MULTIVALUED p-LIENARD SYSTEMS

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We examine p-Lienard systems driven by the vector p-Laplacian differential operator and having a multivalued nonlinearity. We consider Dirichlet systems. Using a fixed point principle for set-valued maps and a nonuniform nonresonance condition, we establish the existence of solutions.

1. Introduction

In this paper, we use fixed point theory to study the following multivalued p-Lienard system:

$$\begin{gathered}\left(\|x'(t)\|^{p-2}x'(t)\right)' + \frac{d}{dt}\nabla G(x(t)) + F(t,x(t),x'(t)) \ni 0 \quad \text{a.e. on } T=[0,b],\\ x(0)=x(b)=0,\ 1<p<\infty.\end{gathered} \tag{1.1}$$

In the last decade, there have been many papers dealing with second-order multivalued boundary value problems. We mention the works of Erbe and Krawcewicz [5, 6], Frigon [7, 8], Halidias and Papageorgiou [9], Kandilakis and Papageorgiou [11], Kyritsi et al. [12], Palmucci and Papalini [17], and Pruszko [19]. In all the above works, with the exception of Kyritsi et al. [12], $p=2$ (linear differential operator), $G=0$, and $g=0$. Moreover, in Frigon [7, 8] and Palmucci and Papalini [17], the inclusions are scalar (i.e., $N=1$). Finally we should mention that recently single-valued p-Lienard systems were studied by Mawhin [14] and Manásevich and Mawhin [13].

In this work, for problem (1.1), we prove an existence theorem under conditions of nonuniform nonresonance with respect to the first weighted eigenvalue of the negative vector ordinary p-Laplacian with Dirichlet boundary conditions [15, 20]. Our approach is based on the multivalued version of the Leray-Schauder alternative principle due to Bader [1] (see Section 2).

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2. Mathematical background

In this section, we recall some basic definitions and facts from multivalued analysis, the spectral properties of the negative vector p-Laplacian, and the multivalued fixed point principles mentioned in the introduction. For details, we refer to Denkowski et al. [3] and Hu and Papageorgiou [10] (for multivalued analysis), to Denkowski et al. [2] and Zhang [20] (for the spectral properties of the p-Laplacian), and to Bader [1] (for the multivalued fixed point principle; similar results can also be found in O'Regan and Precup [16] and Precup [18]).

Let (Ω,Σ) be a measurable space and X a separable Banach space. We introduce the following notations:

$$\begin{gathered} P_{f(c)}(X) = \{A \subseteq X : \text{nonempty, closed (and convex)}\}, \\ P_{(w)k(c)}(X) = \{A \subseteq X : \text{nonempty, (weakly) compact (and convex)}\}. \end{gathered} \tag{2.1}$$

A multifunction $F:\Omega \to P_f(X)$ is said to be measurable if, for all $x \in X$, $\omega \to d(x, F(\omega)) = \inf[\|x-y\| : y \in F(\omega)]$ is measurable. A multifunction $F:\Omega \to 2^X\setminus\{\emptyset\}$ is said to be "graph measurable" if $\mathrm{GrF} = \{(\omega,x) \in \Omega\times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability and the converse is true if Σ is complete (i.e., $\Sigma = \hat{\Sigma}$ = the universal σ-field). Let μ be a finite measure on (Ω,Σ), $1 \le p \le \infty$, and $F:\Omega \to 2^X\setminus\{\emptyset\}$. We introduce the set $S_F^p = \{f \in L^p(\Omega,X) : f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. This set may be empty. For a graph-measurable multifunction, it is nonempty if and only if $\inf[\|y\| : y \in F(\omega)] \le \varphi(\omega)$ μ-a.e. on Ω, with $\varphi \in L^p(\Omega)_+$.

