COINCIDENCE AND FIXED POINT THEOREMS FOR FUNCTIONS IN S-KKM CLASS ON GENERALIZED CONVEX SPACES

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We establish a coincidence theorem in S-KKM class by means of the basic defining property for multifunctions in S-KKM. Based on this coincidence theorem, we deduce some useful corollaries and investigate the fixed point problem on uniform spaces.

1. Introduction

A multimap $T : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y. If $H, T : X \to 2^Y$, then the coincidence problem for H and T is concerned with conditions which guarantee that $H(\hat{x}) \cap T(\hat{x}) \neq \varnothing$ for some $\hat{x} \in X$. Park [11] established a very general coincidence theorem in the class $\mathbf{U}_c^k$ of admissible functions, which extends and improves many results of Browder [1, 2], Granas and Liu [6].

On the other hand, Huang together with Chang et al. [3] introduced the S-KKM class which is much larger than the class $\mathbf{U}_c^k$. A lot of interesting and generalized results about fixed point theory on locally convex topological vector spaces have been studied in the setting of S-KKM class in [3]. In this paper, we will at first construct a coincidence theorem in S-KKM class on generalized convex spaces by means of the basic defining property for multimaps in S-KKM class. And then based on this coincidence theorem, we deduce some useful corollaries and investigate the fixed point problem on uniform spaces.

2. Preliminaries

Throughout this paper, $\langle Y \rangle$ denotes the class of all nonempty finite subsets of a nonempty set Y. The notation $T : X \multimap Y$ stands for a multimap from a set X into $2^Y \setminus \{\varnothing\}$. For a multimap $T : X \to 2^Y$, the following notations are used:

(a) $T(A) = \bigcup_{x \in A} T(x)$ for $A \subseteq X$;

(b) $T^-(y) = \{x \in X : y \in T(x)\}$ for $y \in Y$;

(c) $T^-(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$ for $B \subseteq Y$.

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All topological spaces are supposed to be Hausdorff. Let X and Y be two topological spaces. A multimap $T : X \to 2^Y$ is said to be

(a) upper semicontinuous (u.s.c.) if $T^-(B)$ is closed in X for each closed subset B of Y;

(b) compact if $T(X)$ is contained in a compact subset of Y;

(c) closed if its graph $\mathrm{Gr}(T) = \{(x,y) : y \in T(x),\ x \in X\}$ is a closed subset of $X \times Y$.

Lemma 2.1 (Lassonde [9, Lemma 1]). *Let X and Y be two topological spaces and $T : X \multimap Y$.*

(a) *If Y is regular and T is u.s.c. with closed values, then T is closed. Conversely, if Y is compact and T is closed, then T is u.s.c. with closed values.*

(b) *If T is u.s.c. and compact-valued, then $T(A)$ is compact for any compact subset A of X.*

Let X be a subset of a vector space and D a nonempty subset of X. Then (X,D) is called a convex space if the convex hull $\mathrm{co}(A)$ of any $A \in \langle D\rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. A subset C of (X,D) is said to be D-convex if $\mathrm{co}(A) \subseteq C$ for any $A \in \langle D\rangle$ with $A \subseteq C$. If $X = D$, then $X = (X,X)$ becomes a convex space in the sense of Lassonde [9]. The concept of convexity is further generalized under an extra condition by Park and Kim [12]. Later, Lin and Park [10] give the following definition by removing the extra condition.

Definition 2.2. A generalized convex space or a G-convex space $(X,D;\Gamma)$ consists of a topological space X, a nonempty subset D of X and a map $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with $|A| = n+1$, there exists a continuous function $\varphi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\varphi_A(\Delta_J) \subseteq \Gamma(J)$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A\rangle$.

A subset K of a G-convex space $(X,D;\Gamma)$ is said to be Γ-convex if for any $A \in \langle K \cap D\rangle$, $\Gamma(A) \subseteq K$.

In what follows we will express $\Gamma(A)$ by Γ_A, and we just say that (X,Γ) is a G-convex space provided that $D = X$.

The c-space introduced by Horvath [7] is an example of G-convex space.

For topological spaces X and Y, $\mathscr{C}(X,Y)$ denote the class of all continuous (single-valued) functions from X to Y.

