EXISTENCE OF ZEROS FOR OPERATORS TAKING THEIR VALUES IN THE DUAL OF A BANACH SPACE

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To Professor Giuseppe Pulvirenti, with affection, on his seventieth birthday

Using continuous selections, we establish some existence results about the zeros of weakly continuous operators from a paracompact topological space into the dual of a reflexive real Banach space.

Throughout the sequel, E denotes a reflexive real Banach space and E^* its topological dual. We also assume that E is locally uniformly convex. This means that for each $x \in E$, with $\|x\| = 1$, and each $\epsilon > 0$, there exists $\delta > 0$ such that, for every $y \in E$ satisfying $\|y\| = 1$ and $\|x - y\| \geq \epsilon$, one has $\|x + y\| \leq 2(1 - \delta)$. Recall that any reflexive Banach space admits an equivalent norm with which it is locally uniformly convex [1, page 289]. For $r > 0$, we set $B_r = \{x \in E : \|x\| \leq r\}$.

Moreover, we fix a topology τ on E, weaker than the strong topology and stronger than the weak topology, such that (E, τ) is a Hausdorff locally convex topological vector space with the property that the τ-closed convex hull of any τ-compact subset of E is still τ-compact and the relativization of τ to B_1 is metrizable by a complete metric. In practice, the most usual choice of τ is either the strong topology or the weak topology provided E is also separable.

The aim of this short paper is to establish the following result and present some of its consequences.

Theorem 1. *Let X be a paracompact topological space and $A : X \to E^*$ a weakly continuous operator. Assume that there exist a number $r > 0$, a continuous function $\alpha : X \to \mathbb{R}$ satisfying*

$$|\alpha(x)| \leq r\|A(x)\|_{E^*} \tag{1}$$

for all $x \in X$, a (possibly empty) closed set $C \subset X$, and a τ-continuous function $g : C \to B_r$ satisfying

$$A(x)(g(x)) = \alpha(x) \tag{2}$$

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for all $x \in C$, in such a way that, for every τ-continuous function $\psi : X \to B_r$ satisfying $\psi_{|C} = g$, there exists $x_0 \in X$ such that

$$A(x_0)(\psi(x_0)) \neq \alpha(x_0). \tag{3}$$

Then, there exists $x^* \in X$ such that $A(x^*) = 0$.

For the reader's convenience, we recall that a multifunction $F : S \to 2^V$, between topological spaces, is said to be lower semicontinuous at $s_0 \in S$ if, for every open set $\Omega \subseteq V$ meeting $F(s_0)$, there is a neighborhood U of s_0 such that $F(s) \cap \Omega \neq \varnothing$ for all $s \in U$. F is said to be lower semicontinuous if it is so at each point of S.

The following well-known results will be our main tools.

Theorem 2 [3]. *Let X be a paracompact topological space and $F : X \to 2^{B_1}$ a τ-lower semicontinuous multifunction with nonempty τ-closed convex values.*

Then, for each closed set $C \subset X$ and each τ-continuous function $g : C \to B_1$ satisfying $g(x) \in F(x)$ for all $x \in C$, there exists a τ-continuous function $\psi : X \to B_1$ such that $\psi_{|C} = g$ and $\psi(x) \in F(x)$ for all $x \in X$.

Theorem 3 [4]. *Let X, Y be two topological spaces, with Y connected and locally connected, and let $f : X \times Y \to \mathbb{R}$ be a function satisfying the following conditions:*

(a) *for each $x \in X$, the function $f(x, \cdot)$ is continuous, changes sign in Y, and is identically zero in no nonempty open subset of Y;*
(b) *the set $\{(y,z) \in Y \times Y : \{x \in X : f(x,y) < 0 < f(x,z)\}$ is open in $X\}$ is dense in $Y \times Y$.*

Then, the multifunction $x \to \{y \in Y : f(x,y) = 0$ and y is not a local extremum for $f(x,\cdot)\}$ is lower semicontinuous and its values are nonempty and closed.