Let Y, Z be Hausdorff topological spaces. A multifunction $G: Y \to 2^Z\setminus\{\emptyset\}$ is said to be "upper semicontinuous" (usc for short) if, for all $C \subseteq Z$ closed, $G^-(C) = \{y \in Y : G(y)\cap C \neq \emptyset\}$ is closed or equivalently for all $U \subseteq Z$ open, $G^+\{y \in Y : G(y) \subseteq U\}$ is open. If Z is a regular space, then a $P_f(Z)$-valued multifunction which is usc has a closed graph. The converse is true if the multifunction G is locally compact (i.e., for every $y \in Y$, there exists a neighborhood U of y such that $\overline{G(U)}$ is compact in Z). A $P_k(Z)$-valued multifunction which is usc maps compact sets to compact sets.

Consider the following weighted nonlinear eigenvalue problem in $\mathbb{R}^N$:

$$\begin{gathered} -\left(||x'(t)||^{p-2}x'(t)\right)' = \lambda\theta(t)||x(t)||^{p-2}x(t) \quad \text{a.e. on } T = [0,b], \\ x(0) = x(b) = 0,\ 1 < p < \infty,\ \theta \in L^\infty(T),\ \left|\{\theta > 0\}\right|_1 > 0,\ \lambda \in \mathbb{R}. \end{gathered} \tag{2.2}$$

Here by $|\cdot|_1$ we denote the 1-dimensional Lebesgue measure. The real parameters λ, for which problem (2.3) has a nontrivial solution, are called eigenvalues of the negative vector p-Laplacian with Dirichlet boundary conditions denoted by $(-\triangle_p, W_0^{1,p}(T, \mathbb{R}^N))$, with weight $\theta \in L^\infty(T)$. The corresponding nontrivial solutions are known as eigenfunctions. We know that the eigenvalues of problem (2.3) are the same as those of the corresponding scalar problem [13]. Then from Denkowski et al. [2] and Zhang [20], we know that there exist two sequences $\{\lambda_n(\theta)\}_{n\ge1}$ and $\{\lambda_{-n}(\theta)\}_{n\ge1}$ such that $\lambda_n(\theta) > 0$, $\lambda_n(\theta) \to +\infty$ and $\lambda_{-n}(\theta) < 0$, $\lambda_{-n}(\theta) \to -\infty$ as $n \to \infty$. Moreover, if $\theta(t) \ge 0$ a.e. on T with strict inequality on a set of positive Lebesgue measure, then we have only the positive sequence $\{\lambda_n(\theta)\}_{n\geq 1}$. Also, for $\lambda_1(\theta) > 0$, we have the following variational characterization:

$$\lambda_1(\theta) = \inf\left[\frac{\|x'\|_p^p}{\int_0^b \theta(t)\|x(t)\|^p dt} : x \in W_0^{1,p}(T,\mathbb{R}^N),\ x \neq 0\right]. \tag{2.3}$$

The infimum is attained at the normalized principal eigenfunction u_1 ($\lambda_1(\theta) > 0$ is simple) and $u_1(t) \neq 0$ a.e. on T. Also, $\lambda_1(\theta)$ is strictly monotone with respect to θ, namely, if $\theta_1(t) \leq \theta_2(t)$ a.e. on T with strict inequality on a set of positive measure, then $\lambda_1(\theta_2) < \lambda_1(\theta_1)$ (see (3.2)).

Finally we state the multivalued fixed point principle that we will use in the study of problem (1.1). So let Y, Z be two Banach spaces and $C \subseteq Y$, $D \subseteq Z$ two nonempty closed and convex sets. We consider multifunctions $G : C \to 2^C \setminus \{\emptyset\}$ which have a decomposition $G = K \circ N$, satisfying the following: $K : D \to C$ is completely continuous, namely, if $z_n \xrightarrow{w} z$ in D, then $K(z_n) \to K(z)$ in C and $N : C \to P_{wkc}(D)$ is usc from C, furnished with the strong topology into D, furnished with the weak topology.