Given a class $\mathscr{L}$ of multimaps, $\mathscr{L}(X,Y)$ denotes the set of multimaps $T : X \to 2^Y$ belonging to $\mathscr{L}$, and $\mathscr{L}_c$ the set of finite composites of multimaps in $\mathscr{L}$. Park and Kim [12] introduced the class $\mathbf{U}$ to be the one satisfying

(a) $\mathbf{U}$ contains the class $\mathscr{C}$ of (single-valued) continuous functions;

(b) each $T \in \mathbf{U}_c$ is upper semicontinuous and compact-valued; and

(c) for any polytope P, each $T \in \mathbf{U}_c(P,P)$ has a fixed point.

Further, Park defined the following

$$
\begin{aligned}
T \in \mathbf{U}_c^k(X,Y) \Longleftrightarrow\ & \text{for any compact subset } K \text{ of } X, \text{ there is a} \\
& \Gamma \in \mathbf{U}_c(X,Y) \text{ such that } \Gamma(x) \subseteq T(x) \text{ for each } x \in K.
\end{aligned}
\tag{2.1}
$$

A uniformity for a set X is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ such that

(a) each member of $\mathcal{U}$ contains the diagonal Δ;

(b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(c) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some V in $\mathcal{U}$;

(d) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and

(e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

If $(X,\mathcal{U})$ is a uniform space the topology $\mathcal{T}$ induced by $\mathcal{U}$ is the family of all subsets W of X such that for each x in W there is U in $\mathcal{U}$ such that $U[x] \subseteq W$, where $U[x]$ is defined as $\{y \in X : (x,y) \in U\}$. For details of uniform spaces we refer to [8].

3. The results

The concept of S-KKM property of [3] can be extented to G-convex spaces.

Definition 3.1. Let X be a nonempty set, $(Y,D;\Gamma)$ a G-convex space and Z a topological space. If $S : X \multimap D$, $T : Y \multimap Z$ and $F : X \multimap Z$ are three multimaps satisfying

$$T(\Gamma_{S(A)}) \subseteq F(A) \tag{3.1}$$

for any $A \in \langle X\rangle$, then F is called a S-KKM mapping with respect to T. If the multimap $T : Y \multimap Z$ satisfies that for any S-KKM mapping F with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class S-KKM(X,Y,Z) is defined to be the set $\{T : X \multimap Y : T$ has the S-KKM property$\}$.

When $D = Y$ is a nonempty convex subset of a linear space with $\Gamma_B = \mathrm{co}(B)$ for $B \in \langle Y\rangle$, the S-KKM(X,Y,Z) is just that as in [3]. In the case that $X = D$ and S is the identity mapping 1_D, S-KKM(X,Y,Z) is abbreviated as KKM(Y,Z), and a 1_D-KKM mapping with respect to T is called a KKM mapping with respect to T, and 1_D-KKM property is called KKM property. Just as [3, Propositions 2.2 and 2.3], for X a nonempty set, $(Y,D;\Gamma)$ a G-convex space, Z a topological space and any $S \multimap D$, one has $T \in$ KKM$(Y,Z) \subseteq S$-KKM(X,Y,Z). By the corollary to [13, Theorem 2], we have $\mathbf{U}_c^k(Y,Z) \subseteq$ KKM(Y,Z), and so $\mathbf{U}_c^k(Y,Z) \subseteq S$-KKM$(X,Y,Z)$.

Here we like to give a concrete multimap T having KKM property on a G-convex space. Let $X = [0,1] \times [0,1]$ be endowed with the Euclidean metric. For any $A = \{\mathbf{x}_1,\dots,\mathbf{x}_n\} \in \langle X\rangle$, define $\Gamma_A = \bigcup_{i=1}^n [\mathbf{0},\mathbf{x}_i]$, where $[\mathbf{0},\mathbf{x}_i]$ denotes the line segment joining $\mathbf{0}$ and $\mathbf{x}_i$. It is easy to see that (X,Γ) is a c-space, and so it is a G-convex space. Let $T : X \multimap X$ be defined by $T(\mathbf{x}) = [(0,0),(0,1)] \cup [(0,0),(1,0)]$. If $F : X \multimap X$ is any KKM mapping with respect to T, then for any $A = \{\mathbf{x}_1,\dots,\mathbf{x}_n\} \in \langle X\rangle$, since $T(\Gamma_A) \subseteq F(A)$ and $(0,0) \in T(0,0)$, we infer that $(0,0) \in T(\mathbf{x}_i) \subseteq F(\mathbf{x}_i)$ for any $i = 1,\dots,n$, so $(0,0) \in \bigcap_{i=1}^n F(\mathbf{x}_i)$. This shows that T has the KKM property.