Proof of Theorem 1. Arguing by contradiction, assume that $A(x) \neq 0$ for all $x \in X$. For each $x \in X$, $y \in B_1$, put

$$\begin{gathered} f(x,y) = A(x)(y) - \frac{\alpha(x)}{r}, \\ F(x) = \{z \in B_1 : f(x,z) = 0\}. \end{gathered} \tag{4}$$

Also, set

$$X_0 = \{x \in X : |\alpha(x)| < r\|A(x)\|_{E^*}\}. \tag{5}$$

Since A is weakly continuous, the function $x \to \|A(x)\|_{E^*}$, as a supremum of a family of continuous functions, is lower semicontinuous. From this, it follows that the set X_0 is open. For each $x \in X_0$, the function $f(x,\cdot)$ is continuous and has no local, nonabsolute extrema, being affine. Moreover, it changes sign in B_1 since $A(x)(B_1) = [-\|A(x)\|_{E^*}, \|A(x)\|_{E^*}]$ (recall that E is reflexive). Since $f(\cdot, y)$ is continuous for all $y \in B_1$, we then realize that the restriction of f to $X_0 \times B_1$ satisfies the hypotheses of Theorem 3, B_1 being considered with the relativization of the strong topology. Hence, the multifunction $F_{|X_0}$ is lower semicontinuous. Consequently, since X_0 is open, the multifunction F is lower semicontinuous at each point of X_0. Now, fix $x_0 \in X \setminus X_0$. So, $|\alpha(x_0)| = r\|A(x_0)\|_{E^*}$. Let $y_0 \in F(x_0)$ and $\epsilon > 0$. Clearly, since y_0 is an absolute extremum of $A(x_0)$ in B_1, one has $\|y_0\| = 1$. Choose $\delta > 0$ so that, for each $y \in E$ satisfying $\|y\| = 1$ and $\|y - y_0\| \geq \epsilon$, one has $\|y + y_0\| \leq 2(1 - \delta)$. By semicontinuity, the function $x \to (\|A(x)\|_{E^*})^{-1}$ is bounded in some neighborhood of x_0, and so, since the functions α and $A(\cdot)(y_0)$ are continuous, it follows that

$$\lim_{x \to x_0} \frac{|A(x)(y_0) - \alpha(x)/r|}{\|A(x)\|_{E^*}} = 0. \tag{6}$$

So, there is a neighborhood U of x_0 such that

$$\frac{|A(x)(y_0) - \alpha(x)/r|}{\|A(x)\|_{E^*}} < \frac{\epsilon\delta}{2} \tag{7}$$

for all $x \in U$. Fix $x \in U$. Pick $z \in E$, with $\|z\| = 1$, in such a way that $|A(x)(z)| = \|A(x)\|_{E^*}$ and

$$\left(A(x)(z) - \frac{\alpha(x)}{r}\right)\left(A(x)(y_0) - \frac{\alpha(x)}{r}\right) \leq 0. \tag{8}$$

From this choice, it follows, of course, that the segment joining y_0 and z meets the hyperplane $(A(x))^{-1}(\alpha(x)/r)$. In other words, there is $\lambda \in [0,1]$ such that

$$A(x)(\lambda z + (1-\lambda)y_0) = \frac{\alpha(x)}{r}. \tag{9}$$

So, if we put $y = \lambda z + (1-\lambda)y_0$, we have $y \in F(x)$ and

$$\|y - y_0\| = \lambda\|z - y_0\|. \tag{10}$$

We claim that $\|y - y_0\| < \epsilon$. This follows at once from (10) if $\lambda < \epsilon/2$. Thus, assume $\lambda \geq \epsilon/2$. In this case, to prove our claim, it is enough to show that

$$2(1-\delta) < \|z + y_0\| \tag{11}$$

since (11) implies $\|z - y_0\| < \epsilon$. To this end, note that by (9), one has

$$\frac{|A(x)(y_0) - \alpha(x)/r|}{\|A(x)\|_{E^*}} = \frac{\lambda|A(x)(z - y_0)|}{\|A(x)\|_{E^*}}, \tag{12}$$

and so, from (7), it follows that

$$\frac{|A(x)(z - y_0)|}{\|A(x)\|_{E^*}} < \delta. \tag{13}$$

Suppose $A(x)(z) = \|A(x)\|_{E^*}$. Then, from (13), we get

$$1 - \delta < \frac{A(x)(y_0)}{\|A(x)\|_{E^*}}. \tag{14}$$

On the other hand, we also have

$$1+\frac{A(x)(y_0)}{||A(x)||_{E^*}}=\frac{A(x)(z+y_0)}{||A(x)||_{E^*}}\leq ||z+y_0||. \tag{15}$$

So, (11) follows from (14) and (15). Now, suppose $A(x)(z) = -\|A(x)\|_{E^*}$. Then, from (13), we get

$$1-\delta<-\frac{A(x)(y_0)}{||A(x)||_{E^*}}. \tag{16}$$

On the other hand, we have

$$1-\frac{A(x)(y_0)}{||A(x)||_{E^*}}=-\frac{A(x)(z+y_0)}{||A(x)||_{E^*}}\leq ||z+y_0||. \tag{17}$$