Theorem 2.1. *If C, D, and $G = K \circ N$ are as above, $0 \in C$, and G is compact (namely, G maps bounded subsets of C into relatively compact subsets of D),* then *one of the following alternatives holds:*

(a) *$S = \{y \in C : y \in \mu G(y)$ for some $\mu \in (0,1)\}$ is unbounded or*
(b) *G has a fixed point, that is, there exists $y \in C$ such that $y \in G(y)$.*

Remark 2.2. Evidently this is a multivalued version of the classical Leray-Schauder alternative principle [2, page 206]. In contrast to previous multivalued extensions of the Leray-Schauder alternative principal [4, page 61], Theorem 2.1 does not require G to have convex values, which is important when dealing with nonlinear problems such as (1.1).

3. Nonuniform nonresonance

In this section, we deal with problem (1.1) using a condition of nonuniform nonresonance with respect to the first eigenvalue $\lambda_1(\theta) > 0$. Our hypotheses on the multivalued nonlinearity $F(t,x,y)$ are as follows.

$(H(F)_1)$ $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x, y \in \mathbb{R}^N$, $t \to F(t,x,y)$ is graph measurable;
(ii) for almost all $t \in T$, $(x,y) \to F(t,x,y)$ is usc;
(iii) for every $M > 0$, there exists $\gamma_M \in L^1(T)_+$ such that, for almost all $t \in T$, all $\|x\|, \|y\| \leq M$, and all $u \in F(t,x,y)$, we have $\|u\| \leq \gamma_M(t)$;
(iv) there exists $\theta \in L^\infty(T)$, $\theta(t) \geq 0$ a.e. on T, with strict inequality on a set of positive measure and

$$\limsup_{\|x\|\to+\infty} \frac{\sup\left[(u,x)_{\mathbb{R}^N} : u \in F(t,x,y),\ y \in \mathbb{R}^N\right]}{\|x\|^p} \leq \theta(t) \tag{3.1}$$

uniformly for almost all $t \in T$ and $\lambda_1(\theta) > 1$.

Remark 3.1. Hypothesis $(H(F)_1)$(iv) is the nonuniform nonresonance condition. In the literature [15, 20], we encounter the condition $\theta(t) \le \lambda_1$ a.e. on T with strict inequality on a set of positive measure. Here $\lambda_1 > 0$ is the principal eigenvalue corresponding to the unit weight $\theta = 1$ (i.e., $\lambda_1 = \lambda_1(1)$). Then by virtue of the strict monotonicity property, we have $\lambda_1(\lambda_1) = 1 < \lambda_1(\theta)$, which is the condition assumed in hypothesis $(H(F)_1)$(iv).

$(H(G)_1)$ $G \in C^2(\mathbb{R}^N, \mathbb{R})$.

Given $h \in L^1(T, \mathbb{R}^N)$, we consider the following Dirichlet problem:

$$-\left(\|x'(t)\|^{p-2}x'(t)\right)' = h(t) \quad \text{a.e. on } T = [0,b], \qquad x(0) = x(b) = 0. \tag{3.2}$$

From Manásevich and Mawhin [13, Lemma 4.1], we know that problem (3.3) has a unique solution $K(h) \in C_0^1(T, \mathbb{R}^N) = \{x \in C^1(T\mathbb{R}^N) : x(0) = x(b) = 0\}$. So we can define the solution map $K : L^1(T, \mathbb{R}^N) \to C_0^1(T, \mathbb{R}^N)$.

Proposition 3.2. *$K : L^1(T, \mathbb{R}^N) \to C_0^1(T, \mathbb{R}^N)$ is completely continuous, that is, if $h_n \xrightarrow{w} h$ in $L^1(T, \mathbb{R}^N)$, then $K(h_n) \to K(h)$ in $C_0^1(T, \mathbb{R}^N)$.*

Proof. Let $h_n \xrightarrow{w} h$ in $L^1(T, \mathbb{R}^N)$ and set $x_n = K(h_n)$, $n \ge 1$. We have

$$-\left(\|x_n'(t)\|^{p-2}x_n'(t)\right)' = h_n(t) \quad \text{a.e. on } T,\ x_n(0) = x_n(b) = 0,\ n \ge 1. \tag{3.3}$$

