REGULARIZATION OF NONLINEAR ILL-POSED EQUATIONS WITH ACCRETIVE OPERATORS

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We study the regularization methods for solving equations with arbitrary accretive operators. We establish the strong convergence of these methods and their stability with respect to perturbations of operators and constraint sets in Banach spaces. Our research is motivated by the fact that the fixed point problems with nonexpansive mappings are namely reduced to such equations. Other important examples of applications are evolution equations and co-variational inequalities in Banach spaces.

1. Introduction

Let E be a real normed linear space with dual E^*. The *normalized duality mapping* $j : E \to 2^{E^*}$ is defined by

$$j(x) := \Big\{x^* \in E^* : \langle x,x^* \rangle = \|x\|^2,\ \big\|x^*\big\|_* = \|x\|\Big\}, \tag{1.1}$$

where $\langle x,\phi\rangle$ denotes the dual product (pairing) between vectors $x \in E$ and $\phi \in E^*$. It is well known that if E^* is strictly convex, then j is single valued. We denote the single valued normalized duality mapping by J.

A map $A : D(A) \subseteq E \to 2^E$ is called *accretive* if for all $x, y \in D(A)$ there exists $J(x-y) \in j(x-y)$ such that

$$\big\langle u-v, J(x-y)\big\rangle \geq 0, \quad \forall u \in Ax,\ \forall v \in Ay. \tag{1.2}$$

If A is single valued, then (1.2) is replaced by

$$\big\langle Ax-Ay, J(x-y)\big\rangle \geq 0. \tag{1.3}$$

A is called *uniformly accretive* if for all $x, y \in D(A)$ there exist $J(x-y) \in j(x-y)$ and a strictly increasing function $\psi : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$, $\psi(0) = 0$ such that

$$\big\langle Ax-Ay, J(x-y)\big\rangle \geq \psi\big(\|x-y\|\big). \tag{1.4}$$

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It is called *strongly accretive* if there exists a constant $k > 0$ such that in (1.4) $\psi(t) = kt^2$. If E is a Hilbert space, accretive operators are also called *monotone*. An accretive operator A is said to be *hemicontinuous* at a point $x_0 \in D(A)$ if the sequence $\{A(x_0 + t_n x)\}$ converges weakly to Ax_0 for any element x such that $x_0 + t_n x \in D(A)$, $0 \le t_n \le t(x_0)$ and $t_n \to 0$, $n \to \infty$. An accretive operator A is said to be *maximal accretive* if it is accretive and the inclusion $G(A) \subseteq G(B)$, with B accretive, where $G(A)$ and $G(B)$ denote graphs of A and B, respectively, implies that $A = B$. It is known (see, e.g., [14]) that an accretive hemicontinuous operator $A : E \to E$ with a domain $D(A) = E$ is maximal accretive. In a smooth Banach space, a maximal accretive operator is strongly-weakly demiclosed on $D(A)$. An accretive operator A is said to be *m-accretive* if $R(A + \alpha I) = E$ for all $\alpha > 0$, where I is the identity operator in E.

Interest in accretive maps stems mainly from their firm connection with fixed point problems, evolution equations and co-variational inequalites in a Banach space (see, e.g. [6, 7, 8, 9, 10, 11, 12, 26]). Recall that each nonexpansive mapping is a continuous accretive operator [7, 19]. It is known that many physically significant problems can be modeled by initial-value problems of the form (see, e.g., [10, 12, 26])

$$x'(t) + Ax(t) = 0, \quad x(0) = x_0, \tag{1.5}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrödinger equations. One of the fundamental results in the theory of accretive operators, due to Browder [11], states that if A is locally Lipschitzian and accretive, then A is m-accretive. This result was subsequently generalized by Martin [23] to the continuous accretive operators. If $x(t)$ in (1.5) is independent of t, then (1.5) reduces to the equation

$$Au = 0, \tag{1.6}$$

whose solutions correspond to the equilibrium points of the system (1.5). Consequently, considerable research efforts have been devoted, especially within the past 20 years or so, to iterative methods for approximating these equilibrium points.

The two well-known iterative schemes for successive approximation of a solution of the equation $Ax = f$, where A is either uniformly accretive or strongly accretive, are *the Ishikawa iteration process* (see, e.g., [20]) and *the Mann iteration process* (see, e.g., [22]). These iteration processes have been studied extensively by various authors and have been successfully employed to approximate solutions of several nonlinear operator equations in Banach spaces (see, e.g., [13, 15, 17]). But all efforts to use the Mann and the Ishikawa schemes to approximate the solution of the equation $Ax = f$, where A is an accretive-type mapping (not necessarily uniformly or strongly accretive), have not provided satisfactory results. The major obstacle is that for this class of operators the solution is not, in general, unique.

Our purpose in this paper is to construct approximations generated by regularization algorithms, which converge strongly to solutions of the equations $Ax = f$ with accretive maps A defined on subsets of Banach spaces. Our theorems are applicable to much larger classes of operator equations in uniformly smooth Banach spaces than previous results (see, e.g., [4]). Furthermore, *the stability* of our methods with respect to perturbation of the operators and constraint sets is also studied.

2. Preliminaries

Let E be a real normed linear space of dimension greater than or equal to 2, and $x, y \in E$. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\right\}. \tag{2.1}$$

A Banach space E is called *uniformly smooth* if

$$\lim_{\tau\to 0} h_E(\tau) := \lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0. \tag{2.2}$$

Examples of uniformly smooth spaces are the Lebesgue L_p, the sequence ℓ_p, and the Sobolev W_p^m spaces for $1 < p < \infty$ and $m \geq 1$ (see, e.g., [2]).

If E is a real uniformly smooth Banach space, then the inequality

$$\begin{aligned} \|x\|^2 &\leq \|y\|^2 + 2\langle x-y, Jx\rangle \\ &\leq \|y\|^2 + 2\langle x-y, Jy\rangle + 2\langle x-y, Jx-Jy\rangle \end{aligned} \tag{2.3}$$

holds for every $x, y \in E$. A further estimation of $\|x\|^2$ needs one of the following two lemmas.

Lemma 2.1 [5]. *Let E be a uniformly smooth Banach space. Then for $x, y \in E$,*

$$\langle x-y, Jx-Jy\rangle \leq 8\|x-y\|^2 + C(\|x\|, \|y\|)\rho_E(\|x-y\|), \tag{2.4}$$

where

$$C(\|x\|, \|y\|) \leq 4\max\{2L, \|x\| + \|y\|\} \tag{2.5}$$

and L is the Figiel constant, $1 < L < 1.7$ [18, 24].

Lemma 2.2 [2]. *In a uniformly smooth Banach space E, for $x, y \in E$,*

$$\langle x-y, Jx-Jy\rangle \leq R^2(\|x\|, \|y\|)\rho_E\left(\frac{4\|x-y\|}{R(\|x\|, \|y\|)}\right), \tag{2.6}$$

where

$$R(\|x\|, \|y\|) = \sqrt{2^{-1}(\|x\|^2 + \|y\|^2)}. \tag{2.7}$$

If $\|x\| \leq R$ and $\|y\| \leq R$, then

$$\langle x-y, Jx-Jy\rangle \leq 2LR^2\rho_E\left(\frac{4\|x-y\|}{R}\right), \tag{2.8}$$

where L is the same as in Lemma 2.1.

