_n)\big) \end{aligned}$$

$$= \lim_{n\to\infty} d(gx, ggx_{n+1}) + d\big(F(x,y), F(gx_n, gy_n)\big)$$
$$\le d(gx, ggx_{n+1}) + \max\big(d(gx, ggx_n), d(gy, ggy_n)\big).$$

As $x_n \to x$ and $y_n \to y$, taking $n \to \infty$ in the last inequality, we have

$$d\big(gx, F(x,y)\big) = 0,$$

and, consequently, $F(x,y) = gx$.

Using a similar argument, it can be proved that $gy = F(y,x)$ and this completes the proof. □

Corollary 2.1 [6] *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that*

$$\varphi\big(d\big(F(x,y), F(u,v)\big)\big) \le \varphi\big(\max\big(d(x,u), d(y,v)\big)\big) - \phi\big(\max\big(d(x,u), d(y,v)\big)\big)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, where φ and ϕ are altering distance functions. Moreover, suppose either

(a) *F is continuous, or*
(b) *X has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Corollary 2.2 [3] *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that*

$$d\big(F(x,y), F(u,v)\big) \le \frac{k}{2}\big[d(x,u) + d(y,v)\big]$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Moreover, suppose either

(a) *F is continuous, or*
(b) *X has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Proof Let φ = identity and $\phi = (1 - \frac{1}{k})\varphi$ and g is the identity function. Then applying Theorem 2.1, we get Corollary 2.2. □

3 Uniqueness of the coupled coincidence point

In this section, we prove the uniqueness of the coupled coincidence point. Note that if $(X, \preceq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x,y),(u,v) \in X \times X$,

$$(x,y) \preceq (u,v) \quad \Leftrightarrow \quad x \preceq u, y \succeq v.$$

Theorem 3.1 *In addition to the hypotheses of Theorem* 2.1, *suppose that for every* (x,y), (z,t) *in* $X \times X$, *there exists a* (u,v) *in* $X \times X$ *that is comparable to* (x,y) *and* (z,t), *then* F *and* g *have a unique coupled coincidence point*.

Proof Suppose that (x,y) and (z,t) are coupled coincidence points of F, that is, $gx = F(x,y)$, $gy = F(x,y)$, $gz = f(z,t)$ and $gt = F(t,z)$.

Let (u,v) be an element of $X \times X$ comparable to (x,y) and (z,t). Suppose that $(x,y) \succeq (u,v)$ (the proof is similar in the other case).

We construct the sequences $\{gu_n\}$ and $\{gv_n\}$ as follows:

$$u_0 = u, \qquad v_0 = v, \qquad gu_{n+1} = F(u_n, v_n), \qquad gv_{n+1} = F(v_n, u_n).$$

We claim that $(x,y) \succeq (u_n, v_n)$ for each $n \in N$. In fact, we will use mathematical induction.

For $n = 0$, as $(x,y) \succeq (u,v)$, this means $u_0 = u \preceq x$ and $y \succeq v = v_0$ and, consequently, $(u_0, v_0) \preceq (x,y)$. Suppose that $(x,y) \succeq (u_n, v_n)$, then since F has the mixed g-monotone property and since g is monotone increasing, we get

$$\begin{aligned} gu_{n+1} &= F(u_n, v_n) \preceq F(x, v_n) \preceq F(x,y) = gx, \\ gv_{n+1} &= F(v_n, u_n) \preceq F(y, u_n) \succeq F(y,x) = gy, \end{aligned}$$

and this proves our claim.

