APPROXIMATING ZERO POINTS OF ACCRETIVE OPERATORS WITH COMPACT DOMAINS IN GENERAL BANACH SPACES

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We prove strong convergence theorems of Mann's type and Halpern's type for resolvents of accretive operators with compact domains and apply these results to find fixed points of nonexpansive mappings in Banach spaces.

1. Introduction

Let E be a real Banach space, let C be a closed convex subset of E, let T be a nonexpansive mapping of C into itself, that is, $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$, and let $A \subset E \times E$ be an accretive operator. For $r > 0$, we denote by J_r the resolvent of A, that is, $J_r = (I + rA)^{-1}$. The problem of finding a solution $u \in E$ such that $0 \in Au$ has been investigated by many authors; for example, see [3, 4, 7, 16, 26]. We know the proximal point algorithm based on a notion of resolvents of accretive operators. This algorithm generates a sequence $\{x_n\}$ in E such that $x_1 = x \in E$ and

$$x_{n+1} = J_{r_n} x_n \quad \text{for } n = 1, 2, \ldots, \tag{1.1}$$

where $\{r_n\}$ is a sequence in $(0, \infty)$. Rockafellar [18] studied the weak convergence of the sequence generated by (1.1) in a Hilbert space; see also the original works of Martinet [12, 13].

On the other hand, Mann [11] introduced the following iterative scheme for finding a fixed point of a nonexpansive mapping T in a Banach space: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \quad \text{for } n = 1, 2, \ldots, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, and studied the weak convergence of the sequence generated by (1.2). Reich [17] also studied the following iterative scheme for finding a fixed point of a nonexpansive mapping $T : x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n \quad \text{for } n = 1, 2, \ldots, \tag{1.3}$$

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where $\{\alpha_n\}$ is a sequence in $[0,1]$; see the original work of Halpern [6]. Wittmann [27] showed that the sequence generated by (1.3) in a Hilbert space converges strongly to the point of $F(T)$, the set of fixed points of T, which is the nearest to x if $\{\alpha_n\}$ satisfies $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$. Since then, many authors have studied the iterative schemes of Mann's type and Halpern's type for nonexpansive mappings and families of various mappings; for example, see [1, 2, 19, 20, 21, 22, 23, 24, 14, 15].

Motivated by two iterative schemes of Mann's type and Halpern's type, Kamimura and Takahashi [8, 9] introduced the following iterative schemes for finding zero points of m-accretive operators in a uniformly convex Banach space: $x_1 = x \in E$ and

$$\begin{aligned} x_{n+1} &= \alpha_n x + (1-\alpha_n)J_{r_n}x_n \quad \text{for } n = 1,2,\dots, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n \quad \text{for } n = 1,2,\dots, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{r_n\}$ is a sequence in $(0,\infty)$. They studied the strong and weak convergence of the sequences generated by (1.4). Such iterative schemes for accretive operators with compact domains in a strictly convex Banach space have also been studied by Kohsaka and Takahashi [10].

In this paper, we first deal with the strong convergence of resolvents of accretive operators defined in compact sets of smooth Banach spaces. Next, we prove strong convergence theorems of Mann's type and Halpern's type for resolvents of accretive operators with compact domains. We apply these results to find fixed points of nonexpansive mappings with compact domains in Banach spaces.

2. Preliminaries

Through this paper, we denote by $\mathbb{N}$ the set of positive integers. We also denote by E a real Banach space with topological dual E^* and by J the duality mapping of E, that is, a multivalued mapping J of E into E^* such that for each $x \in E$,

$$J(x) = \{f \in E^* : f(x) = \|x\|^2 = \|f\|^2\}. \tag{2.1}$$

A Banach space E is said to be *smooth* if the duality mapping J of E is single-valued. We know that if E is smooth, then J is norm to weak-star continuous. Let $S(E)$ be the unit sphere of E, that is, $S(E) = \{x \in E : \|x\| = 1\}$. Then, the norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in S(E)$, the limit

$$\lim_{\lambda\to 0}\frac{\|x+\lambda y\| - \|x\|}{\lambda} \tag{2.2}$$

exists uniformly in $x \in S(E)$. We know that if E has a uniformly Gâteaux differentiable norm, then E is smooth. We also know that if E has a uniformly Gâteaux differentiable norm, then the duality mapping J of E is norm to weak-star uniformly continuous on each bounded subsets of E. For more details, see [25].