We will need the following lemma on the recursive numerical inequalities.

Lemma 2.3 [1]. *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and let $\{\alpha_k\}$ be a sequence of positive numbers satisfying the conditions*

$$\sum_{1}^{\infty} \alpha_n = \infty, \quad \frac{\gamma_n}{\alpha_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.9}$$

Let the recursive inequality

$$\lambda_{n+1} \le \lambda_n - \alpha_n \phi(\lambda_n) + \gamma_n, \quad n = 1,2,\ldots, \tag{2.10}$$

be given where $\phi(\lambda)$ is a continuous and nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that it is positive on $\mathbb{R}^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) \ge c > 0$. Then $\lambda_n \to 0$ as $n \to \infty$.

We will also use the concept of a sunny nonexpansive retraction [19].

Definition 2.4. Let G be a nonempty closed convex subset of E. A mapping $Q_G : E \to G$ is said to be

(i) a retraction onto G if $Q_G^2 = Q_G$;

(ii) a nonexpansive retraction if it also satisfies the inequality

$$||Q_G x - Q_G y|| \le \|x - y\|, \quad \forall x, y \in E; \tag{2.11}$$

(iii) a sunny retraction if for all $x \in E$ and for all $0 \le t < \infty$,

$$Q_G(Q_G x + t(x - Q_G x)) = Q_G x. \tag{2.12}$$

Definition 2.5. If Q_G satisfies (i)–(iii) of Definition 2.4, then the element $\tilde{x} = Q_G x$ is said to be a sunny nonexpansive retractor of $x \in E$ onto G.

Proposition 2.6. *Let E be a uniformly smooth Banach space, and let G be a nonempty closed convex subset of E. A mapping $Q_G : E \to G$ is a sunny nonexpansive retraction if and only if for all $x \in E$ and for all $\xi \in G$,*

$$\langle x - Q_G x, J(Q_G x - \xi)\rangle \ge 0. \tag{2.13}$$

Denote by $\mathcal{H}_E(G_1, G_2)$ the Hausdorff distance between sets G_1 and G_2 in the space E, that is,

$$\mathcal{H}_E(G_1, G_2) = \max\left\{ \sup_{z_1 \in G_1} \inf_{z_2 \in G_2} ||z_1 - z_2||, \sup_{z_1 \in G_2} \inf_{z_2 \in G_1} ||z_1 - z_2|| \right\}. \tag{2.14}$$

Lemma 2.7 [7]. *Let E be a uniformly smooth Banach space, and let Ω_1 and Ω_2 be closed convex subsets of E such that the Hausdorff distance $\mathcal{H}_E(\Omega_1, \Omega_2) \le \sigma$. If Q_{Ω_1} and Q_{Ω_2} are the sunny nonexpansive retractions onto the subsets Ω_1 and Ω_2, respectively, then*

$$||Q_{\Omega_1} x - Q_{\Omega_2} x||^2 \le 16R(2r + q) h_E(16LR^{-1}\sigma), \tag{2.15}$$

where $h_E(\tau) = \tau^{-1}\rho_E(\tau)$, L is the Figiel constant, $r = \|x\|$, $q = \max\{q_1, q_2\}$, and $R = 2(2r + q) + \sigma$. Here $q_i = \operatorname{dist}(\theta, \Omega_i)$, $i = 1,2$, and θ is the origin of the space E.

3. Operator regularization method

We will deal with accretive operators $A : E \to E$ and operator equation

$$Ax = f \tag{3.1}$$

given on a closed convex subset $G \subset D(A) \subseteq E$, where $D(A)$ is a domain of A.

In the sequel, we understand a solution of (3.1) in the sense of a solution of the co-variational inequality (see, e.g., [9])

$$\langle Ax - f, J(y - x)\rangle \geq 0, \quad \forall y \in G,\ x \in G. \tag{3.2}$$

The following statement is a motivation of this approach [25].

Theorem 3.1. *Suppose that E is a reflexive Banach space with strictly convex dual space E^*. Let $A : E \to E$ be a hemicontinuous operator. If for fixed $x^* \in E$ and $f \in E$ the co-variational inequality*

$$\langle Ax - f, J(x - x^*)\rangle \geq 0, \quad \forall x \in E, \tag{3.3}$$

holds, then $Ax^* = f$.

In fact, the following more general theorem was proved in [8].

Theorem 3.2. *Let E be a smooth Banach space and let $A : E \to 2^E$ be an accretive operator. Then the following statements are equivalent:*

(i) *x^* satisfies the covariational inequality*

$$\langle z - f, J(x - x^*)\rangle \geq 0, \quad \forall z \in Ax,\ \forall x \in E; \tag{3.4}$$

(ii) $0 \in R(Ax^* - f)$.

We present the following two definitions of a solution of the operator equation (3.1) on G.

Definition 3.3. An element $x^* \in G$ is said to be a generalized solution of the operator equation (3.1) on G if there exists $z \in Ax^*$ such that

$$\langle z - f, J(y - x^*)\rangle \geq 0, \quad \forall y \in G. \tag{3.5}$$

Definition 3.4. An element $x^* \in G$ is said to be a total solution of the operator equation (3.1) on G if

$$\langle z - f, J(y - x^*)\rangle \geq 0, \quad \forall y \in G,\ \forall z \in Ay. \tag{3.6}$$

Lemma 3.5 [6]. *Suppose that E is a reflexive Banach space with strictly convex dual space E^*. Let A be an accretive operator. If an element $x^* \in G$ is the generalized solution of (3.1) on G characterized by the inequality (3.5), then it satisfies also the inequality (3.6), that is, it is a total solution of (3.1).*

Lemma 3.6 [6]. *Suppose that E is a reflexive Banach space with strictly convex dual space E^*. Let an operator A be either hemicontinuous or maximal accretive. If $G \subset \operatorname{int} D(A)$, then Definitions 3.3 and 3.4 are equivalent.*

Lemma 3.7. *Under the conditions of Lemma 3.6, the set of solutions of the operator equation (3.1) on G is closed.*

The proof follows from the fact that J is continuous in smooth reflexive Banach spaces and any hemicontinuous or maximal accretive operator is demiclosed in such spaces.

For finding a solution x^* of (3.1), we consider the regularized equation

$$Az_\alpha + \alpha z_\alpha = f, \tag{3.7}$$

where α is a positive parameter.

