NOTE ON SOME RESULTS FOR ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS AND ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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We discuss convergence theorems of modified Ishikawa and Mann iterative sequences with errors for asymptotically pseudocontractive and asymptotically nonexpansive mappings in Banach spaces, and the boundedness of the domain and range can be dropped, generalizing theorems of Chang.

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1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the topological dual space of E, $\langle \cdot, \cdot \rangle$ is the dual between E and E^*, $D(T)$ and $F(T)$ denote the domain of T and the set of all fixed points of T, respectively, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \ \|f\| = \|x\| \}, \quad x \in E. \tag{1.1}$$

Definition 1.1. Let $T : D(T) \subset E \to E$ be a mapping.

(1) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $(0,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\| T^n x - T^n y \right\| \leq k_n \|x - y\| \tag{1.2}$$

for all $x, y \in D(T)$ and $n = 1, 2, \ldots$.

(2) T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}$ in $(0,\infty)$ with $\lim_{n\to\infty} k_n = 1$, and for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\left\langle T^n x - T^n y, \ j(x-y) \right\rangle \leq k_n \|x - y\|^2 \tag{1.3}$$

for all $n = 1, 2, \ldots$.

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(3) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that

$$||T^n x - T^n y|| \leq L\|x - y\| \tag{1.4}$$

for all $x, y \in D(T)$ and $n = 1, 2, \ldots$.

The following proposition follows from Definition 1.1 immediately.

Proposition 1.2. (1) *If $T : D(T) \subset E \to E$ is nonexpansive, then T is an asymptocially nonexpansive mapping with a constant sequence $\{1\}$.*

(2) *If $T : D(T) \subset E \to E$ is asymptotically nonexpansive, then T is a uniformly L-Lipschitzian, where $L = \sup_{n\geq 1}\{k_n\}$ and asymptotically pseudocontractive mapping.*

Definition 1.3. (1) Let $T : D(T) \subset E \to E$ be a mapping, let $D(T)$ be a nonempty convex subset of E, let $x_0 \in D(T)$ be a given point, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be four sequences in $[0,1]$. Then the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n y_n + \gamma_n u_n, \\ y_n &= (1 - \beta_n - \delta_n)x_n + \beta_n T^n x_n + \delta_n v_n, \end{aligned} \qquad \forall n \geq 0, \tag{1.5}$$

is called the modified Ishikawa iterative sequence with errors of T, where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in $D(T)$.

(2) In (1.5) if $\beta_n = 0$ and $\delta_n = 0$, $n = 0, 1, 2, \ldots$, then $y_n = x_n$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n x_n + \gamma_n u_n, \quad \forall n \geq 0, \tag{1.6}$$

is called the modified Mann iterative sequence with errors of T.

In this paper, we discuss convergence theorems of modified Ishikawa and Mann iterative sequences with errors for asymptotically pseudocontractive and asymptotically nonexpansive mappings in Banach spaces, and the boundedness of the domain and range can be dropped, generalizing theorems of Chang [1].

Lemma 1.4 [4]. *Let $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$A_{n+1} \leq (1 + B_n)A_n + C_n, \quad \forall n \geq 0. \tag{1.7}$$

If $\sum_{n=0}^{\infty} B_n < +\infty$ and $\sum_{n=0}^{\infty} C_n < +\infty$, then $\lim_{n\to\infty} A_n$ exists.

2. Main results

Lemma 2.1. *Let E be an arbitrary real Banach space, let D be a nonempty closed convex subset of E, let $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\sum_{n=0}^{\infty}(k_n - 1) < +\infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and*

$\{\delta_n\}$ *be four sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \le 1$, $\beta_n + \delta_n \le 1$;

(ii) $\beta_n \le \alpha_n$, $\delta_n \le \gamma_n$, *for all* $n \ge 0$;

(iii) $\sum_{n=0}^{\infty} \gamma_n < +\infty$;

(iv) $\sum_{n=0}^{\infty} \alpha_n^2 < +\infty$.