So, in the present case, (11) is a consequence of (16) and (17). In such a manner, we have proved that F is lower semicontinuous at x_0. Hence, it remains proved that F is lower semicontinuous in X with respect to the strong topology and so, a fortiori, with respect to τ. Since F is also with nonempty τ-closed convex values and g/r is a τ-continuous selection of it over the closed set C, by Theorem 2, F admits a τ-continuous selection ω in X such that $\omega_{|C} = g/r$. At this point, if we put $\psi = r\omega$, it follows that ψ is a τ-continuous function, from X into B_r, such that $\psi_{|C} = g$ and $A(x)(\psi(x)) = \alpha(x)$ for all $x \in X$, against the hypotheses. This concludes the proof. □

Remark 4. From the proof, it clearly follows that if the assumption $|\alpha(x)| \leq r\|A(x)\|_{E^*}$ for all $x \in X$ is replaced by the more restrictive $|\alpha(x)| < r\|A(x)\|_{E^*}$ for all $x \in X \setminus A^{-1}(0)$, then the restrictions made on E and its norm become superfluous and, furthermore, the continuity assumption on A can be weakened to supposing that the function $x \to A(x)(y)$ is continuous for each y in a dense subset of E. Likewise, essentially the same proof gives the following version of Theorem 1, for $r = \infty$.

THEOREM 5. *Let X be a paracompact topological space, Y a real Banach space, and $A : X \to Y^*$ an operator such that the set*

$$\{y \in Y : x \longrightarrow A(x)(y) \text{ is continuous}\} \tag{18}$$

is dense in Y. Assume that there exist a continuous function $\alpha : X \to \mathbb{R}$, a (possibly empty) closed set $C \subset X$, and a continuous function $g : C \to Y$ satisfying $A(x)(g(x)) = \alpha(x)$ for all $x \in C$, in such a way that, for every continuous function $\psi : X \to Y$ satisfying $\psi_{|C} = g$, there exists $x_0 \in X$ such that $A(x_0)(\psi(x_0)) \neq \alpha(x_0)$. Then, there exists $x^* \in X$ such that $A(x^*) = 0$.

Sketch of proof. Arguing by contradiction, assume that $A^{-1}(0) = \emptyset$. For each $x \in X$, put

$$F(x) = \{y \in Y : A(x)(y) = \alpha(x)\}. \tag{19}$$

Thanks to Theorem 3, the multifunction F is lower semicontinuous. Since F is also with nonempty closed convex values and g is a continuous selection of it over the closed set C, by Michael's theorem, F admits a continuous selection ψ in X such that $\psi_{|C} = g$, against the hypotheses. $\square$

We now point out an interesting alternative coming from Theorem 5. The spaces $C^0(X,Y)$ and $C^0(X)$ that will appear are considered with the sup-norm. We recall that a subset D of a topological space S is a retract of S if there exists a continuous function $h : S \to D$ such that $h(s) = s$ for all $s \in D$.

Theorem 6. *Let X be a compact Hausdorff topological space, Y a real Banach space, with* $\dim(Y) \geq 2$, *and $A : X \to Y^*$ a continuous operator.*

Then, at least one of the following assertions holds:

(a) *there exists $x^* \in X$ such that $A(x^*) = 0$;*

(b) *there exists $\epsilon > 0$ such that, for every Lipschitzian operator $J : C^0(X,Y) \to C^0(X)$, with Lipschitz constant less than ϵ, the set*

$$\{\psi \in C^0(X,Y) : A(x)(\psi(x)) = J(\psi)(x)\ \forall x \in X\} \tag{20}$$

is an unbounded retract of $C^0(X,Y)$.

Proof. Assume that $A(x) \neq 0$ for all $x \in X$. For each $\psi \in C^0(X,Y)$ and $x \in X$, put

$$T(\psi)(x) = A(x)(\psi(x)). \tag{21}$$

Since A is continuous and bounded (due to the compactness of X), the function $T(\psi)(\cdot)$ is continuous (see the proof of Theorem 12). So, T turns out to be a continuous linear operator from $C^0(X,Y)$ into $C^0(X)$. Due to Theorem 5 (applied taking $C = \emptyset$), $A^{-1}(0) \neq \emptyset$ if (and only if) the operator T is not surjective. Thus, since we are supposing that $A^{-1}(0) = \emptyset$, the operator T is surjective. Furthermore, note that T is not injective. Indeed, if we fix any $x_0 \in X$ and choose $y_0 \in Y \setminus \{0\}$ so that $A(x_0)(y_0) = 0$ (recall that $\dim(Y) \geq 2$), by Theorem 5 again (applied taking $C = \{x_0\}$), there is $\psi \in C^0(X,Y)$ such that $T(\psi) = 0$ and $\psi(x_0) = y_0$. Finally, set

$$\epsilon = \frac{1}{\sup_{\|\varphi\|_{C^0(X)} \leq 1} \operatorname{dist}(0, T^{-1}(\varphi))}. \tag{22}$$