Let z_α^0 be a generalized solution of (3.7) on G, that is, there exists $\zeta_\alpha^0 \in Az_\alpha^0$ such that

$$\langle \zeta_\alpha^0 + \alpha z_\alpha^0 - f, J(x - z_\alpha^0)\rangle \geq 0, \quad \forall x \in G. \tag{3.8}$$

Theorem 3.8. *Assume that E is a reflexive Banach space with strictly convex dual space E^* and with origin θ, A is a hemicontinuous or maximal accretive operator with domain $D(A) \subseteq E$, $G \subset \operatorname{int} D(A)$ is convex and closed, (3.1) has a nonempty generalized solution set $N \subset G$. Then $\|z_\alpha^0\| \leq 2\|\bar{x}^*\|$, where $\bar{x}^*$ is an element of N with minimal norm. If the normalized duality mapping J is sequentially weakly continuous on E, then $z_\alpha^0 \to \widetilde{x}^*$ as $\alpha \to 0$, where $\widetilde{x}^* \in N$ is a sunny nonexpansive retractor of θ onto N, that is, a (necessarily unique) solution of the inequality*

$$\langle \widetilde{x}^*, J(x^* - \widetilde{x}^*)\rangle \geq 0, \quad \forall x^* \in N. \tag{3.9}$$

Proof. First, we show that z_α^0 is the *unique* solution of (3.7). Suppose that u_α^0 is another solution of this equation. Then along with (3.8), we have for some $\xi_\alpha^0 \in Au_\alpha^0$ that

$$\langle \xi_\alpha^0 + \alpha u_\alpha^0 - f, J(x - u_\alpha^0)\rangle \geq 0, \quad \forall x \in G. \tag{3.10}$$

Since $z_\alpha^0 \in G$ and $u_\alpha^0 \in G$, we have the inequalities

$$\begin{aligned} &\langle \zeta_\alpha^0 + \alpha z_\alpha^0 - f, J(u_\alpha^0 - z_\alpha^0)\rangle \geq 0, \\ &\langle \xi_\alpha^0 + \alpha u_\alpha^0 - f, J(z_\alpha^0 - u_\alpha^0)\rangle \geq 0. \end{aligned} \tag{3.11}$$

Summing these inequalities, we obtain

$$0 \geq \langle \xi_\alpha^0 - \zeta_\alpha^0, J(z_\alpha^0 - u_\alpha^0)\rangle \geq \alpha\langle z_\alpha^0 - u_\alpha^0, J(z_\alpha^0 - u_\alpha^0)\rangle = \alpha\left|\left|z_\alpha^0 - u_\alpha^0\right|\right|^2. \tag{3.12}$$

From this the claim follows.

Next, we prove that the sequence $\{z_\alpha^0\}$ is bounded. Observe that the covariational inequality (3.8) implies that

$$\langle \zeta_\alpha^0 + \alpha z_\alpha^0 - f, J(x^* - z_\alpha^0)\rangle \geq 0, \quad \forall x^* \in N, \tag{3.13}$$

because $x^* \in G$. At the same time, since x^* is a generalized solution of (3.1), there exists $\xi^* \in Ax^*$ such that

$$\langle \xi^* - f, J(z_\alpha^0 - x^*)\rangle \geq 0. \tag{3.14}$$

Then (3.13) and (3.14) together give

$$\langle \zeta_\alpha^0 - \xi^* + \alpha z_\alpha^0, J(x^* - z_\alpha^0)\rangle = \langle \zeta_\alpha^0 - \xi^*, J(x^* - z_\alpha^0)\rangle + \alpha\langle z_\alpha^0, J(x^* - z_\alpha^0)\rangle \geq 0. \tag{3.15}$$

By accretiveness of A, one gets

$$\langle z_\alpha^0, J(x^* - z_\alpha^0)\rangle \geq 0. \tag{3.16}$$

The obtained inequality yields the estimates

$$\|x^* - z_\alpha^0\|^2 \leq \langle x^*, J(x^* - z_\alpha^0)\rangle \leq \|x^*\|\|x^* - z_\alpha^0\|. \tag{3.17}$$

Hence, $\|z_\alpha^0\| \leq 2\|x^*\|$ for all $x^* \in N$, that is, $\|z_\alpha^0\| \leq 2\|\bar{x}^*\|$. Note that $\bar{x}^*$ exists because N is closed and E is reflexive.

Show now that $\|z_\alpha^0 - \widetilde{x}^*\| \to 0$ as $\alpha \to 0$. Since $\{z_\alpha^0\}$ is bounded, there exist a subsequence $z_\beta^0 \subset z_\alpha^0$ and an element $\widetilde{x} \in E$ such that $z_\beta^0 \rightharpoonup \widetilde{x}$ as $\beta \to 0$. Since $z_\beta^0 \in G$ and G is weakly closed (since it is closed and convex), we conclude that $\widetilde{x} \in G$. Due to Lemma 3.6, the inequality (3.8) is equivalent to the following one:

$$\langle w + \alpha x - f, J(x - z_\alpha^0)\rangle \geq 0, \quad \forall x \in G,\ \forall w \in Ax. \tag{3.18}$$

Therefore

$$\langle w + \beta x - f, J(x - z_\beta^0)\rangle \geq 0, \quad \forall x \in G,\ \forall w \in Ax. \tag{3.19}$$

Passing to the limit in (3.19) as $\beta \to 0$ and using the weak continuity of J, one gets

$$\langle w - f, J(x - \widetilde{x})\rangle \geq 0, \quad \forall x \in G,\ \forall w \in Ax. \tag{3.20}$$

By Lemma 3.6 again, it follows that $\widetilde{x}$ is a total (consequently, generalized) solution of (3.1) on G.

We now show that $\widetilde{x} = \widetilde{x}^* = Q_N\theta$ and $\widetilde{x}^*$ is unique. This will mean that $z_\alpha^0 \rightharpoonup \widetilde{x}^*$ as we presumed above. Consider (3.17) on $\{z_\beta^0\}$ with $x^* = \widetilde{x}$. It is clear that $\|\widetilde{x} - z_\beta^0\| \to 0$. Then we deduce from (3.16) that

$$\langle \widetilde{x}, J(x^* - \widetilde{x})\rangle \geq 0, \quad \forall x^* \in N. \tag{3.21}$$

This means that $\widetilde{x} = Q_N\theta$.

We prove that $\widetilde{x}$ is a unique solution of the last inequality. Suppose that $\widetilde{x}_1 \in N$ is its another solution. Then

$$\langle \widetilde{x}_1, J(x^* - \widetilde{x}_1)\rangle \geq 0, \quad \forall x^* \in N. \tag{3.22}$$

We have

$$\langle \tilde{x}, J(\tilde{x}_1 - \tilde{x}) \rangle \geq 0, \\ \langle \tilde{x}_1, J(\tilde{x} - \tilde{x}_1) \rangle \geq 0. \tag{3.23}$$

Their combination gives

$$\langle \tilde{x} - \tilde{x}_1, J(\tilde{x}_1 - \tilde{x}) \rangle \geq 0, \tag{3.24}$$

which contradicts the fact that $\|\tilde{x} - \tilde{x}_1\| \geq 0$. Thus, the claim is true.