Taking the inner product with $x_n(t)$, integrating over T, and performing integration by parts, we obtain

$$\|x_n'\|_p^p \le \|h_n\|_1 \|x_n\|_\infty \le c_1 \|x_n'\|_p \text{ for some } c_1 > 0 \text{ and all } n \ge 1. \tag{3.4}$$

Here we have used Hölder and Poincare inequalities. It follows that

$$\begin{aligned} &\{x_n'\}_{n\ge1} \subseteq L^p(T, \mathbb{R}^N) \text{ is bounded (since } p > 1) \\ &\Longrightarrow \{x_n\}_{n\ge1} \subseteq W_0^{1,p}(T, \mathbb{R}^N) \text{ is bounded (by the Poincare inequality).} \end{aligned} \tag{3.5}$$

So from (3.22) we infer that

$$\begin{aligned} &\{\|x_n'\|^{p-2}x_n'\}_{n\ge1} \subseteq W^{1,q}(T, \mathbb{R}^N) \left(\frac{1}{p} + \frac{1}{q} = 1\right) \text{ is bounded} \\ &\Longrightarrow \{\|x_n'\|^{p-2}x_n'\}_{n\ge1} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact} \end{aligned} \tag{3.6}$$

(recall that $W^{1,q}(T, \mathbb{R}^N)$ is embedded compactly in $C(T, \mathbb{R}^N)$). The map $\varphi_p : \mathbb{R}^N \to \mathbb{R}^N$, defined by $\varphi_p(y) = \|y\|^{p-2}y$, $y \in \mathbb{R}^N \setminus \{\varnothing\}$, and $\varphi_p(0) = 0$, is a homeomorphism and so $\hat{\varphi_p}^{-1} : C(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$, defined by $\hat{\varphi_p}^{-1}(y)(\cdot) = \varphi_p^{-1}(y(\cdot))$, is continuous and bounded. Thus it follows that

$$\begin{aligned} &\{x_n'\}_{n\ge1} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact} \\ &\Longrightarrow \{x_n\}_{n\ge1} \subseteq C_0^1(T, \mathbb{R}^N) \text{ is relatively compact.} \end{aligned} \tag{3.7}$$

Therefore we may assume that $x_n \to x$ in $C_0^1(T,\mathbb{R}^N)$. Also $\{\|x_n'\|^{p-2}x_n'\}_{n\geq 1} \subseteq W^{1,q}(T,\mathbb{R}^N)$ is bounded and so we may assume that $\|x_n'\|^{p-2}x_n' \xrightarrow{w} u$ in $W^{1,q}(T,\mathbb{R}^N)$ and $\|x_n'\|^{p-2}x_n' \to u$ in $C(T,\mathbb{R}^N)$ (because $W^{1,q}(T,\mathbb{R}^N)$ is embedded compactly in $C(T,\mathbb{R}^N)$). It follows that $u = \|x'\|^{p-2}x'$. Hence if in (3.22) we pass to the limit as $n \to \infty$, we obtain

$$\begin{aligned} &-\left(\|x'(t)\|^{p-2}x'(t)\right)' = h(t) \quad \text{a.e. on } T = [0,b],\ x(0) = x(b) = 0 \\ &\Longrightarrow K(h) = x. \end{aligned} \tag{3.8}$$

Since every subsequence of $\{x_n\}_{n\geq 1}$ has a further subsequence which converges to x in $C_0^1(T,\mathbb{R}^N)$, we conclude that the original sequence converges too. This proves the complete continuity of K. □

Let $N_F : C_0^1(T,\mathbb{R}^N) \to 2^{L^1(T,\mathbb{R}^N)}$ be the multivalued Nemitsky operator corresponding to F, that is,

$$N_F(x) = \{u \in L^1(T,\mathbb{R}^N) : u(t) \in F(t,x(t),x'(t)) \text{ a.e. on } T\}. \tag{3.9}$$

Also let $N : C_0^1(T,\mathbb{R}^N) \to 2^{L^1(T,\mathbb{R}^N)}$ be defined by

$$N(x) = \frac{d}{dx}\nabla G(x(\cdot)) + N_F(x). \tag{3.10}$$

This multifunction has the following structure.