Let D be a subset of C and let P be a retraction of C onto D, that is, $Px = x$ for each $x \in D$. Then P is said to be *sunny* [16] if for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$,

$$P\big(Px + t(x - Px)\big) = Px. \tag{2.3}$$

A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction P of C onto D. We know that if E is smooth and P is a retraction of C onto D, then P is sunny and nonexpansive if and only if for each $x \in C$ and $z \in D$,

$$\langle x - Px, J(z - Px)\rangle \leq 0. \tag{2.4}$$

For more details, see [25].

Let $A \subset E \times E$ be a multivalued operator. We denote by $D(A)$ and $A^{-1}0$ the effective domain of A, that is, $D(A) = \{x \in E : Ax \neq \emptyset\}$ and the set of zeros of A, that is, $A^{-1}0 = \{x \in E : 0 \in Ax\}$, respectively. An operator A is said to be *accretive* if for each $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j\rangle \geq 0. \tag{2.5}$$

Such an operator was first studied by Kato and Browder, independently. We know that for each $(x_1, y_1), (x_2, y_2) \in A$ and $r > 0$,

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|. \tag{2.6}$$

Let C be a closed convex subset of E such that $C \subset \bigcap_{r>0} R(I + rA)$, where I denotes the identity mapping of E and $R(I + rA)$ is the range of $I + rA$, that is, $R(I + rA) = \bigcup\{(I + rA)x : x \in D(A)\}$. Then, for each $r > 0$, we define a mapping J_r on C by $J_r = (I + rA)^{-1}$. Such a mapping J_r is called the *resolvent* of A. We know that the resolvent J_r of A is single-valued. For each $r > 0$, we define the Yosida approximation A_r of A by $A_r = r^{-1}(I - J_r)$. We know that for each $x \in C$, $(J_r x, A_r x) \in A$. We also know that for each $x \in C \cap D(A)$, $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for each $r > 0$ and A is also said to be *maximal* if the graph of A is not properly contained in the graph of any other accretive operator. We know from [5, page 181] that if A is an m-accretive operator, then A is maximal.

We need the following theorem [14], which is crucial in the proofs of main theorems.

Theorem 2.1. *Let C be a compact convex subset of a smooth Banach space E, let S be a commutative semigroup with identity, let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C, and let $F(\mathcal{S})$ be the set of common fixed points of $\mathcal{S}$. Then $F(\mathcal{S})$ is a sunny nonexpansive retract of C, and a sunny nonexpansive retraction of C onto $F(\mathcal{S})$ is unique. In particular, if T is a nonexpansive mapping of C into itself, then $F(T)$ is a sunny nonexpansive retract of C and a sunny nonexpansive retraction of C onto $F(T)$ is unique.*

3. Main results

Let E be a Banach space and let $A \subset E \times E$ be an accretive operator. In this section, we study the existence of a sunny nonexpansive retraction onto $A^{-1}0$ and the convergence of resolvents of A.

Theorem 3.1. *Let C be a compact convex subset of a smooth Banach space E and let $A \subset E \times E$ be an accretive operator such that $D(A) \subset C \subset \bigcap_{r>0} R(I+rA)$. Then the set $A^{-1}0$ is a nonempty sunny nonexpansive retract of C and a sunny nonexpansive retraction P of C onto $A^{-1}0$ is unique. In this case, for each $x \in C$, $\lim_{t\to\infty} J_t x = Px$.*

Proof. Since $C \subset R(I+rA)$ for each $r > 0$, the resolvent J_r of A is well defined on C. We know that J_r is a nonexpansive mapping of C into itself and $A^{-1}0 = F(J_r)$, where $F(J_r)$ denotes the set of fixed points of J_r. Then, by Theorem 2.1, $A^{-1}0$ is a sunny nonexpansive retract of C and a sunny nonexpansive retraction P of C onto $A^{-1}0$ is unique.

Next, we will show that for each $x \in C$, $\lim_{t\to\infty} J_t x$ exists and $\lim_{t\to\infty} J_t x = Px$. Let $x \in C$ be fixed. Since C is compact, there exist a sequence $\{t_n\}$ of positive real numbers and $z \in C$ such that $\lim_{n\to\infty} t_n = \infty$ and $\{J_{t_n}x\}$ converges strongly to z. Then, z is contained in $A^{-1}0$. Indeed, we have, for each $r > 0$,

$$\begin{aligned}
\left\|J_r J_{t_n}x - J_{t_n}x\right\| &= \left\|(J_r - I)J_{t_n}x\right\| = r\left\|A_r J_{t_n}x\right\| \\
&\le r \inf\{\|y\| : y \in AJ_{t_n}x\} \\
&\le r\left\|A_{t_n}x\right\| = r\left\|\frac{x - J_{t_n}x}{t_n}\right\| \\
&\le \frac{r}{t_n}\left(\|x\| + \left\|J_{t_n}x\right\|\right)
\end{aligned} \tag{3.1}$$