Now, since $u_n \ \preceq x$ and $u_n \succeq y$, using (3), we get

$$\begin{aligned} \varphi\bigl(d(gx, gu_n)\bigr) &= \varphi\bigl(d\bigl(F(x,y), F(u_{n-1}, v_{n-1})\bigr)\bigr) \\ &\leq \varphi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr) \\ &\quad - \phi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr) \\ &\leq \varphi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr). \end{aligned} \tag{29}$$

In the same way, we have

$$\begin{aligned} \varphi\bigl(d(gy, gv_n)\bigr) &= \varphi\bigl(d\bigl(F(y,x), F(v_{n-1}, u_{n-1})\bigr)\bigr) \\ &= \varphi\bigl(d\bigl(F(v_{n-1}, u_{n-1}), F(y,x)\bigr)\bigr) \\ &\leq \varphi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr) \\ &\quad - \phi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr) \\ &\leq \varphi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr). \end{aligned} \tag{30}$$

Using (29) and (30) and the fact that ϕ is non-decreasing, we get

$$\begin{aligned} \varphi\bigl(\max\bigl(d(gx, gu_n), d(gy, gv_n)\bigr)\bigr) &= \max\bigl(\varphi\bigl(d(gx, gu_n), d(gy, gv_n)\bigr)\bigr) \\ &\leq \varphi(\max\bigl(d\bigl(gx, gu_{n-1}, d(gy, gv_{n-1})\bigr)\bigr) \end{aligned}$$

$$\begin{aligned} & - \phi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr) \\ & \le \varphi\bigl(\max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr)\bigr). \end{aligned} \tag{31}$$

Using the last inequality and the fact that φ is non-decreasing, we have

$$\max\bigl(d(gx, gu_n), d(gy, gv_n)\bigr) \le \max\bigl(d(gx, gu_{n-1}), d(gy, gv_{n-1})\bigr).$$

Thus the sequence $(\max(d(gx, gu_n), d(gy, gv_n)))$ is decreasing and non-negative, and hence, for certain $r \geq 0$,

$$\lim_{n\to\infty}\bigl(\max\bigl(d(gx, gu_n), d(gy, gv_n)\bigr)\bigr) = r. \tag{32}$$

Using (32) and letting $n \to \infty$ in (31), we have

$$\varphi(r) \le \varphi(r) - \phi(r) < \varphi(r).$$

This gives $\phi(r) = 0$ and hence $r = 0$.

Finally, since $\lim_{n\to\infty}(\max(d(gx, gu_n), d(gy, gv_n))) = 0$, we have $gu_n \to gx$ and $gv_n \to gy$. Using a similar argument for (z, t), we can get $gu_n \to gz$ and $gv_n \to gt$, and the uniqueness of the limit gives $gx = gz$ and $gy = gt$. This completes the proof. □

Theorem 3.2 *Under the assumptions of Theorem* 2.1, *suppose that* x_0 *and* y_0 *are comparable, then the coupled coincidence point* $(x, y) \in X \times X$ *satisfies* $x = y$.

Proof Assume $x_0 \preceq y_0$ (a similar argument applies to $y_0 \preceq x_0$).

We claim that $x_n \preceq y_n$ for all n, where $gx_{n_1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$.

Obviously, the inequality is satisfied for $n = 0$. Suppose $x_n \preceq y_n$. Using the mixed g-monotone property of F, we have

$$gx_{n+1} = F(x_n, y_n) \preceq F(y_n, y_n) \preceq F(y_n, x_n) = gy_{n+1},$$

and since g is non-decreasing, this proves our claim.

Now, using (3) and $x_n \preceq y_n$, we get

$$\begin{aligned} \varphi\bigl(d(gx_{n+1}, gy_{n+1})\bigr) &= \varphi\bigl(d(gy_{n+1}, gx_{n+1})\bigr) = \varphi\bigl(d\bigl(F(y_n, x_n), F(x_n, y_n)\bigr)\bigr) \\ &\le \varphi\bigl(d(gx_n, gy_n)\bigr) - \phi\bigl(d(gx_n, gy_n)\bigr) \le \varphi\bigl(d(gx_n, gy_n)\bigr), \end{aligned} \tag{33}$$

and since φ is non-decreasing, we get

$$d(gx_{n+1}, gy_{n+1}) \le d(gx_n, gy_n).$$

We notice that the sequence $d(gx_n, gy_n)$ is decreasing. Thus, $\lim_{n\to\infty} d(gx_n, gy_n) = r$ for certain $r > 0$. Hence,

$$\varphi(r) \le \varphi(r) - \phi(r) \le \varphi(r),$$

and this gives us $r = 0$.