Finally, the first inequality in (3.17) implies the strong convergence of $\{z_\alpha^0\}$ to $\bar{x}^*$. The proof is accomplished. In particular, the theorem is valid if N is a singleton. □

Next we will study an operator regularization method for (3.1) with a perturbed right-hand side, perturbed constraint set, and perturbed operator. Assume that, instead of f, G, and A, we have the sequences $\{f^\delta\} \in E$, $\{G_\sigma\} \in E$, and $\{A^\omega\}$, $A^\omega : G_\sigma \to E$, such that

$$||f^\delta - f|| \leq \delta, \\ \mathcal{H}_E(G_\sigma, G) \leq \sigma, \tag{3.25}$$

where $\mathcal{H}_E(G_1, G_2)$ is the Hausdorff distance (2.14), and

$$||A^\omega x - Ax|| \leq \omega\zeta(\|x\|), \quad \forall x \in D, \tag{3.26}$$

where $\zeta(t)$ is a positive and bounded function defined on $\mathbb{R}^+$ and $D = D(A) = D(A^\omega)$. Thus, in reality, the equations

$$A^\omega y = f^\delta \tag{3.27}$$

are given on G_σ, $\sigma \geq 0$. Consider the following regularized equation on G_σ:

$$A^\omega z + \alpha z = f^\delta. \tag{3.28}$$

Let z_α^γ with $\gamma = (\delta, \sigma, \omega)$ be its (unique) generalized solution. This means that there exists $y_\alpha^\gamma \in A^\omega z_\alpha^\gamma$ such that

$$\langle y_\alpha^\gamma + \alpha z_\alpha^\gamma - f^\delta, J(x - z_\alpha^\gamma) \rangle \geq 0, \quad \forall x \in G_\sigma. \tag{3.29}$$

Theorem 3.9. *Assume that*

(i) *in real uniformly smooth Banach space E with the modulus of smoothness $\rho_E(\tau)$, all the conditions of Theorem 3.8 are fulfilled;*
(ii) *(3.28) has bounded generalized solutions z_α^γ for all $\delta \geq 0$, $\sigma \geq 0$, $\omega \geq 0$, and $\alpha > 0$;*
(iii) *the operators A^ω are accretive and bounded (i.e., they carry bounded sets of E to bounded sets of E);*
(iv) *$G \subset D$ and $G_\sigma \subset D$ are convex and closed sets;*
(v) *the estimates (3.25) and (3.26) are satisfied for $\delta \geq 0$, $\sigma \geq 0$, and $\omega \geq 0$.*

If

$$\frac{\delta+\omega+h_E(\sigma)}{\alpha} \longrightarrow 0 \quad as\ \alpha \longrightarrow 0, \tag{3.30}$$

then $z_\alpha^\gamma \to \bar{x}^*$, *where* $\bar{x}^*$ *is a sunny nonexpansive retractor of* θ *onto* N.

Proof. Write the obvious inequality

$$||z_\alpha^\gamma - \bar{x}^*|| \leq ||z_\alpha^0 - \bar{x}^*|| + ||z_\alpha^\gamma - z_\alpha^0||, \tag{3.31}$$

where z_α^0 is a generalized solution of (3.7). The limit relation $\|z_\alpha^0 - \bar{x}^*\| \to 0$ has been already established in Theorem 3.8. At the same time, the result $\|z_\alpha^\gamma - z_\alpha^0\| \to 0$ immediately follows from Lemma 4.1 proved in the next section. The condition (3.30) is sufficient for this conclusion. □

Remark 3.10. We do not suppose that in the operator equation (3.28) every operator A^ω has been defined on every set G_σ. Only possibility for the parameters ω and σ to be simultaneously rushed to zero is required.

4. Proximity lemma

We further present *the proximity lemma* between solutions of two regularized equations

$$T_1 z_1 + \alpha_1 z_1 = f_1, \quad \alpha_1 > 0, \tag{4.1}$$

$$T_2 z_2 + \alpha_2 z_2 = f_2, \quad \alpha_2 > 0, \tag{4.2}$$

on G_1 and G_2, respectively, provided their intersection $G_1 \bigcap G_2$ is not empty.

Lemma 4.1 (cf. [3]). *Suppose that*

(i) *E is a real uniformly smooth Banach space with the modulus of smoothness $\rho_E(\tau)$;*
(ii) *the solution sequences $\{z_1\}$ and $\{z_2\}$ of (4.1) and (4.2), respectively, are bounded, that is, there exists a constant $M_1 > 0$ such that $\|z_1\| \leq M_1$ and $\|z_2\| \leq M_1$;*
(iii) *the operators T_1 and T_2 are accretive and bounded on the sequences $\{z_1\}$ and $\{z_2\}$, that is, there exist constants $M_2 > 0$ and $M_3 > 0$ such that $\|T_1 z_1\| \leq M_2$ and $\|T_2 z_2\| \leq M_3$;*
(iv) *$G_1 \subset D$ and $G_2 \subset D$ are convex and closed subsets of E and $D = D(T_1) = D(T_2)$;*
(v) *the estimates $\|f_1 - f_2\| \leq \delta$, $\mathcal{H}_E(G_1, G_2) \leq \sigma$, and $\|T_1 z - T_2 z\| \leq \omega\zeta(\|z\|)$, $\forall z \in D$, are fulfilled. Then*

$$||z_1 - z_2|| \leq \zeta(M_1)\frac{\omega}{\alpha_1} + \frac{\delta}{\alpha_1} + M_1\frac{|\alpha_1 - \alpha_2|}{\alpha_1} + \sqrt{\frac{c_1 h_E(c_2\sigma)}{\alpha_1}}, \tag{4.3}$$

where

$$c_1 = 8R(2\alpha_1 M_1 + M_2 + M_3 + ||f_1|| + ||f_2||), \qquad c_2 = 16LR^{-1}, \qquad R = 2M_1 + \sigma. \tag{4.4}$$

Proof. Solutions $z_1 \in G_1$ and $z_2 \in G_2$ of the operator equations (4.1) and (4.2) are defined by the following co-variational inequalities, respectively:

$$\langle T_1 z_1 + \alpha_1 z_1 - f_1, J(x - z_1)\rangle \geq 0, \quad \forall x \in G_1,\ \alpha_1 > 0, \tag{4.5}$$

$$\langle T_2 z_2 + \alpha_2 z_2 - f_2, J(x - z_2)\rangle \geq 0, \quad \forall x \in G_2,\ \alpha_2 > 0. \tag{4.6}$$

Estimate a dual product

$$B = \langle T_1 z_1 + \alpha_1 z_1 - f_1 - T_2 z_2 - \alpha_2 z_2 + f_2, J(z_1 - z_2)\rangle. \tag{4.7}$$

Obviously,

$$B = \langle T_1 z_1 - T_1 z_2 + \alpha_1 (z_1 - z_2) + T_1 z_2 - T_2 z_2 + (\alpha_1 - \alpha_2) z_2 + f_2 - f_1, J(z_1 - z_2)\rangle. \tag{4.8}$$