and hence $\lim_{n\to\infty} \|J_r J_{t_n}x - J_{t_n}x\| = 0$. Then, from

$$\begin{aligned}
\left\|J_r z - z\right\| &\le \left\|J_r z - J_r J_{t_n}x\right\| + \left\|J_r J_{t_n}x - J_{t_n}x\right\| + \left\|J_{t_n}x - z\right\| \\
&\le 2\left\|J_{t_n}x - z\right\| + \left\|J_r J_{t_n}x - J_{t_n}x\right\|,
\end{aligned} \tag{3.2}$$

we have $J_r z = z$. This implies that $z \in F(J_r) = A^{-1}0$.

Let $\{J_{t_n}x\}$ and $\{J_{s_n}x\}$ be subsequences of $\{J_t x\}$ such that $\{J_{t_n}x\}$ and $\{J_{s_n}x\}$ converge strongly to y and z as $t_n \to \infty$ and $s_n \to \infty$, respectively. From $z \in A^{-1}0$, we have

$$\begin{aligned}
0 &\le \left\langle A_{t_n}x - 0, J\left(J_{t_n}x - z\right)\right\rangle \\
&= \frac{1}{t_n}\left\langle \left(I - J_{t_n}\right)x, J\left(J_{t_n}x - z\right)\right\rangle
\end{aligned} \tag{3.3}$$

and hence $\langle J_{t_n}x - x, J(J_{t_n}x - z)\rangle \le 0$. Thus, we have $\langle y - x, J(y - z)\rangle \le 0$. Similarily, we have $\langle z - x, J(z - y)\rangle \le 0$ and hence $y = z \in A^{-1}0$.

Let y be the limit $\lim_{t\to\infty} J_t x$. By a similar argument, we have

$$\left\langle y - x, J(y - Px)\right\rangle \le 0. \tag{3.4}$$

Thus, since P is a sunny nonexpansive retraction of C onto $A^{-1}0$, we have

$$\begin{aligned}
\|y - Px\|^2 &= \left\langle y - Px, J(y - Px)\right\rangle \\
&= \left\langle y - x, J(y - Px)\right\rangle + \left\langle x - Px, J(y - Px)\right\rangle \\
&\le \left\langle y - x, J(y - Px)\right\rangle \le 0.
\end{aligned} \tag{3.5}$$

This implies that $y = Px$. This completes the proof. □

Next, we prove a strong convergence theorem of Mann's type for resolvents of an m-accretive operator in a Banach space.

THEOREM 3.2. *Let C be a compact convex subset of a smooth Banach space E and let $A \subset E \times E$ be an m-accretive operator such that $D(A) \subset C$. Let $x_1 = x \in C$ and define an iterative sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n \quad \text{for } n = 1, 2, \dots, \tag{3.6}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

Proof. Let $u \in A^{-1}0$. Since for each $n \in \mathbb{N}$,

$$\begin{aligned} ||x_{n+1} - u|| &\le \alpha_n ||x_n - u|| + (1-\alpha_n) ||J_{r_n} x_n - u|| \\ &\le \alpha_n ||x_n - u|| + (1-\alpha_n) ||x_n - u|| \\ &= ||x_n - u||, \end{aligned} \tag{3.7}$$

the limit $\lim_{n\to\infty} \|x_n - u\|$ exists.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $v \in C$. Since for each $n \in \mathbb{N}$,

$$||x_{n+1} - J_{r_n} x_n|| = \alpha_n ||x_n - J_{r_n} x_n|| \tag{3.8}$$

and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty} ||x_{n+1} - J_{r_n} x_n|| = 0. \tag{3.9}$$

Then, $J_{r_{n_k-1}} x_{n_k-1}$ converges strongly to $v \in C$. Since A is accretive, we have, for each $(y,z) \in A$ and $n \in \mathbb{N}$,

$$\langle z - A_{r_n} x_n, J(y - J_{r_n} x_n) \rangle \ge 0. \tag{3.10}$$

We also have

$$\lim_{n\to\infty} ||A_{r_n} x_n|| = \lim_{n\to\infty} r_n^{-1} ||x_n - J_{r_n} x_n|| = 0. \tag{3.11}$$

Thus, we have, for each $(y,z) \in A$,

$$\langle z, J(y - v) \rangle \ge 0. \tag{3.12}$$

We know that an m-accretive operator A is maximal. For the sake of completeness, we will give the proof. Let $B \subset E \times E$ be an accretive operator such that $A \subset B$ and let $(x,u) \in B$. Since A is m-accretive, there exists $y \in D(A)$ such that $x + u \in (I + A)y$. Choose $v \in Ay$ such that $x + u = y + v$. Since B is accretive, we have

$$\|x - y\| \le \|x - y + u - v\| = 0 \tag{3.13}$$

and hence $x = y \in D(A)$ and $u = v \in R(A)$. This implies that $(x,u) \in A$. So, A is maximal.