Since $gx_n \to x$, $gy_n \to y$ and $\lim_{n\to\infty} d(gx_n, gy_n) = 0$, we have

$$0 = \lim_{n\to\infty} d(gx_n, gy_n) = d(gx_n, gy_n) = d\Big(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n\Big) = d(x,y)$$

and thus $x = y$. This completes the proof. □

4 Example

The following example illustrates our main result.

Example 4.1 Let $X = [0,1]$. Then $(X, \leq)$ is a partially ordered set with the natural ordering of real numbers. Let

$$d(x,y) = |x-y| \quad \text{for } x, y \in [0,1].$$

Then (X,d) is a complete metric space. Let $g : X \to X$ be defined as

$$g(x) = x^2 \quad \text{for all } x \in X,$$

and let $F : X \times X \to X$ be defined as

$$F(x,y) = \begin{cases} \frac{x^2-y^2}{5} & \text{if } x, y \in [0,1], x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then, F satisfies the mixed g-monotone property.

Let $\varphi : [0,\infty) \to [0,\infty)$ be defined as

$$\varphi(t) = \frac{1}{3}t \quad \text{for } t \in [0,\infty),$$

and let $\phi : [0,\infty) \to [0,\infty)$ be defined as

$$\phi(t) = \frac{1}{5}t \quad \text{for } t \in [0,\infty).$$

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = a$, $\lim_{n\to\infty} gx_n = a$, $\lim_{n\to\infty} F(y_n, x_n) = b$ and $\lim_{n\to\infty} gy_n = b$. Then, obviously, $a = 0$ and $b = 0$. Now, for all $n \geq 0$,

$$g(x_n) = x_n^2, \qquad g(y_n) = y_n^2,$$

$$F(x_n, y_n) = \begin{cases} \frac{x_n^2-y_n^2}{5} & \text{if } x_n \geq y_n, \\ 0 & \text{if } x_n < y_n. \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} \frac{y_n^2-x_n^2}{5} & \text{if } y_n \geq x_n, \\ 0 & \text{if } y_n < x_n. \end{cases}$$

Then it follows that

$$\lim_{n\to\infty} d\big(g\big(F(x_n,y_n)\big),F(gx_n,gy_n)\big) = 0$$

and

$$\lim_{n\to\infty} d\big(g\big(F(y_n,x_n)\big),F(gy_n,gx_n)\big) = 0.$$

Hence, the mappings F and g are compatible in X. Also, $x_0 = 0$ and $y_0 = c$ ($c > 0$) are two points in X such that

$$g(x_0) = g(0) = 0 = F(0,c) = F(x_0,y_0)$$

and

$$g(y_0) = g(c) = c^2 \geq \frac{c^2}{5} = F(c,0) = F(y_0,x_0).$$

We next verify the contraction of Theorem 2.1. We take $x, y, u, v, \in X$ such that $gx \geq gu$ and $gy \leq gv$, that is, $x^2 \geq u^2$ and $y^2 \leq v^2$.

We consider the following cases.

Case 1. $x \geq y$, $u \geq v$. Then

$$\begin{aligned}
\varphi\big(d\big(F(x,y),F(u,v)\big)\big) &= \frac{1}{3}\big[d(F(x,y),F(u,v)\big] \\
&= \frac{1}{3}\left[d\left(\frac{x^2-y^2}{5},\frac{u^2-v^2}{5}\right)\right] \\
&= \frac{1}{3}\left|\frac{(x^2-y^2)-(u^2-v^2)}{5}\right| \\
&\leq \frac{1}{3}\frac{|x^2-u^2|+|y^2-v^2|}{5} \\
&= \frac{2}{15}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&\leq \frac{2}{15}\big[\max(d(gx,gu),d(gy,gv)\big] \\
&\leq \frac{1}{3}\big[\max(d(gx,gu),d(gy,gv)\big] \\
&\quad - \frac{1}{5}\big[\max(d(gx,gu),d(gy,gv)\big] \\
&= \varphi\big(\max\big\{d(gx,gu),d(gy,gv)\big\}\big) \\
&\quad - \phi\big(\max\big\{d(gx,gu),d(gy,gv)\big\}\big).
\end{aligned}$$