Let $x_0 \in D$ be any given point and let $\{x_n\}$ and $\{y_n\}$ be the modified Ishikawa iterative sequence with errors defined by (1.5). If $F(T) \ne \emptyset$, then for any given $q \in F(T)$, $\lim_{n\to\infty} \|x_n - q\|$ exists.

Proof. Set $M = \max\{\sup_{n\ge 0} \|u_n - q\|, \sup_{n\ge 0} \|v_n - q\|\}$.

Since T is asymptotically pseudocontractive, for all $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y,\ j(x-y)\rangle \le k_n \|x-y\|^2. \tag{2.1}$$

Then from inequality (2.1), we obtain

$$\langle (k_n I - T^n)x - (k_n I - T^n)y,\ j(x-y)\rangle = k_n\|x-y\|^2 - \langle T^n x - T^n y,\ j(x-y)\rangle \ge 0, \tag{2.2}$$

and it follows from Kato [2] that

$$\|x-y\| \le \big\|x - y + \lambda[(k_n I - T^n)x - (k_n I - T^n)y]\big\|, \quad \forall x, y \in D,\ \lambda > 0. \tag{2.3}$$

Set $a_n := \alpha_n + \gamma_n$. Then from the recursive formula (1.5), we have $x_{n+1} = (1-a_n)x_n + a_n T^n y_n - \gamma_n(T^n y_n - u_n)$. It follows that

$$\begin{aligned} x_n = {} & (1+a_n)x_{n+1} + a_n(k_n I - T^n)x_{n+1} - a_n k_n x_n + a_n^2(1+k_n)(x_n - T^n y_n) \\ & + a_n(T^n x_{n+1} - T^n y_n) + \gamma_n[1 + a_n(1+k_n)](T^n y_n - u_n). \end{aligned} \tag{2.4}$$

Observe that

$$q = (1+a_n)q + a_n(k_n I - T^n)q - a_n k_n q. \tag{2.5}$$

So that

$$\begin{aligned} x_n - q = {} & (1+a_n)(x_{n+1} - q) + a_n[(k_n I - T^n)x_{n+1} - (k_n I - T^n)q] - a_n k_n (x_n - q) \\ & + a_n^2(1+k_n)(x_n - T^n y_n) + a_n(T^n x_{n+1} - T^n y_n) + \gamma_n[1 + a_n(1+k_n)](T^n y_n - u_n). \end{aligned} \tag{2.6}$$

Hence

$$
\begin{aligned}
\|x_n-q\| \geq{} & (1+a_n)\left\|x_{n+1}-q+\frac{a_n}{1+a_n}\left[(k_nI-T^n)x_{n+1}-(k_nI-T^n)q\right]\right\|-a_nk_n\|x_n-q\| \\
& -a_n^2(1+k_n)\|x_n-T^ny_n\|-a_n\|T^nx_{n+1}-T^ny_n\| \\
& -\gamma_n\left[1+a_n(1+k_n)\right]\|T^ny_n-u_n\| \\
\geq{} & (1+a_n)\|x_{n+1}-q\|-a_nk_n\|x_n-q\|-a_n^2(1+k_n)\|x_n-T^ny_n\| \\
& -a_n\|T^nx_{n+1}-T^ny_n\|-\gamma_n\left[1+a_n(1+k_n)\right]\|T^ny_n-u_n\|.
\end{aligned}
\tag{2.7}
$$

So

$$
\begin{aligned}
\|x_{n+1}-q\| \leq{} & \left[1+\frac{a_n}{1+a_n}(k_n-1)\right]\|x_n-q\|+a_n^2(1+k_n)\|x_n-T^ny_n\| \\
& +a_n\|T^nx_{n+1}-T^ny_n\|+\gamma_n\left[1+a_n(1+k_n)\right]\|T^ny_n-u_n\|.
\end{aligned}
\tag{2.8}
$$