The operator T_1 is accretive, therefore,

$$\langle T_1 z_1 - T_1 z_2, J(z_1 - z_2)\rangle \geq 0. \tag{4.9}$$

Then

$$B \geq \alpha_1 ||z_1 - z_2||^2 - (||T_1 z_2 - T_2 z_2|| + |\alpha_1 - \alpha_2|\,||z_2|| + ||f_1 - f_2||)||z_1 - z_2||. \tag{4.10}$$

Since $\|z_2\| \leq M_1$, we conclude in conformity with (v) that

$$B \geq -c||z_1 - z_2|| + \alpha_1 ||z_1 - z_2||^2, \tag{4.11}$$

where

$$c = \omega\zeta(M_1) + M_1 |\alpha_1 - \alpha_2| + \delta. \tag{4.12}$$

Next, if $\mathcal{H}_E(G_1, G_2) \leq \sigma$, then for every $z_2 \in G_2$ there exists $\tilde{z} \in G_1$ such that $\|z_2 - \tilde{z}\| \leq \sigma$ and

$$\begin{aligned}
&\langle T_1 z_1 + \alpha_1 z_1 - f_1, J(z_1 - z_2)\rangle \\
&\quad = \langle T_1 z_1 + \alpha_1 z_1 - f_1, J(z_1 - z_2) + J(z_1 - \tilde{z}) - J(z_1 - \tilde{z})\rangle \\
&\quad = \langle T_1 z_1 + \alpha_1 z_1 - f_1, J(z_1 - \tilde{z})\rangle \\
&\qquad + \langle T_1 z_1 + \alpha_1 z_1 - f_1, J(z_1 - z_2) - J(z_1 - \tilde{z})\rangle.
\end{aligned} \tag{4.13}$$

By (4.5),

$$\langle T_1 z_1 + \alpha_1 z_1 - f_1, J(z_1 - \tilde{z})\rangle \leq 0. \tag{4.14}$$

Estimate the last term in (4.13). For this recall that if $\|x\| \leq R$ and $\|y\| \leq R$, then (see [2, page 38])

$$||J(x) - J(y)||_* \leq 8R h_E(16LR^{-1}\|x - y\|). \tag{4.15}$$

For $\|z_1 - z_2\| \le 2M_1$ and $\|z_1 - \tilde{z}\| \le 2M_1 + \sigma = R$, this implies that

$$\begin{aligned} &\langle T_1 z_1 + \alpha_1 z_1 - f_1, J(z_1 - z_2) - J(z_1 - \tilde{z})\rangle \\ &\quad \le \|T_1 z_1 + \alpha_1 z_1 - f_1\| \|J(z_1 - z_2) - J(z_1 - \tilde{z})\|_* \\ &\quad \le 8R(\alpha_1\|z_1\| + \|T_1 z_1\| + \|f_1\|) h_E(16LR^{-1}\|z_2 - \tilde{z}\|) \\ &\quad \le 8R(\alpha_1 M_1 + M_2 + \|f_1\|) h_E(16LR^{-1}\sigma). \end{aligned} \tag{4.16}$$

Analogously to the previous chain of inequalities,

$$\langle T_2 z_2 + \alpha_2 z_2 - f_2, J(z_2 - z_1)\rangle \le 8R(\alpha_1 M_1 + M_3 + \|f_2\|) h_E(16LR^{-1}\sigma). \tag{4.17}$$

Therefore,

$$B \le c_1 h_E(c_2\sigma). \tag{4.18}$$

Finally, combining (4.11) with (4.18), one gets

$$c_1 h_E(c_2\sigma) + c\|z_1 - z_2\| \ge \alpha_1 \|z_1 - z_2\|^2. \tag{4.19}$$

This quadratic inequality gives

$$\|z_1 - z_2\| \le \frac{c + \sqrt{c^2 + 4\alpha_1 c_1 h_E(c_2\sigma)}}{2\alpha_1} \le \frac{c}{\alpha_1} + \sqrt{\frac{c_1 h_E(c_2\sigma)}{\alpha_1}}, \tag{4.20}$$

because $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for all $a, b \ge 0$. Thus, (4.3) holds. □

From Theorem (3.10) and Lemma 4.1 we obtain the following corollary.

Corollary 4.2. *If, in the conditions of Lemma 4.1, $\omega = \delta = \sigma = 0$, that is, $T_1 = T_2$, $f_1 = f_2$, and $G_1 = G_2$, then*

$$\|z_1 - z_2\| \le 2\|x^*\| \frac{|\alpha_1 - \alpha_2|}{\alpha_1}. \tag{4.21}$$

5. Iterative regularization methods

5.1. We begin by considering iterative regularization with exact given data.

Theorem 5.1. *Let E be a real uniformly smooth Banach space with the modulus of smoothness $\rho_E(\tau)$, let $A : E \to E$ be a bounded accretive operator with $D(A) \subseteq E$, and let $G \subset \operatorname{int} D(A)$ be a closed convex set. Suppose that (3.1) has a generalized solution x^* on G. Let $\{\epsilon_n\}$ and $\{\alpha_n\}$ be real sequences such that $\epsilon_n \le 1$, $\alpha_n \le 1$. Starting from arbitrary $x_0 \in G$ define the sequence $\{x_n\}$ as follows:*

$$x_{n+1} := Q_G(x_n - \epsilon_n(Ax_n + \alpha_n x_n - f)), \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

where Q_G is a nonexpansive retraction of E onto G. Then there exists $1 > d > 0$ such that whenever

$$\epsilon_n \le d, \quad \frac{\rho_E(\epsilon_n)}{\epsilon_n \alpha_n} \le d^2 \tag{5.2}$$

for all $n \ge 0$, the sequence $\{x_n\}$ is bounded.

Proof. Denote by $B_r(x^*)$ the closed ball of radius r with the center in x^*. Choose $r > 0$ sufficiently large such that $r \ge 2\|x^*\|$ and $x_0 \in B_r(x^*)$. Construct the set $S = B_r(x^*) \cap G$ and let

$$M := \frac{3}{2}r + \|f\| + \sup\{\|Ax\| : x \in S\}. \tag{5.3}$$

We claim that $\{x_n\}$ is bounded in our circumstances. Show by induction that $x_n \in S$ for all positive integers. Actually, $x_0 \in S$ by the assumption. Hence, for given $n > 0$, we may presume the inclusion $x_n \in S$ and prove that $x_{n+1} \in S$. Suppose that x_{n+1} does not belong to S. Since $x_{n+1} \in G$, this means that $\|x_{n+1} - x^*\| > r$. By (5.1) and due to the nonexpansiveness of Q_G, we have

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||Q_G(x_n - \epsilon_n(Ax_n + \alpha_n x_n - f)) - Q_G x^*|| \\ &\le ||x_n - x^* - \epsilon_n(Ax_n + \alpha_n x_n - f)|| \\ &\le ||x_n - x^*|| + \epsilon_n ||Ax_n + \alpha_n x_n - f|| \\ &\le ||x_n - x^*|| + ||Ax_n|| + ||x_n - x^*|| + ||x^*|| + \|f\| \\ &\le r + \sup_{x \in S} \|Ax\| + r + \frac{1}{2}r + \|f\| = r + M = \overline{M}. \end{aligned} \tag{5.4}$$