Case 2. $x \geq y$, $u < v$ Then

$$\begin{aligned}
\varphi\big(d\big(F(x,y),F(u,v)\big)\big) &= \frac{1}{3}\big[d(F(x,y),F(u,v)\big] \\
&= \frac{1}{3}\left[d\left(\frac{x^2-y^2}{5},0\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3}\frac{|x^2 - y^2|}{5} \\
&\leq \frac{1}{3}\frac{|v^2 + x^2 - y^2 - u^2|}{5} \\
&= \frac{1}{3}\frac{|(v^2 - y^2) - (u^2 - x^2)|}{5} \\
&\leq \frac{1}{3}\frac{|v^2 - y^2| + |u^2 - x^2|}{5} \\
&= \frac{1}{3}\frac{|u^2 - x^2| + |y^2 - v^2|}{5} \\
&= \frac{2}{15}\left(\frac{|u^2 - x^2| + |y^2 - v^2|}{2}\right) \\
&= \frac{2}{15}\left(\frac{d(gx,gu) + d(gy,gv)}{2}\right) \\
&\leq \frac{2}{15}\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr) \\
&= \frac{1}{3}\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr) \\
&\quad - \frac{1}{5}\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr) \\
&= \varphi\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr) \\
&\quad - \phi\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr).
\end{aligned}$$

Case 3. $x < y$ and $u \geq v$. Then

$$\begin{aligned}
\varphi\bigl(d\bigl(F(x,y), F(u,v)\bigr)\bigr) &= \frac{1}{3}\left[d\left(0, \frac{u^2 - v^2}{5}\right)\right] \\
&= \frac{1}{3}\frac{|u^2 - v^2|}{5} \\
&= \frac{1}{3}\frac{|u^2 + x^2 - v^2 - x^2|}{5} \\
&= \frac{1}{3}\frac{|(x^2 - v^2) + (u^2 - x^2)|}{5} \quad \text{(since } y > x\text{)} \\
&\leq \frac{1}{3}\frac{|y^2 - v^2| + |u^2 - x^2|}{5} \\
&= \frac{2}{15}\left(\frac{|u^2 - x^2| + |y^2 - v^2|}{2}\right) \\
&= \frac{2}{15}\left(\frac{d(gx,gu) + d(gy,gv)}{2}\right) \\
&\leq \frac{2}{15}\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr) \\
&= \frac{1}{3}\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr) \\
&\quad - \frac{1}{5}\bigl(\max\bigl\{d(gx,gu), d(gy,gv)\bigr\}\bigr)
\end{aligned}$$

$$\begin{aligned} &= \varphi\bigl(\max\bigl\{d(gx,gu),d(gy,gv)\bigr\}\bigr) \\ &\quad - \phi\bigl(\max\bigl\{d(gx,gu),d(gy,gv)\bigr\}\bigr). \end{aligned}$$

Case 4. $x < y$ and $u < v$ with $x^2 \leq u^2$ and $y^2 \geq v^2$. Then $F(x,y) = 0$ and $F(u,v) = 0$, that is,

$$\varphi\bigl(d\bigl(F(x,y),F(u,v)\bigr)\bigr) = \varphi\bigl(d(0,0)\bigr) = \varphi(0) = 0.$$

Obviously, the contraction of Theorem 2.1 is satisfied.

Competing interests
The author declares that he has no competing interests.

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