Furthermore, set $b_n := \beta_n+\delta_n$. Then from recursive formula (1.5), we have $y_n=(1-b_n)x_n+b_nT^nx_n-\delta_n(T^nx_n-v_n)$. By condition (ii), we have $b_n \leq a_n$, for all $n \geq 0$. We make the following estimates:

$$
\begin{aligned}
\|y_n-q\| &= \|(1-b_n)(x_n-q)+b_n(T^nx_n-q)-\delta_n(T^nx_n-v_n)\| \\
&\leq \left[1+b_n(L-1)\right]\|x_n-q\|+\delta_n\|T^nx_n-v_n\| \\
&\leq L\|x_n-q\|+\delta_nL\|x_n-q\|+\delta_nM \\
&= L(1+\delta_n)\|x_n-q\|+\delta_nM,
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
\|x_n-T^ny_n\| &\leq \|x_n-q\|+L\|y_n-q\| \\
&\leq \|x_n-q\|+L^2(1+\delta_n)\|x_n-q\|+L\delta_nM \\
&\leq \left[1+L^2(1+\delta_n)\right]\|x_n-q\|+L\delta_nM, \\
\|T^ny_n-u_n\| &\leq L\|y_n-q\|+\|u_n-q\| \\
&\leq L^2(1+\delta_n)\|x_n-q\|+(1+L\delta_n)M, \\
\|T^nx_{n+1}-T^ny_n\| &\leq L\|x_{n+1}-y_n\| \\
&= L\|x_n-y_n+a_n(T^ny_n-x_n)-\gamma_n(T^ny_n-u_n)\| \\
&\leq L\|x_n-y_n\|+La_n\|T^ny_n-x_n\|+L\gamma_n\|T^ny_n-u_n\| \\
&= L\|b_n(x_n-T^nx_n)+\delta_n(T^nx_n-v_n)\| \\
&\quad +La_n\|T^ny_n-x_n\|+L\gamma_n\|T^ny_n-u_n\|
\end{aligned}
$$

$$\begin{aligned}
&\leq Lb_n\|x_n - T^n x_n\| + L\delta_n\|T^n x_n - v_n\| \\
&\quad + La_n\|T^n y_n - x_n\| + L\gamma_n\|T^n y_n - u_n\| \\
&\leq La_n(1+L)\|x_n - q\| + L^2\delta_n\|x_n - q\| \\
&\quad + L\delta_n M + La_n[1+L^2(1+\delta_n)]\|x_n - q\| \\
&\quad + L^2 a_n\delta_n M + L^3\gamma_n(1+\delta_n)\|x_n - q\| + L\gamma_n(1+L\delta_n)M.
\end{aligned} \tag{2.10}$$

Using (2.9) and (2.10) in (2.8), we obtain the following estimation:

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \left[1 + \frac{a_n}{1+a_n}(k_n - 1)\right]\|x_n - q\| + a_n^2(1+k_n)[1+L^2(1+\delta_n)]\|x_n - q\| \\
&\quad + a_n^2(1+k_n)L\delta_n M + La_n^2(1+L)\|x_n - q\| + L^2 a_n\delta_n\|x_n - q\| + La_n\delta_n M \\
&\quad + La_n^2[1+L^2(1+\delta_n)]\|x_n - q\| + L^2 a_n^2\delta_n M + L^3 a_n\gamma_n(1+\delta_n)\|x_n - q\| \\
&\quad + La_n\gamma_n(1+L\delta_n)M + \gamma_n[1+a_n(1+k_n)]L^2(1+\delta_n)\|x_n - q\| \\
&\quad + \gamma_n[1+a_n(1+k_n)](1+L\delta_n)M \\
&= \Big\{1 + \frac{a_n}{1+a_n}(k_n - 1) + a_n^2(1+k_n)[1+L^2(1+\delta_n)] + La_n^2(1+L) + L^2 a_n\delta_n \\
&\qquad + La_n^2[1+L^2(1+\delta_n)] + L^3 a_n\gamma_n(1+\delta_n) \\
&\qquad + \gamma_n[1+a_n(1+k_n)]L^2(1+\delta_n)\Big\}\|x_n - q\| \\
&\quad + \Big\{a_n^2(1+k_n)L\delta_n M + La_n\delta_n M + L^2 a_n^2\delta_n M \\
&\qquad + La_n\gamma_n(1+L\delta_n)M + \gamma_n[1+a_n(1+k_n)](1+L\delta_n)M\Big\}.
\end{aligned} \tag{2.11}$$