In the next calculations, we apply Lemma 2.2 with $x = x_{n+1} - x^*$ and $y = x_n - x^*$. It is easy to see that

$$\begin{aligned} \|x\| = ||x_{n+1} - x^*|| \le \overline{M}, \qquad \|y\| = ||x_n - x^*|| \le r, \\ \|x - y\| = ||x_{n+1} - x_n|| \le \varepsilon_n ||Ax_n + \alpha_n x_n - f|| \le \varepsilon_n M. \end{aligned} \tag{5.5}$$

Thus, $\max\{\|x\|, \|y\|\} \le \overline{M}$, and we have

$$\langle x_{n+1} - x_n, J(x_{n+1} - x^*) - J(x_n - x^*)\rangle \le 2L\overline{M}^2 \rho_E(4M\overline{M}^{-1}\epsilon_n), \tag{5.6}$$

because the function $\rho_E(\tau)$ is nondecreasing [18, 21]. Besides, the function $\rho_E(\tau)$ is convex, therefore, $\rho_E(c\tau) \le c\rho_E(\tau)$, for all $c \le 1$. Since $M\overline{M}^{-1} \le 1$, (5.6) yields

$$\langle x_{n+1} - x_n, J(x_{n+1} - x^*) - J(x_n - x^*)\rangle \le 2LM\overline{M}\rho_E(4\varepsilon_n). \tag{5.7}$$

Then using the facts that $\rho_E(\tau)$ is continuous, $0 \le \varepsilon_n \le 1$, and by [16],

$$2 \le \lim_{\tau \to 0} \frac{\rho_E(4\tau)}{\rho_E(2\tau)} \le 4, \tag{5.8}$$

we conclude that there is a constant $C > 1$ such that

$$\langle x_{n+1} - x_n, J(x_{n+1} - x^*) - J(x_n - x^*) \rangle \le 8LCM\overline{M}\rho_E(\varepsilon_n). \tag{5.9}$$

Moreover, by (2.3), (5.1), (5.6) and by the inclusion $x_n \in S$, one gets

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\le ||x_n - x^* - \epsilon_n(Ax_n + \alpha_n x_n - f)||^2 \\ &\le ||x_n - x^*||^2 - 2\epsilon_n\langle Ax_n - f, J(x_n - x^*)\rangle \\ &\quad - 2\epsilon_n\alpha_n\langle x_n, J(x_n - x^*)\rangle + 16LCM\overline{M}\rho_E(\varepsilon_n). \end{aligned} \tag{5.10}$$

Since x^* is a generalized solution of (3.1) on G and $x_n \in G$ for all $n \ge 0$, we can write

$$\langle Ax_n - f, J(x_n - x^*)\rangle \ge 0. \tag{5.11}$$

Then (5.10) implies the inequality

$$||x_{n+1} - x^*||^2 \le ||x_n - x^*||^2 - 2\epsilon_n\alpha_n\langle x_n, J(x_n - x^*)\rangle + 16LC\overline{M}\rho_E(\varepsilon_n). \tag{5.12}$$

Choose $K > 0$ such that

$$K \le \frac{r^2}{4(\sqrt{D} + M)^2}, \tag{5.13}$$

where $D = 8LCM\overline{M}$. Set $d := \sqrt{K}$. It is not difficult to verify that $1 > d > 0$. By virtue of our assumption, $\|x_{n+1} - x^*\| > \|x_n - x^*\|$. This allows us to deduce from (5.12) the following estimate:

$$\epsilon_n\alpha_n\langle x_n, J(x_n - x^*)\rangle \le 8LCM\overline{M}\rho_E(\epsilon_n). \tag{5.14}$$

It gives the inequality

$$\langle x_n, J(x_n - x^*)\rangle \le DK \tag{5.15}$$

because of the assumption that

$$\frac{\rho_E(\epsilon_n)}{\alpha_n\epsilon_n} \le K = d^2, \quad \forall n \ge 0. \tag{5.16}$$

Now adding $\langle -x^*, J(x_n - x^*)\rangle$ to both sides of (5.15), we get

$$\begin{aligned} ||x_n - x^*||^2 &\le KD + \langle -x^*, J(x_n - x^*)\rangle \\ &\le KD + ||x^*||||x_n - x^*|| \le KD + \frac{r}{2}||x_n - x^*||. \end{aligned} \tag{5.17}$$

Solving this quadratic inequality for $\|x_n - x^*\|$ and using the estimate

$$\sqrt{\frac{r^2}{16} + KD} \le \frac{r}{4} + \sqrt{KD}, \tag{5.18}$$

we derive that

$$||x_n - x^*|| \le \frac{r}{2} + \sqrt{KD}. \tag{5.19}$$

In any case,

$$||x_{n+1} - x^*|| \le ||x_n - x^*|| + \epsilon_n ||Ax_n + \alpha_n x_n - f||, \tag{5.20}$$

so that

$$||x_{n+1} - x^*|| \le \frac{r}{2} + \sqrt{KD} + \epsilon_n M < r, \tag{5.21}$$

by the original choices of K and ϵ_n, and this contradicts the assumption that x_{n+1} is not in S. Therefore $x_n \in S$ for any integers $n \ge 0$. Thus $\{x_n\}$ is bounded, say, $\|x_n\| \le \widetilde{C}$. □

In what follows, we suppose that the normalized duality mapping J is continuous and sequentially weakly continuous in the ball $B_r(\theta)$ with $r = \widetilde{C}$. We show that $x_n \to \bar{x}^*$, where $\bar{x}^*$ is a unique solution of (3.9).

Theorem 5.2. *Assume that all the conditions of Theorems 3.8 and 5.1 are fulfilled. In addition, let $\alpha_n \to 0$ as $n \to \infty$,*

$$\frac{\epsilon_n}{\alpha_n} \longrightarrow 0, \qquad \frac{|\alpha_n - \alpha_{n+1}|}{\epsilon_n \alpha_n^2} \longrightarrow 0, \qquad \frac{\rho_E(\epsilon_n)}{\epsilon_n \alpha_n} \longrightarrow 0. \tag{5.22}$$

Then the sequence $\{x_n\}$ generated by (5.1) converges strongly to $\bar{x}^*$ as $n \to \infty$.