From (3.12) and the maximality of A, we have $v \in A^{-1}0$. Thus, we have

$$\lim_{n\to\infty} \|x_n - v\| = \lim_{k\to\infty} \|x_{n_k} - v\| = 0. \tag{3.14}$$

This completes the proof. □

The following is a strong convergence theorem of Halpern's type for resolvents of an accretive operator in a Banach space.

Theorem 3.3. *Let C be a compact convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let $A \subset E \times E$ be an accretive operator such that $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $x_1 = x \in C$ and define an iterative sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n \quad \text{for } n = 1, 2, \ldots, \tag{3.15}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} r_n = \infty. \tag{3.16}$$

Then $\{x_n\}$ converges strongly to Px, where P denotes a unique sunny nonexpansive retraction of C onto $A^{-1}0$.

Proof. We know from Theorem 3.1 that there exists a unique sunny nonexpansive retraction P of C onto $A^{-1}0$. For $x_1 = x \in C$, we define $\{x_n\}$ by (3.15). First, we will show that

$$\limsup_{n\to\infty} \langle x - Px, J(J_{r_n} x_n - Px) \rangle \le 0. \tag{3.17}$$

Let $\epsilon > 0$ and let $z_t = J_t x$ for each $t > 0$. Since A is accretive and $t^{-1}(x - z_t) \in Az_t$, we have

$$\langle A_{r_n} x_n - t^{-1}(x - z_t), J(J_{r_n} x_n - z_t) \rangle \ge 0 \tag{3.18}$$

and hence,

$$\langle x - z_t, J(J_{r_n} x_n - z_t) \rangle \le t \langle A_{r_n} x_n, J(J_{r_n} x_n - z_t) \rangle. \tag{3.19}$$

Then, from $\lim_{n\to\infty} A_{r_n} x_n = 0$, we have

$$\limsup_{n\to\infty} \langle x - z_t, J(J_{r_n} x_n - z_t) \rangle \le 0 \tag{3.20}$$

for each $t > 0$. From Theorem 3.1, we have $\lim_{n\to\infty} z_t = Px$. Since the norm of E is uniformly Gâteaux differentiable, there exists $t_0 > 0$ such that for each $t > t_0$ and $n \in \mathbb{N}$,

$$\begin{aligned} |\langle Px - z_t, J(J_{r_n} x_n - z_t) \rangle| &\le \frac{\epsilon}{2}, \\ |\langle x - Px, J(J_{r_n} x_n - z_t) - J(J_{r_n} x_n - Px) \rangle| &\le \frac{\epsilon}{2}. \end{aligned} \tag{3.21}$$

Thus, we have, for each $t > t_0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\left| \langle x - z_t, J(J_{r_n}x_n - z_t) \rangle - \langle x - Px, J(J_{r_n}x_n - Px) \rangle \right| \\
&\quad \le \left| \langle x - z_t, J(J_{r_n}x_n - z_t) \rangle - \langle x - Px, J(J_{r_n}x_n - z_t) \rangle \right| \\
&\qquad + \left| \langle x - Px, J(J_{r_n}x_n - z_t) \rangle - \langle x - Px, J(J_{r_n}x_n - Px) \rangle \right| \\
&\quad = \left| \langle Px - z_t, J(J_{r_n}x_n - z_t) \rangle \right| \\
&\qquad + \left| \langle x - Px, J(J_{r_n}x_n - z_t) - J(J_{r_n}x_n - Px) \rangle \right| \\
&\quad \le \epsilon.
\end{aligned} \tag{3.22}$$

This implies that

$$\limsup_{n\to\infty} \langle x - Px, J(J_{r_n}x_n - Px) \rangle \le \limsup_{n\to\infty} \langle x - z_t, J(J_{r_n}x_n - z_t) \rangle + \epsilon \le \epsilon. \tag{3.23}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} \langle x - Px, J(J_{r_n}x_n - Px) \rangle \le 0. \tag{3.24}$$

From $x_{n+1} - J_{r_n}x_n = \alpha_n(x - J_{r_n}x_n)$ and $\lim_{n\to\infty} \alpha_n = 0$, we have $x_{n+1} - J_{r_n}x_n \to 0$. Since the norm of E is uniformly Gâteaux differentiable, we also have

$$\limsup_{n\to\infty} \langle x - Px, J(x_{n+1} - Px) \rangle \le 0. \tag{3.25}$$