Due to this choice, by [5, Théorème 2], for every Lipschitzian operator $J : C^0(X,Y) \to C^0(X)$, with Lipschitz constant less than ϵ, the set

$$\Gamma := \{\psi \in C^0(X,Y) : T(\psi) = J(\psi)\} \tag{23}$$

turns out to be a retract of $C^0(X,Y)$. Moreover, from the proof of [5, Théorème 2], it follows that the multifunction $\psi \to T^{-1}(J(\psi))$ is a multivalued contraction, and so, since its values are closed and unbounded, the set of its fixed points (which agrees with Γ) is unbounded too by [7, Corollary 9]. $\square$

We now indicate two reasonable ways to apply Theorem 1. The first one is based on the Tychonoff fixed point theorem.

Theorem 7. *Assume that E is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $r > 0$ and let $A : B_r \to E$ be a continuous operator from the weak to the strong topology. Assume that there exist a weakly continuous function $\alpha : B_r \to \mathbb{R}$ satisfying $|\alpha(x)| \leq r\|A(x)\|$ for all $x \in B_r$, and a weakly continuous function $g : C \to B_r$ such that*

$$\langle A(x), g(x) \rangle = \alpha(x), \quad g(x) \neq x, \tag{24}$$

for all $x \in C$, where

$$C = \{x \in B_r : \langle A(x), x \rangle = \alpha(x)\}. \tag{25}$$

Then, there exists $x^* \in B_r$ such that $A(x^*) = 0$.

Proof. Identifying E with E^*, we apply Theorem 1 taking $X = B_r$, with the relativization of the weak topology of E, and taking τ as the weak topology of E. Due to the kind of continuity we are assuming for A, the function $x \to \langle A(x), x \rangle$ turns out to be weakly continuous (see the proof of Theorem 12), and so the set C is weakly closed. Now, let $\psi : B_r \to B_r$ be any weakly continuous function such that $\psi_{|C} = g$. By the Tychonoff fixed point theorem, there is $x_0 \in B_r$ such that $\psi(x_0) = x_0$. Since g has no fixed points in C, it follows that $x_0 \notin C$, and so

$$\langle A(x_0), \psi(x_0) \rangle = \langle A(x_0), x_0 \rangle \neq \alpha(x_0). \tag{26}$$

Hence, all the assumptions of Theorem 1 are satisfied and the conclusion follows from it. $\square$

It is worth noticing the following consequences of Theorem 7.

Theorem 8. *Let E and A be as in Theorem 7. Assume that for each $x \in B_r$, with $\|A(x)\| > r$,*

$$\left\| A\left(\frac{rA(x)}{||A(x)||} \right) \right\| \leq r. \tag{27}$$

Then, the operator A has either a zero or a fixed point.

Proof. Define the function $\alpha : B_r \to \mathbb{R}$ by

$$\alpha(x) = \begin{cases} ||A(x)||^2 & \text{if } ||A(x)|| \leq r, \\ r||A(x)|| & \text{if } ||A(x)|| > r. \end{cases} \tag{28}$$

Clearly, the function α is weakly continuous and satisfies $|\alpha(x)| \leq r\|A(x)\|$ for all $x \in B_r$. Put $C = \{x \in B_r : \langle A(x), x \rangle = \alpha(x)\}$. Note that if $x \in C$, then $\|A(x)\| \leq r$. Indeed, otherwise, we would have $\langle A(x), x \rangle = r\|A(x)\|$, and so, necessarily, $x = rA(x)/\|A(x)\|$, against (27). Hence, we have $\langle A(x), A(x) \rangle = \alpha(x)$ for all $x \in C$. At this point, the conclusion follows at once from Theorem 7, taking $g = A_{|C}$. $\square$

Remark 9. It would be interesting to know whether Theorem 8 can be improved assuming that A is a compact operator (i.e., continuous and with relatively compact range).