Proposition 3.3. *If hypotheses $(H(F)_1)$ and $(H(G)_1)$ hold, then N has values in $P_{wkc}(L^1(T,\mathbb{R}^N))$ and it is usc from $C_0^1(T,\mathbb{R}^N)$ with the norm topology into $L^1(T,\mathbb{R}^N)$ with the weak topology.*

Proof. Clearly N has closed, convex values which are uniformly integrable (see hypothesis $(H(F)_1)$(iii)). Therefore for every $x \in C_0^1(T,\mathbb{R}^N)$, $N(x)$ is convex and w-compact in $L^1(T,\mathbb{R}^N)$. What is not immediately clear is that $N(x) \neq \emptyset$, since hypotheses $(H(F)_1)$(i) and (ii) in general do not imply the graph measurability of $(t,x,y) \to F(t,x,y)$ [10, page 227]. To see that $N(x) \neq \emptyset$, we proceed as follows. Let $\{s_n\}_{n\geq 1}$, $\{r_n\}_{n\geq 1}$ be step functions such that $s_n \to x$ and $r_n \to x'$ a.e. on T and $\|s_n(t)\| \leq \|x(t)\|$, $\|r_n(t)\| \leq \|x'(t)\|$ a.e. on T, $n \geq 1$. Then by virtue of hypothesis $(H(F)_1)$(i), for every $n \geq 1$, the multifunction $t \to F(t,s_n(t),r_n(t))$ is measurable and so by the Yankon-von Neumann-Aumann selection theorem [10, page 158], we can find $u_n : T \to \mathbb{R}^N$ a measurable map such that $u_n(t) \in F(t,s_n(t),r_n(t))$ for all $t \in T$. Note that $\|s_n\|_\infty, \|r_n\|_\infty \leq M_1$ for some $M_1 > 0$ and all $n \geq 1$. So $\|u_n(t)\| \leq \gamma_{M_1}(t)$ a.e. on T, with $\gamma_{M_1} \in L^1(T)_+$ (see hypothesis $(H(F)_1)$(iii)). Thus by virtue of the Dunford-Pettis theorem, we may assume that $u_n \xrightarrow{w} u$ in $L^1(T,\mathbb{R}^N)$ as $n \to \infty$. From Hu and Papageorgiou [10, page 694], we have

$$u(t) \in \overline{\text{conv}} \limsup_{n\to\infty} F(t,s_n(t),r_n(t)) \subseteq F(t,x(t),x'(t)) \quad \text{a.e. on } T, \tag{3.11}$$

with the last inclusion being a consequence of hypothesis $(H(F)_1)$(ii). So we have $u \in S^q_{F(\cdot,x(\cdot),x'(\cdot))}$, hence $N(x) \neq \emptyset$.

Next we check the upper semicontinuity of N into $L^1(T,\mathbb{R}^N)_w$ ($L^1(T,\mathbb{R}^N)_w$ equals the Banach space $L^1(T,\mathbb{R}^N)$ furnished with the weak topology). Because of hypothesis $(H(F)_1)$(iii), N is locally compact into $L^1(T,\mathbb{R}^N)_w$ (recall that uniformly integrable sets are relatively compact in $L^1(T,\mathbb{R}^N)_w$). Also on weakly compact subsets of $L^1(T,\mathbb{R}^N)$, the relative weak topology is metrizable. Therefore to check the upper semicontinuity of N, it suffices to show that GrN is sequentially closed in $C_0^1(T,\mathbb{R}^N)\times L^1(T,\mathbb{R}^N)_w$ (see Section 2). To this end, let $(x_n,f_n)\in \mathrm{GrN}$, $n\geq 1$, and suppose that $x_n\to x$ in $C_0^1(T,\mathbb{R}^N)$ and $f_n\xrightarrow{w} f$ in $L^1(T,\mathbb{R}^N)$. For every $n\geq 1$, we have

$$f_n(t)=\frac{d}{dt}\nabla G(x_n(t))+u_n(t) \quad \text{a.e. on } T, \text{ with } u_n\in S^1_{F(\cdot,x_n(\cdot),x_n'(\cdot))}. \tag{3.12}$$