A subset B of a topological space Z is said to be compactly open if for any compact subset K of Z, $K \cap B$ is open in K. We begin with the following coincidence theorem.

Theorem 3.2. *Let X be any nonempty set, $(Y,D;\Gamma)$ a G-convex space and Z a topological space. Suppose $s : X \to D$, $W : D \to 2^Z$, $H : Y \to 2^Z$ and $T \in s$-KKM(X,Y,Z) satisfy the* *following conditions:*

(3.2.1) *T is compact;*

(3.2.2) *for any $y \in D$, $W(y) \subseteq H(y)$ and $W(y)$ is compactly open in Z;*

(3.2.3) *for any $z \in T(Y)$, $M \in \langle W^{-}(z)\rangle$ implies that $\Gamma_M \subseteq H^{-}(z)$;*

(3.2.4) *$\overline{T(Y)} \subseteq \bigcup_{x\in X} W(s(x))$.*

Then T and H have a coincidence point.

Proof. We prove the theorem by contradiction. Assume that $T(y) \cap H(y) = \varnothing$ for any $y \in Y$. Put $K = \overline{T(Y)}$. By (3.2.1), K is a compact subset of Z. Define $F : X \to 2^Z$ by

$$F(x) = K \setminus W(s(x)) \tag{3.2}$$

for $x \in X$. Since $W(s(x))$ is compactly open, $F(x)$ is closed for each $x \in X$. The assumption that $T(y) \cap H(y) = \varnothing$ for any $y \in Y$ implies that $T(s(x)) \cap H(s(x)) = \varnothing$ for any $x \in X$, so

$$\begin{aligned} \varnothing \neq T(s(x)) &\subseteq K \setminus H(s(x)) \\ &\subseteq K \setminus W(s(x)) \\ &= F(x). \end{aligned} \tag{3.3}$$

Hence F is a nonempty and compact-valued multimap. Since

$$\begin{aligned} \bigcap_{x\in X} F(x) &= \bigcap_{x\in X} \left(K \setminus W(s(x))\right) \\ &= K \setminus \bigcup_{x\in X} W(s(x)) \\ &\subseteq K \setminus K \quad \text{by (3.2.4)} \\ &= \varnothing, \end{aligned} \tag{3.4}$$

F is not a s-KKM mapping with respect to T. Hence there is $A = \{x_1,\dots,x_n\} \in \langle X\rangle$ such that

$$T\left(\Gamma_{\{s(x_1),\dots,s(x_n)\}}\right) \nsubseteq \bigcup_{i=1}^{n} F(x_i). \tag{3.5}$$

Choose $\hat{y} \in \Gamma_{\{s(x_1),\dots,s(x_n)\}}$ and $\hat{z} \in T(\hat{y})$ such that $\hat{z} \notin \bigcup_{i=1}^{n} F(x_i)$. It follows from

$$\begin{aligned} \hat{z} &\in K \setminus \bigcup_{i=1}^{n} F(x_i) \\ &= \bigcap_{i=1}^{n} \left(K \setminus F(x_i)\right) \\ &\subseteq \bigcap_{i=1}^{n} W(s(x_i)) \\ &\subseteq \bigcap_{i=1}^{n} H(s(x_i)) \end{aligned} \tag{3.6}$$

that $s(x_i) \in W^-(\hat{z}) \subseteq H^-(\hat{z})$ for any $i \in \{1,\ldots,n\}$. Therefore by (3.2.3), $\Gamma_{\{s(x_1),\ldots,s(x_n)\}} \subseteq H^-(\hat{z})$. In particular, $\hat{y} \in H^-(\hat{z})$, and so $\hat{z} \in H(\hat{y}) \cap T(\hat{y})$, a contradiction. This completes the proof. □

Corollary 3.3. *Let (Y,D) be a convex space and Z a topological space. Suppose $H : Y \to 2^Z$ and $T \in \mathrm{KKM}(Y,Z)$ satisfy the following conditions:*

(3.3.1) *T is compact;*

(3.3.2) *for any $z \in T(Y)$, $H^-(z)$ is D-convex;*

(3.3.3) $\overline{T(Y)} \subseteq \bigcup_{y\in D} \mathrm{Int}(H(y))$.