Remark 10. Note that Theorem 8 can be compared with the classical Rothe's theorem which assures the existence of a fixed point of A provided that it is compact and maps ∂B_r into B_r. Theorem 8 tells us that, under a more severe continuity assumption (see, however, Remark 9) and the condition $A^{-1}(0) = \emptyset$, the key Rothe's condition can be remarkably weakened to

$$A\left(\bigcup_{\lambda>0} \lambda A(A^{-1}(E \setminus B_r)) \cap \partial B_r\right) \subseteq B_r. \tag{29}$$

Theorem 11. *Let E and A be as in Theorem 7. Assume that there exists $w \in B_r$, with $\langle A(w), w\rangle \neq 0$, such that $\langle A(x), w\rangle = 0$ for all $x \in B_r$ satisfying $\langle A(x), x\rangle = 0$.*

Then, there exists $x^* \in B_r$ such that $A(x^*) = 0$.

Proof. Apply Theorem 7 taking $\alpha(x) = 0$ and $g(x) = w$ for all $x \in B_r$. □

The second application of Theorem 1 is based on connectedness arguments. For other results of this type, we refer to [6] (see also [2]).

Theorem 12. *Let X be a connected paracompact topological space and $A : X \to E^*$ a weakly continuous and locally bounded operator. Assume that there exist $r > 0$, a closed set $C \subset X$, a continuous function $g : C \to B_r$, and an upper semicontinuous function $\beta : X \to \mathbb{R}$, with $|\beta(x)| \leq r\|A(x)\|_{E^*}$ for all $x \in X$, such that $g(C)$ is disconnected,*

$$\beta(x) \leq A(x)(g(x)) \tag{30}$$

for all $x \in C$, and

$$A(x)(y) < \beta(x) \tag{31}$$

for all $x \in X \setminus C$ and $y \in B_r \setminus g(C)$.

Then, there exists $x^* \in C$ such that $A(x^*) = 0$.

Proof. First, note that the function $x \to A(x)(g(x))$ is continuous in C. To see this, let $x_1 \in C$ and let $\{x_\gamma\}_{\gamma \in D}$ be any net in C converging to x_1. By assumption, there are $M > 0$ and a neighborhood U of x_1 such that $\|A(x)\|_{E^*} \leq M$ for all $x \in U$. Let $\gamma_0 \in D$ be such that $x_\gamma \in U$ for all $\gamma \geq \gamma_0$. Thus, for each $\gamma \geq \gamma_0$, one has

$$\begin{aligned} &|A(x_\gamma)(g(x_\gamma)) - A(x_1)(g(x_1))| \\ &\quad \leq M\|g(x_\gamma) - g(x_1)\| + |A(x_\gamma)(g(x_1)) - A(x_1)(g(x_1))| \end{aligned} \tag{32}$$

from which, of course, it follows that $\lim_\gamma A(x_\gamma)(g(x_\gamma)) = A(x_1)(g(x_1))$. Next, observe that the multifunction $x \to [\beta(x), r\|A(x)\|_{E^*}]$ is lower semicontinuous and that the function $x \to A(x)(g(x))$ is a continuous selection of it in C. Hence, by Michael's theorem, there is a continuous function $\alpha : X \to \mathbb{R}$ such that $\alpha(x) = A(x)(g(x))$ for all $x \in C$ and $\beta(x) \leq \alpha(x) \leq r\|A(x)\|_{E^*}$ for all $x \in X$. Now, let $\psi : X \to B_r$ be any continuous function such that $\psi_{|C} = g$. Since X is connected, $\psi(X)$ is connected too. But then, since $g(C)$ is disconnected and $g(C) \subset \psi(X)$, there exists $y_0 \in \psi(X) \setminus g(C)$. Let $x_0 \in X \setminus C$ be such that $\psi(x_0) = y_0$.

So, by hypothesis, we have

$$A(x_0)(\psi(x_0)) = A(x_0)(y_0) < \beta(x_0) \leq \alpha(x_0). \tag{33}$$

Hence, taking τ as the strong topology of E, all the assumptions of Theorem 1 are satisfied and the conclusion follows from it. □

Remark 13. Observe that when X is first-countable, the local boundedness of A follows automatically from its weak continuity. This follows from the fact that, in a Banach space, any weakly convergent sequence is bounded.

It is worth noticing the corollary of Theorem 12 which comes out taking $X = B_r, \beta = 0$, and $g =$ identity.

Theorem 14. *Let E be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $r > 0$ and let $A : B_r \to E$ be a continuous operator from the strong to the weak topology. Assume that the set $C = \{x \in B_r : \langle A(x), x\rangle \geq 0\}$ is disconnected and that, for each $x, y \in B_r \setminus C$, $\langle A(x), y\rangle < 0$.*

Then, there exists $x^* \in C$ such that $A(x^*) = 0$.

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