Because of hypothesis $(H(F)_1)$(iii), we may assume (at least for a subsequence) that $u_n\xrightarrow{w} u$ in $L^1(T,\mathbb{R}^N)$. As before, from Hu and Papageorgiou [10, page 694], we have

$$u(t)\in \overline{\mathrm{conv}}\limsup_{n\to\infty} F(t,x_n(t),x_n'(t))\subseteq F(t,x(t),x'(t)) \quad \text{a.e. on } T \tag{3.13}$$

(again the last inclusion follows from hypothesis $(H(F)_1)$(ii)). So $u\in S^1_{F(\cdot,x(\cdot),x'(\cdot))}$. Also by virtue of hypothesis $(H(G)_1)$, we have

$$\begin{aligned}&\frac{d}{dt}\nabla G(x_n(t))=G''(x_n(t))x_n'(t)\longrightarrow G''(x(t))x'(t)=\frac{d}{dt}\nabla G(x(t)), \quad \forall t\in T\\ &\Longrightarrow \frac{d}{dt}\nabla G(x_n(\cdot))\longrightarrow \frac{d}{dt}\nabla G(x(\cdot)) \quad \text{in } L^1(T,\mathbb{R}^N)\\ &\qquad \text{(by the dominated convergence theorem).}\end{aligned} \tag{3.14}$$

So in the limit as $n\to\infty$, we have

$$\begin{aligned}&f=\frac{d}{dt}\nabla G(x(\cdot))+u \quad \text{with } u\in N_F(x)\\ &\Longrightarrow (x,f)\in \mathrm{GrN}.\end{aligned} \tag{3.15}$$

This proves the desired upper semicontinuity of N. □

Proposition 3.4. *There exists $\xi>0$ such that, for all $x\in W_0^{1,p}(T,\mathbb{R}^N)$,*

$$\|x'\|_p^p-\int_0^b \theta(t)\|x(t)\|^p\,dt\geq \xi\|x'\|_p^p. \tag{3.16}$$

Proof. Let $\eta: W_0^{1,p}(T,\mathbb{R}^N)\to\mathbb{R}$ be the functional defined by

$$\eta(x)=\|x'\|_p^p-\int_0^b\theta(t)\|x(t)\|^p\,dt. \tag{3.17}$$

From the variational characterization of $\lambda_1(\theta)>1$, we see that $\eta(x)>0$ for all $x\in W_0^{1,p}(T,\mathbb{R}^N)$, $x\neq 0$. Suppose that the proposition was not true. Then by virtue of the p-homogeneity of η, we can find $\{x_n\}_{n\geq1}\subseteq W_0^{1,p}(T,\mathbb{R}^N)$ such that $\|x_n'\|_p=1$ and $\eta(x_n)\downarrow 0$.

By the Poincare inequality, the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(T,\mathbb{R}^N)$ is bounded and so we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(T,\mathbb{R}^N), \qquad x_n \longrightarrow x \quad \text{in } C_0(T,\mathbb{R}^N). \tag{3.18}$$

Also exploiting the weak lower semicontinuity of the norm functional in a Banach space, we obtain

$$\|x'\|_p^p \leq \int_0^b \theta(t)\|x(t)\|^p dt \Longrightarrow \lambda_1(\theta) \leq 1, \tag{3.19}$$

a contradiction to our hypothesis that $\lambda_1(\theta) > 1$. □

We introduce the set

$$S = \{x \in C_0^1(T,\mathbb{R}^N) : x \in \lambda KN(x),\ 0 < \lambda < 1\}. \tag{3.20}$$