Then T and H have a coincidence point.

Proof. Putting $X = D$, $s : X \to D$ be the identity mapping 1_D and $W : D \to 2^Z$ be defined by $W(y) = \mathrm{Int}(H(y))$ in the above theorem, the result follows immediately. □

Here we like to mention that Corollary 3.3 is an improvement for Theorem 4 of Chang and Yen [4], where except the conditions (3.3.1) $\sim$ (3.3.3), they require T be closed. For $\mathbf{U}_c^k(Y,Z)$ instead of $\mathrm{KKM}(Y,Z)$, Corollary 3.3 is due to Park [11]. We now give a concrete example showing that Corollary 3.3 extends both of [4, Theorem 4] and [11, Theorem 2] properly. Let $X = [0,1]$ and V be any convex open subset of 0 in $\mathbb{R}$. Define $T : X \multimap X$ by $T(x) = \{1\}$ for $x \in [0,1)$; and $[0,1)$ for $x = 1$, and $H : X \multimap X$ by $H(x) = (x+V) \cap X$. Then we have

(a) T belongs to $\mathrm{KKM}(X,X)$ and is compact;

(b) $H^-(y)$ is convex for each $y \in X$, and

(c) each $H(x)$ is open and $\overline{T(X)} \subseteq \bigcup_{x\in X} H(x)$.

Thus, Corollary 3.3 guarantees that $T(\hat{x}) \cap H(\hat{x}) \neq \varnothing$ for some $\hat{x} \in [0,1]$. But, Theorem 4 of Chang and Yen [4] is not applicable in this case because T is not closed. On the other hand, if $T \in \mathbf{U}_c^k(X,X)$, then there would exist $\Gamma \in \mathbf{U}_c(X,X)$ such that $\Gamma(x) \subseteq T(x)$ for each $x \in [0,1]$. Since X is a polytope, Γ must have a fixed a point which is impossible by noting that T has no fixed point. Consequently, $T \notin \mathbf{U}_c^k(X,X)$, and hence we can not apply Theorem 2 of Park [11] to conclude that T and H have a coincidence point.

Corollary 3.4. *Let X be any nonempty set, (Y,D) a convex space and Z a topological space. Suppose $s : X \to D$, $H : Y \to 2^Z$ and $T \in s\text{-}\mathrm{KKM}(X,Y,Z)$ satisfy the following conditions:*

(3.4.1) *T is compact;*

(3.4.2) *for any $z \in T(Y)$, $H^-(z)$ is D-convex;*

(3.4.3) $\overline{T(Y)} \subseteq \bigcup_{x\in X} \mathrm{Int}(H(s(x)))$.

Then T and H have a coincidence point.

Proof. In Theorem 3.2, putting $W : D \to 2^Z$ be $W(y) = \mathrm{Int}(H(y))$ for each $y \in Y$, the result follows immediately. □

Lemma 3.5 (Lassonde [9, Lemma 2]). *Let Y be a nonempty subset of a topological vector space E, $T : Y \to 2^E$ a compact and closed multimap and $i : Y \to E$ the inclusion map. Then for each closed subset B of Y, $(T - i)(B)$ is closed in E.*

Corollary 3.6. *Let X be any nonempty set and Y, C be two nonempty convex subsets of a locally convex topological vector space E. Suppose $s : X \to Y$ and $T \in s\text{-}\mathrm{KKM}(X,Y,Y+C)$ satisfy the following conditions (3.6.1), (3.6.2) and any one of (3.6.3), (3.6.3)′ and (3.6.3)″.*

(3.6.1) *T is compact and closed.*
(3.6.2) $\overline{T(Y)} \subseteq s(X) + C$.
(3.6.3) *Y is closed and C is compact.*
(3.6.3)$'$ *Y is compact and C is closed.*
(3.6.3)$''$ $C = \{0\}$.
Then there is $\hat{y} \in Y$ such $(\hat{y} + C) \cap T(\hat{y}) \neq \varnothing$.