From (3.15) and [25, page 99], we have, for each $n \in \mathbb{N}$,

$$(1-\alpha_n)^2 \|J_{r_n}x_n - Px\|^2 - \|x_{n+1} - Px\|^2 \ge -2\alpha_n \langle x - Px, J(x_{n+1} - Px) \rangle. \tag{3.26}$$

Hence, we have

$$\|x_{n+1} - Px\|^2 \le (1-\alpha_n)\|J_{r_n}x_n - Px\|^2 + 2\alpha_n \langle x - Px, J(x_{n+1} - Px) \rangle. \tag{3.27}$$

Let $\epsilon > 0$. Then, there exists $m \in \mathbb{N}$ such that

$$\langle x - Px, J(x_n - Px) \rangle \le \frac{\epsilon}{2} \tag{3.28}$$

for each $n \ge m$. We have, for each $n \ge m$,

$$\begin{aligned}
\|x_{n+1} - Px\|^2 &\le (1-\alpha_n)\|x_n - Px\|^2 + \epsilon(1-(1-\alpha_n)) \\
&\le (1-\alpha_n)\Big((1-\alpha_{n-1})\|x_{n-1} - Px\|^2 + \epsilon(1-(1-\alpha_{n-1}))\Big) \\
&\quad + \epsilon(1-(1-\alpha_n)) \\
&\le (1-\alpha_n)(1-\alpha_{n-1})\|x_{n-1} - Px\|^2 \\
&\quad + \epsilon(1-(1-\alpha_n)(1-\alpha_{n-1})) \\
&\le \prod_{k=m}^{n}(1-\alpha_k)\|x_m - Px\|^2 + \epsilon\left(1 - \prod_{k=m}^{n}(1-\alpha_k)\right).
\end{aligned} \tag{3.29}$$

Thus, we have

$$\limsup_{n\to\infty} \|x_n - Px\|^2 \le \prod_{k=m}^{\infty} (1-\alpha_k)\|x_m - Px\|^2 + \epsilon\left(1 - \prod_{k=m}^{\infty}(1-\alpha_k)\right). \tag{3.30}$$

From $\sum_{n=1}^{\infty}\alpha_n = \infty$, we have $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$. So, we have

$$\limsup_{n\to\infty} \|x_n - Px\|^2 \le \epsilon. \tag{3.31}$$

Since $\epsilon > 0$ is arbitrary, we have $\lim_{n\to\infty}\|x_n - Px\|^2 = 0$. This completes the proof. □

4. Applications

Using convergence theorems in Section 3, we prove two convergence theorems for finding a fixed point of a nonexpansive mapping in a Banach space.

Theorem 4.1. *Let C be a compact convex subset of a smooth Banach space E and let T be a nonexpansive mapping of C into itself. Let $x_1 = x \in C$ and define an iterative sequence $\{x_n\}$ by*

$$x_n = \frac{1}{1+r_n}x + \frac{r_n}{1+r_n}Tx_n \quad \text{for } n = 1,2,\ldots, \tag{4.1}$$

where $\{r_n\} \subset (0,\infty)$ satisfies $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to Px, where P denotes a unique sunny nonexpansive retraction of C onto $F(T)$.

Proof. We define a mapping A of C into E by $A = I - T$. For $r > 0$, we denote by J_r the resolvent of A. Then, A is an accretive operator which satisfies $D(A) = C \subset \bigcap_{r>0} R(I + rA)$. From (4.1), we have, for each $n \in \mathbb{N}$,

$$x_n + r_n(I-T)x_n = x \tag{4.2}$$

and hence $x_n = J_{r_n}x$. It follows from Theorem 3.1 that $\{x_n\}$ converges strongly to Px. This completes the proof. □

As in the proof of Theorem 4.1, from Theorem 3.3, we obtain the following convergence theorem for finding a fixed point of a nonexpansive mapping.

Theorem 4.2. *Let C be a compact convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let T be a nonexpansive mapping of C into itself. Let $x_1 = x \in C$ and define an iterative sequence $\{x_n\}$ by*

$$\begin{aligned} u_n &= \frac{1}{1+r_n}x_n + \frac{r_n}{1+r_n}Tu_n, \\ x_{n+1} &= \alpha_n x + (1-\alpha_n)u_n, \end{aligned} \tag{4.3}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} r_n = \infty. \tag{4.4}$$

Then $\{x_n\}$ *converges strongly to* Px*, where* P *denotes a unique sunny nonexpansive retraction of* C *onto* $F(T)$.

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