Proposition 3.5. *If hypotheses $(H(F)_1)$ and $(H(G)_1)$ hold,* then $S \subseteq C_0^1(T,\mathbb{R}^N)$ *is bounded.*

Proof. Let $x \in S$. We have

$$\begin{aligned}
&\frac{1}{\lambda}x \in KN(x) \quad \text{with } 0<\lambda<1 \\
&\Longrightarrow \frac{1}{\lambda^{p-1}}\left(\|x'(t)\|^{p-2}x'(t)\right)' + \frac{d}{dt}\nabla G(x(t)) + u(t) = 0 \quad \text{a.e. on } T, \text{ with } u \in S^1_{F(\cdot,x(\cdot),x'(\cdot))} \\
&\Longrightarrow \left(\|x'(t)\|^{p-2}x'(t)\right)' + \lambda^{p-1}\frac{d}{dt}\nabla G(x(t)) + \lambda^{p-1}u(t) = 0 \quad \text{a.e. on } T.
\end{aligned} \tag{3.21}$$

Taking the inner product with $x(t)$, integrate over T, and perform integration by parts, we obtain

$$-\|x'\|_p^p - \lambda^{p-1}\int_0^b (\nabla G(x(t)),x'(t))_{\mathbb{R}^N}dt + \lambda^{p-1}\int_0^b (u(t),x(t))_{\mathbb{R}^N}dt = 0. \tag{3.22}$$

Remark that

$$\int_0^b (\nabla G(x(t)),x'(t))_{\mathbb{R}^N}dt = \int_0^b \frac{d}{dt}G(x(t))dt = G(x(b)) - G(x(0)) = 0. \tag{3.23}$$

By virtue of hypotheses $(H(F)_1)$(iii) and (iv), given $\varepsilon > 0$, we can find $\gamma_\varepsilon \in L^1(T)_+$ such that for almost all $t \in T$, all $x, y \in \mathbb{R}^N$, and all $u \in F(t,x,y)$, we have

$$(u,x)_{\mathbb{R}^N} \leq (\theta(t)+\varepsilon)\|x\|^p + \gamma_\varepsilon(t). \tag{3.24}$$

So we have

$$\int_0^b (u(t),x(t))_{\mathbb{R}^N}dt \leq \int_0^b \theta(t)\|x(t)\|^p dt + \varepsilon\|x\|_p^p + \|\gamma_\varepsilon\|_1. \tag{3.25}$$

Using (3.24) and (3.27) in (3.23), we obtain

$$\begin{aligned}
&\|x'\|_p^p \le \int_0^b \theta(t)\|x(t)\|^p dt + \varepsilon\|x\|_p^p + \|\gamma_\varepsilon\|_1 \\
&\Longrightarrow \xi\|x'\|_p^p - \frac{\varepsilon}{\lambda_1}\|x'\|_p^p \le \|\gamma_\varepsilon\|_1
\end{aligned} \tag{3.26}$$

(see Proposition 3.5 and recall that $\lambda_1\|x\|_p^p \le \|x'\|_p^p$, $\lambda_1 = \lambda_1(1)$).

Choose $\varepsilon > 0$ so that $\varepsilon < \lambda_1\xi$. Then from the last inequality, we infer that

$$\begin{aligned}
&\{x'\}_{x\in S} \subseteq L^p(T,\mathbb{R}^N) \text{ is bounded} \\
&\Longrightarrow S \subseteq W_0^{1,p}(T,\mathbb{R}^N) \text{ is bounded (by Poincare's inequality)} \\
&\Longrightarrow S \subseteq C_0(T,\mathbb{R}^N) \text{ is relatively compact.}
\end{aligned} \tag{3.27}$$