Proof. Let V be any convex open neighborhood of $0 \in E$ and $K = \overline{T(Y)}$. Define $H : Y \to 2^{Y+C}$ to be $H(y) = (y + C + V) \cap K$ for each $y \in Y$. Each $H(y)$ is open in K and $H^-(z) = (z - C - V) \cap Y$ is convex for any $z \in K$. Moreover,

$$\begin{aligned} \bigcup_{x \in X} H(s(x)) &= \bigcup_{x \in X} \big((s(x) + C + V) \cap K\big) \\ &= \big(s(X) + C + V\big) \cap K \\ &= \overline{T(Y)} \quad \text{by (3.6.2)}. \end{aligned} \tag{3.7}$$

Therefore, it follows from Corollary 3.4 that there are $y_V \in Y$ and $z_V \in K$ such that $z_V \in T(y_V) \cap H(y_V)$. Then in view of the definition of H, $z_V - y_V \in C + V$. Up to now, we have proved the assertion.

($*$) For each convex open neighborhood V of 0 in E, $(T - i)(Y) \cap (C + V) \neq \varnothing$, where $i : Y \to E$ is the inclusion map.

Now take into account of conditions (3.6.3), (3.6.3)$'$ and (3.6.3)$''$. Suppose (3.6.3) holds. Since Y is closed, so is $(T - i)(Y)$ by Lemma 3.5, and then the assertion ($*$) in conjunction with the compactness of C and the regularity of E implies that $(T - i)(Y) \cap C \neq \varnothing$, that is, there exists a $\hat{y} \in Y$ such that $T(\hat{y}) \cap (\hat{y} + C) \neq \varnothing$. In case that (3.6.3)$'$ holds, since $(T - i)(Y)$ is compact by Lemma 2.1 and since C is closed, the conclusion follows as the previous case. Finally, assume that (3.6.3)$''$ holds. By ($*$), for every convex open neighborhood V of 0, there are y_V and z_V in Y such that $z_V \in T(y_V)$ and $z_V - y_V \in V$. Since $\overline{T(Y)}$ is compact, we may assume that $z_V \to \hat{y}$ for some $\hat{y} \in \overline{T(Y)}$. Then we also have that $y_V \to \hat{y}$. The closedness of T implies that $\hat{y} \in T(\hat{y})$. This completes the proof. □

The above corollary extends Park [11, Theorem 3], which in turn is a generalization to Lassonde [9, Theorem 1.6 and Corollary 1.18].

We now turn to investigate the fixed point problem on uniform spaces. At first we apply Theorem 3.2 to establish a useful lemma.

Lemma 3.7. *Let X be any nonempty set, $(Y, D; \Gamma)$ be a G-convex space whose topology is induced by a uniformity $\mathcal{U}$. Suppose $s : X \to D$ and $T \in s$-KKM(X, Y, Y) satisfy that*
(3.7.1) *T is compact; and*
(3.7.2) $\overline{T(Y)} \subseteq s(X)$.
If $V \in \mathcal{U}$ is symmetric and satisfies that $V[y]$ is Γ-convex for any $y \in Y$, then there is $y_V \in Y$ such that

$$V[y_V] \cap T(y_V) \neq \varnothing. \tag{3.8}$$

Proof. Define $H : Y \to 2^Y$ to be $H(y) = V[y]$ for any $y \in Y$. By symmetry of V it is easy to see that $H^-(z) = V[z]$ for any $z \in Y$, and so $H^-(z)$ is Γ-convex. Also, it follows from condition (3.6.2) that for any $z \in \overline{T(Y)}$, there is $x_0 \in s(X)$ such that $z = s(x_0)$. Then in view of $(s(x_0), s(x_0)) \in V$ we see that $z = s(x_0) \in V[s(x_0)] = H(s(x_0))$, and hence $z \in \bigcup_{x\in X} H(s(x))$, that is $\overline{T(Y)} \subseteq \bigcup_{x\in X} H(s(x))$. Finally, noting H is open-valued and putting $W : D \to 2^Y$ to be $W(y) = H(y)$ for any $y \in D$, we see that all the requirements of Theorem 3.2 are satisfied. Thus there is $y_V \in Y$ such that $H(y_V) \cap T(y_V) \neq \varnothing$, that is $V[y_V] \cap T(y_V) \neq \varnothing$. □

Definition 3.8 [14]. A G-convex space $(X, D; \Gamma)$ is said to be a locally G-convex uniform space if the topology of X is induced by a uniformity $\mathcal{U}$ which has a base $\mathcal{N}$ consisting of symmetric entourages such that for any $V \in \mathcal{N}$ and $x \in X$, $V[x]$ is Γ-convex.