Proof. So, by Theorem 5.1, $\{x_n\}$ is bounded by a constant $\widetilde{C}$. Let z_n and z_{n+1} be generalized solutions of the equation

$$Az + \alpha_k z = f \tag{5.23}$$

on G for $k = n$ and $k = n + 1$, respectively. It follows from (4.3) and (5.22) that there exists a constant $d > 0$ such that $\|z_n - z_{n+1}\| \le d$. Put

$$p_n = x_n - \epsilon_n (Ax_n + \alpha_n x_n - f). \tag{5.24}$$

Then by (5.1) and by convexity of the functional $\|x\|^2$, we have that

$$\begin{aligned} ||x_{n+1} - z_{n+1}||^2 &= ||Q_G p_n - Q_G z_{n+1}||^2 \\ &\le ||p_n - z_{n+1}||^2 \\ &\le ||p_n - z_n||^2 + 2\langle z_{n+1} - z_n, J(z_{n+1} - p_n)\rangle \\ &\le ||p_n - z_n||^2 + 2\langle z_{n+1} - z_n, J(z_n - p_n)\rangle \\ &\quad + 2\langle z_{n+1} - z_n, J(z_{n+1} - p_n) - J(z_n - p_n)\rangle. \end{aligned} \tag{5.25}$$

We continue the estimation of (5.25) using Lemma 2.1. It is easy to see that if H is a Hilbert space and $\tau \le \bar{\tau}$, then [18, 21]

$$\rho_E(\tau) \ge \rho_H(\tau) = \left(1+\tau^2\right)^{1/2} - 1 \ge \bar{c}\tau^2, \tag{5.26}$$

where

$$\bar{c} = \left(\sqrt{1+\bar{\tau}^2}+1\right)^{-1}. \tag{5.27}$$

One gets

$$\begin{aligned} ||x_{n+1}-z_{n+1}||^2 \le{}& ||p_n-z_n||^2 + 2||z_n-p_n|| \cdot ||z_{n+1}-z_n|| \\ &+ 16||z_{n+1}-z_n||^2 + C_1(n)\rho_E(||z_{n+1}-z_n||), \end{aligned} \tag{5.28}$$

where

$$\begin{aligned} C_1(n) &= 8\max\{2L, ||z_n-p_n|| + ||z_{n+1}-p_n||\} \\ &\le 8\max\{2L, \widetilde{C}+M+2||x^*||\} = C_1, \end{aligned} \tag{5.29}$$

where M is defined by (5.3). Therefore, due to Corollary 4.2,

$$\begin{aligned} ||x_{n+1}-z_{n+1}||^2 \le{}& ||p_n-z_n||^2 + 4\frac{|\alpha_n-\alpha_{n+1}|}{\alpha_n} \cdot ||x^*|| \cdot ||p_n-z_n|| \\ &+ (16\bar{c}^{-1}+C_1)\rho_E\left(\frac{2(\alpha_n-\alpha_{n+1})}{\alpha_n}||x^*||\right), \end{aligned} \tag{5.30}$$

and in (5.27) $\bar{\tau} = d \ge \|z_{n+1}-z_n\| = \tau$.

Now we evaluate $\|p_n-z_n\|^2$. The convexity inequality (2.3) yields

$$\begin{aligned} ||p_n-z_n||^2 &= ||x_n-\epsilon_n(Ax_n+\alpha_n x_n-f)-z_n||^2 \\ &\le ||x_n-z_n||^2 - 2\epsilon_n\langle Ax_n+\alpha_n x_n-f, J(p_n-z_n)\rangle \\ &= ||x_n-z_n||^2 - 2\epsilon_n\langle Ax_n+\alpha_n x_n-f, J(x_n-z_n)\rangle \\ &\quad + 2\langle p_n-x_n, J(p_n-z_n)-J(x_n-z_n)\rangle. \end{aligned} \tag{5.31}$$

Since

$$\langle Az_n+\alpha_n z_n-f, J(x_n-z_n)\rangle \ge 0, \tag{5.32}$$

and by the accretiveness property of A,

$$\langle Ax_n-Az_n, J(x_n-z_n)\rangle \ge 0, \tag{5.33}$$

we deduce

$$\begin{aligned}||p_n - z_n||^2 &\le ||x_n - z_n||^2 - 2\epsilon_n\alpha_n||x_n - z_n||^2 \\ &\quad + 2\langle p_n - x_n, J(p_n - z_n) - J(x_n - z_n)\rangle \\ &\le ||x_n - z_n||^2 - 2\epsilon_n\alpha_n||x_n - z_n||^2 + 16\epsilon_n^2||Ax_n + \alpha_n x_n - f||^2 \\ &\quad + C_2(n)\rho_E(\epsilon_n||Ax_n + \alpha_n x_n - f||),\end{aligned} \tag{5.34}$$

where

$$\begin{aligned}C_2(n) &= 4\max\{2L, ||z_n - p_n|| + ||x_n - z_n||\} \\ &\le 4\max\{2L, 2\widetilde{C} + M + 2||x^*||\} = C_2.\end{aligned} \tag{5.35}$$

Substituting (5.34) for (5.30) and using the fact that $\rho_E(\tau) \le \tau$, we obtain

$$\begin{aligned}||x_{n+1} - z_{n+1}||^2 &\le ||x_n - z_n||^2 - 2\epsilon_n\alpha_n||x_n - z_n||^2 + 16\epsilon_n^2 M^2 \\ &\quad + C_2\rho_E(\epsilon_n M) + C_3\frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n}, C_1)\rho_E\left(\frac{(\alpha_n - \alpha_{n+1})}{\alpha_n}M\right),\end{aligned} \tag{5.36}$$

where

$$C_3 = MC_1 + 4||x^*||(\widetilde{C} + M + 2||x^*||) + 32M\bar{c}^{-1}. \tag{5.37}$$

Therefore, by Lemma 2.3 and by hypothesis (5.22), we conclude that $\|x_n - z_n\| \to 0$. In addition, by Theorem 3.8,

$$||x_n - \bar{x}^*|| \le ||x_n - z_n|| + ||z_n - \bar{x}^*|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{5.38}$$

which implies that $\{x_n\}$ converges strongly to $\bar{x}^*$. □

5.2. In this subsection, we study an iterative regularization method for (3.1) with a perturbed operator and perturbed right-hand side. Assume that, instead of f and A, we have the sequences $\{f_n\}$, $f_n \in E$, and $\{A_n\}$, $A_n : D(A_n) \subseteq E \to E$, such that

$$\begin{aligned}||f_n - f|| &\le \delta_n, \\ ||A_n x - Ax|| &\le \omega_n\zeta(\|x\|) + \mu_n, \quad \forall x \in G,\end{aligned} \tag{5.39}$$

where $\zeta(t)$ is a positive and bounded function defined on $\mathbb{R}^+$, $G \subset D(A_n)$ and $G \subset D(A)$. Thus, in reality, the following equations are given:

$$A_n y = f_n, \tag{5.40}$$

which may not have a solution. Consider the regularizing iterative algorithm

$$y_{n+1} = Q_G(y_n - \epsilon_n(A_n y_n + \alpha_n y_n - f_n)), \quad n = 0, 1, 2, \ldots, \tag{5.41}$$

where Q_G is a nonexpansive retraction of E onto G.