Also we have

$$\begin{aligned}
&\|(\|x'(t)\|^{p-2}x'(t))'\| \\
&\quad \le \|G''(x(t))\|_{\mathscr{L}}\|x'(t)\| + \|u(t)\| \text{ a.e. on } T \\
&\quad \le M_2(\|x'(t)\| + \theta(t) + \varepsilon + \gamma_\varepsilon(t)) \text{ a.e. on } T \text{ for some } M_2 > 0 \text{ (see (3.25))} \\
&\quad \Longrightarrow \{\|x'\|^{p-2}x'\}_{x\in S} \subseteq W^{1,1}(T,\mathbb{R}^N) \text{ is bounded} \\
&\quad \Longrightarrow \{\|x'\|^{p-2}x'\}_{x\in S} \subseteq C(T,\mathbb{R}^N) \text{ is bounded} \\
&\qquad\quad \text{(since } W^{1,1}(T,\mathbb{R}^N) \text{ is embedded continuously but not compactly in } C(T,\mathbb{R}^N)) \\
&\quad \Longrightarrow \{x'\}_{x\in S} \subseteq C(T,\mathbb{R}^N) \text{ is bounded.}
\end{aligned} \tag{3.28}$$

From (3.28) and (3.29), we conclude that $S \subseteq C_0^1(T,\mathbb{R}^N)$ is bounded. □

Propositions 3.2, 3.3, and 3.5 permit the use of Theorem 2.1. So we obtain the following existence result for problem (1.1).

THEOREM 3.6. If *hypotheses* $(H(F)_1)$ *and* $(H(G)_1)$ *hold,* then *problem (1.1) has a solution* $x \in C_0^1(T,\mathbb{R}^N)$ *with* $\|x'\|^{p-2}x' \in W^{1,1}(T,\mathbb{R}^N)$.

As an application of this theorem, we consider the following system:

$$\begin{aligned}
(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}Ax(t) + F(t,x(t)) \ni e(t) \quad \text{a.e. on } T = [0,b], \\
x(0) = x(b) = 0,\ e \in L^1(T,\mathbb{R}^N).
\end{aligned} \tag{3.29}$$

Our hypotheses on the data of problem (3.29) are the following.

$(H(A))$ A is an $N \times N$ matrix such that for all $x \in \mathbb{R}^N$ we have $(Ax,x)_{\mathbb{R}^N} \le \theta\|x\|^2$ with $\theta < (\pi_p/b)^p$.

Remark 3.7. The quantity π_p is defined by $\pi_p = 2(p-1)^{1/p}\int_0^1(1/(1-t)^{1/p})dt = 2(p-1)^{1/p}((\pi/p)/\sin(\pi/p))$. If $p = 2$, then $\pi_2 = \pi$. Recall that the eigenvalues of $(-\triangle_p, W_0^{1,p}(T,\mathbb{R}^N))$ are $\lambda_n = (n\pi_p/b)^p$, $n \ge 1$ [13]. So in hypothesis $(H(A))$, we have $\theta < \lambda_1$.

$(H(F)_1')$ $F: T\times\mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x\in\mathbb{R}^N$, $t\to F(t,x)$ is graph measurable;

(ii) for almost all $t\in T$, $x\to F(t,x)$ is usc;

(iii) for every $M>0$, there exists $\gamma_M\in L^1(T)_+$ such that for almost all $t\in T$, all $\|x\|\le M$, and all $u\in F(t,x)$, we have $\|u\|\le\gamma_M(t)$;

(iv) $\lim_{\|x\|\to\infty}((u,x)_{\mathbb{R}^N}/\|x\|^p)=0$ uniformly for almost all $t\in T$ and all $u\in F(t,x)$.

Invoking Theorem 3.6, we obtain the following existence result for problem (3.29).

Theorem 3.8. If *hypotheses* $(H(A))$ *and* $(H(F)_1')$ *hold,* then *for every* $e\in L^1(T,\mathbb{R}^N)$, *problem (3.29) has a solution* $x\in C_0^1(T,\mathbb{R}^N)$ *with* $\|x'\|^{p-2}x'\in W^{1,1}(T,\mathbb{R}^N)$.

Remark 3.9. Theorem 3.8 extends Theorem 7.1 of Manásevich and Mawhin [13].

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