Recall that the concepts of *l.c.* space and *l.c.* metric space in Horvath [7]. If $D = X$ and $\Gamma_x = \{x\}$ for any $x \in X$, then it is obvious that both of them are examples of locally G-convex uniform space.

Theorem 3.9. *Let X be any nonempty set, $(Y, D; \Gamma)$ a locally G-convex space. Suppose $s : X \to D$ and $T \in s$-KKM(X, Y, Y) satisfy that*
(3.9.1) *T is compact and closed;*
(3.9.2) $\overline{T(Y)} \subseteq s(X)$.
Then T has a fixed point.

Proof. By Lemma 3.7, for any $V \in \mathcal{N}$ there is $y_V \in Y$ such that $V[y_V] \cap T(y_V) \neq \varnothing$. Choose $z_V \in V[y_V] \cap T(y_V)$. Then $(y_V, z_V) \in V \cap \mathrm{Gr}(T)$. Since T is compact, we may assume that $\{z_V\}_{V\in\mathcal{N}}$ converges to z_0. For any $W \in \mathcal{N}$, choose $U \in \mathcal{N}$ such that $U \circ U \subseteq W$. Since $\{z_V\}_{V\in\mathcal{N}}$ converges to z_0, there is $V_0 \in \mathcal{N}$ such that $V_0 \subseteq U$ and

$$z_V \in U[z_0], \quad \forall V \in \mathcal{N} \text{ with } V \subseteq V_0, \tag{3.9}$$

that is,

$$(z_V, z_0) \in U, \quad \forall V \in \mathcal{N} \text{ with } V \subseteq V_0. \tag{3.10}$$

Thus, for $V \in \mathcal{N}$ with $V \subseteq V_0$, it follows from

$$(y_V, z_V) \in V \subseteq U, \qquad (z_V, z_0) \in U \tag{3.11}$$

that $(y_V, z_0) \in U \circ U \subseteq W$. Hence $y_V \in W[z_0]$. This shows that $\{y_V\}_{V\in\mathcal{N}}$ converges to z_0. Since T is closed, we conclude that $z_0 \in T(z_0)$, completing the proof. □

For a topological space X and locally G-convex uniform space (Y, Γ), define

$$\begin{aligned} T \in \mathcal{K}(X,Y) \Longleftrightarrow T : X \longrightarrow Y &\text{ is a Kakutani map, that is,} \\ &T \text{ is u.s.c. with nonempty compact } \Gamma\text{-convex values.} \end{aligned} \tag{3.12}$$

$\mathcal{K}_c(X,Y)$ denotes the set of finite composites of multimaps in $\mathcal{K}$ of which ranges are contained in locally G-convex uniform spaces (Y_i, Γ_i) $(i = 0, \dots, n)$ for some n.

Lemma 3.10 (Watson [14]). *Let* (X,Γ) *be a compact locally* G*-convex uniform space. Then any u.s.c.* $T : X \multimap X$ *with closed* Γ*-convex values has a fixed point.*

By the above lemma, we see that, in the setting of locally G-convex uniform spaces, the class $\mathcal{K}$ is an example of the Park's class **U**. Therefore, for any locally G-convex uniform space (X,Γ), $\mathcal{K}_c(X,X) \subseteq \mathrm{KKM}(X,X)$, and so we have the following theorem.

Theorem 3.11. *Suppose* (X,Γ) *is a locally* G*-convex uniform space. If* $T \in \mathcal{K}_c(X,X)$ *is compact, then it has a fixed point.*

Proof. Since X is regular by Kelley [8, Corollary 6.17 on page 188] and $T \in \mathcal{K}_c(X,X)$, it is u.s.c. and compact-valued, and so it is closed. Now due to that $\mathcal{K}_c(X,X) \subseteq \mathrm{KKM}(X,X)$, we have $T \in \mathrm{KKM}(X,X)$. Since T is compact and closed, it follows from Theorem 3.9 that T has a fixed point. $\square$

Since any metric space is regular, we infer that for any *l.c.* metric space (X,d) satisfying that $\Gamma_x = \{x\}$, if $T \in \mathcal{K}_c(X,X)$ is compact, then T has a fixed point. This generalizes the famous Fan-Glicksberg fixed point theorem [5].

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