Set

$$\begin{aligned}
A_n &:= \|x_n - q\|, \\
B_n &:= \frac{a_n}{1+a_n}(k_n - 1) + a_n^2(1+k_n)[1+L^2(1+\delta_n)] + La_n^2(1+L) + L^2 a_n\delta_n \\
&\quad + La_n^2[1+L^2(1+\delta_n)] + L^3 a_n\gamma_n(1+\delta_n) + \gamma_n[1+a_n(1+k_n)]L^2(1+\delta_n), \\
C_n &:= a_n^2(1+k_n)L\delta_n M + La_n\delta_n M + L^2 a_n^2\delta_n M + La_n\gamma_n(1+L\delta_n)M \\
&\quad + \gamma_n[1+a_n(1+k_n)](1+L\delta_n)M,
\end{aligned} \tag{2.12}$$

then inequality (2.11) is equal to

$$A_{n+1} \le (1+B_n)A_n + C_n. \tag{2.13}$$

By conditions (ii), (iii), and (iv), we know $\sum_{n=0}^{\infty} B_n < +\infty$, $\sum_{n=0}^{\infty} C_n < +\infty$. By Lemma 1.4, we know $\lim_{n\to\infty} \|x_n - q\|$ exists. □

Theorem 2.2. *Let E be a real uniformly smooth Banach space, let D be a nonempty closed convex subset of E, and let $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $\sum_{n=0}^{\infty}(k_n - 1) < +\infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be four sequences in $[0,1]$ satisfying the conditions (i)–(iv) in Lemma 2.1.*

Let $x_0 \in D$ be any given point and let $\{x_n\}$, $\{y_n\}$ be the modified Ishikawa iterative sequence with errors defined by (1.5).

(1) If $\{x_n\}$ converges strongly to a fixed point q of T in D, then there exists a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ such that

$$\langle T^n y_n - q, J(y_n - q)\rangle \le k_n \|y_n - q\|^2 - \phi(\|y_n - q\|), \tag{$*$}$$

for all $n \ge 0$.

(2) Conversely, if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ satisfying condition $(*)$, then $x_n \to q \in F(T)$.

Proof. Since E is uniformly smooth, the normalized duality mapping $J : E \to E^*$ is single-valued and uniformly continous on any bounded subset of E.

(1) Let $x_n \to q \in F(T)$. From conditions (ii)–(iv) in Lemma 2.1, we have $\beta_n \to 0$, $\delta_n \to 0$. Besides noticing $\|y_n - q\| \le L(1+\delta_n)\|x_n - q\| + \delta_n M$ in (2.9) of Lemma 2.1, we have

$$y_n \longrightarrow q \quad (n \longrightarrow \infty). \tag{2.14}$$

The rest of the proof is the same as Chang's [1, Theorem 2.1].

(2) By Lemma 2.1, we know $\lim_{n\to\infty} \|x_n - q\|$ exists. So $\{x_n\}$ is bounded. And by the proof of Lemma 2.1, we can also get $\{T^n y_n - y_n\}$, $\{x_n - T^n x_n\}$, $\{x_n - v_n\}$, and $\{u_n - y_n\}$; all are bounded. So the rest of the proof is the same as Chang's [1, Theorem 2.1]. □

Remark 2.3. (1) Theorem 2.2 removes the restriction on D which is bounded in Chang [1, Theorem 2.1].

(2) Respectively, we can get Chang [1, Theorems 2.2, 2.3, and 2.4], but without the restriction on D which is bounded.

(3) In Osilike and Akuchu [3], they discussed the common fixed points of a family of asymptotically pseudocontractive maps. Our paper is different from it.

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