Theorem 5.3. *Assume that all the conditions of Theorems 3.8 and 5.1 are fulfilled. Suppose that there exist sequences of positive numbers ω_n, μ_n, and δ_n converging to zero as $n \to \infty$, such that (5.39) holds. Suppose that*

$$\alpha_n \longrightarrow 0, \qquad \frac{\omega_n + \mu_n + \delta_n}{\alpha_n} \longrightarrow 0, \qquad \frac{\rho_E(\epsilon_n)}{\epsilon_n \alpha_n} \longrightarrow 0, \qquad \frac{|\alpha_n - \alpha_{n+1}|}{\epsilon_n \alpha_n^2} \longrightarrow 0 \tag{5.42}$$

as $n \to \infty$. Starting from arbitrary $y_0 \in G$ define the sequence $\{y_n\}$ by (5.41). Then there exists $1 > d > 0$ such that whenever

$$\epsilon_n \le d, \quad \frac{\rho_E(\epsilon_n)}{\epsilon_n \alpha_n} \le d^2 \tag{5.43}$$

for all $n \ge 0$, the sequence $\{y_n\}$ is bounded and converges in norm to the solution $\bar{x}^*$, where $\bar{x}^*$ is the unique solution of inequality (3.9).

Proof. First of all, as in the proof of Theorem 5.1, we aim at showing that $\{y_n\}$ is bounded. To this end, introduce again a closed ball $B_r(x^*)$ with sufficiently large radius $r > 0$ such that $r \ge 2\|x^*\|$ and $y_0 \in B_r(x^*)$. And construct again the set $S = B_r(x^*) \cap G$. Without loss of generality, according to (5.42), put $\omega_n \le \bar{\omega}$, $\mu_n \le \bar{\mu}$, $\delta_n \le \bar{\delta}$, and

$$M := \frac{3}{2}r + \|f\| + \sup\{\|Ay\| : y \in S\} + \bar{\omega}M_\zeta + \bar{\mu} + \bar{\delta}, \tag{5.44}$$

where

$$M_\zeta = \sup\{\zeta(\|y\|) : y \in S\}. \tag{5.45}$$

We claim that $\{y_n\}$ is bounded.

Assume that $y_n \in S$ and show that $y_{n+1} \in S$. We denote

$$p_n = y_n - \epsilon_n(A_n y_n + \alpha_n y_n - f_n). \tag{5.46}$$

The operator Q_G is nonexpansive, therefore, by (5.41) we have

$$\begin{aligned}
\|y_{n+1} - x^*\| &\le \|Q_G p_n - Q_G x^*\| \\
&\le \|y_n - x^* - \epsilon_n(A_n y_n + \alpha_n y_n - f_n)\| \\
&\le \|y_n - x^*\| + \epsilon_n\|A_n y_n + \alpha_n y_n - f_n\| \\
&\le \|y_n - x^*\| + \|Ay_n\| + \|A_n y_n - Ay_n\| \\
&\quad + \|y_n - x^*\| + \|x^*\| + \|f\| + \|f_n - f\|.
\end{aligned} \tag{5.47}$$

Then

$$\begin{aligned}
\|y_{n+1} - x^*\| &\le r + \sup_{y\in S}\|Ay\| + \omega_n \sup_{y\in S}\zeta(\|y\|) + \mu_n + r + \frac{1}{2}r + \|f\| + \delta_n \\
&\le r + M = \overline{M},
\end{aligned} \tag{5.48}$$

$$\|y_{n+1} - y_n\| = \varepsilon_n\|A_n y_n + \alpha_n y_n - f_n\| \le \varepsilon_n M. \tag{5.49}$$

Moreover, by (2.3), (2.6), and (5.41) one gets similarly to (5.10) that

$$\begin{aligned} \|y_{n+1}-x^*\|^2 &\leq \|y_n-x^*-\epsilon_n(A_ny_n+\alpha_ny_n-f_n)\|^2 \\ &\leq \|y_n-x^*\|^2-2\epsilon_n\langle A_ny_n-f_n,J(y_n-x^*)\rangle \\ &\quad -2\epsilon_n\alpha_n\langle y_n,J(y_n-x^*)\rangle+16LCM\overline{M}\rho_E(\varepsilon_n) \\ &\leq \|y_n-x^*\|^2-2\epsilon_n\langle Ay_n-f,J(y_n-x^*)\rangle \\ &\quad -2\epsilon_n\langle A_ny_n-Ay_n,J(y_n-x^*)\rangle-2\epsilon_n\langle f_n-f,J(y_n-x^*)\rangle \\ &\quad -2\epsilon_n\alpha_n\langle y_n,J(y_n-x^*)\rangle+16LCM\overline{M}\rho_E(\varepsilon_n). \end{aligned} \tag{5.50}$$

Since x^* is a solution of the equation $Ax=f$ on G and $y_n\in G$ for all $n\geq 0$, we can write

$$\langle Ay_n-f,J(y_n-x^*)\rangle\geq 0. \tag{5.51}$$

Then (5.50) implies the inequality

$$\begin{aligned} \|y_{n+1}-x^*\|^2 &\leq \|y_n-x^*\|^2-2\epsilon_n\alpha_n\langle y_n,J(y_n-x^*)\rangle \\ &\quad +2\epsilon_n(\omega_nM_\zeta+\mu_n+\delta_n)\|y_n-x^*\|+16LMC\overline{M}\rho_E(\varepsilon_n). \end{aligned} \tag{5.52}$$

Denoting $D=8LCM\overline{M}+rM_\zeta^+$, where $M_\zeta^+=\max\{1,M_\zeta\}$, we obtain

$$\langle y_n,J(y_n-x^*)\rangle\leq 8LCM\overline{M}\frac{\rho_E(\varepsilon_n)}{\epsilon_n\alpha_n}+\frac{r(\omega_nM_\zeta+\mu_n+\delta_n)}{\alpha_n}\leq DK, \tag{5.53}$$

because of the given inequalities

$$\frac{\rho_E(\epsilon_n)}{\alpha_n\epsilon_n}\leq K=d^2,\qquad \frac{\omega_n+\mu_n+\delta_n}{\alpha_n}\leq K=d^2,\quad \forall n\geq 0. \tag{5.54}$$

The rest of the boundedness proof of $\{y_n\}$ follows as in the proof of Theorem 5.1. Thus, there exists $\widetilde{C}$ such that $\|y_n\|\leq\widetilde{C}$.

We present next the convergence analysis of (5.41). By convexity of $\|x\|^2$, we obtain as in (5.25) the following:

$$\begin{aligned} \|y_{n+1}-z_{n+1}\|^2 &\leq \|y_{n+1}-z_n\|^2+2\langle z_{n+1}-z_n,J(z_{n+1}-y_{n+1})\rangle \\ &\leq \|y_{n+1}-z_n\|^2+2\langle z_{n+1}-z_n,J(z_n-y_{n+1})\rangle \\ &\quad +2\langle z_{n+1}-z_n,J(z_{n+1}-y_{n+1})-J(z_n-y_{n+1})\rangle \\ &\leq \|y_{n+1}-z_n\|^2+2\|z_n-y_{n+1}\|\cdot\|z_{n+1}-z_n\| \\ &\quad +(16\bar{c}^{-1}+C_1)\rho_E(\|z_{n+1}-z_n\|), \end{aligned} \tag{5.55}$$

where C_1 is defined by (5.29). Moreover,

$$\begin{aligned}
\|y_{n+1}-z_n\|^2 &= \|Q_G p_n - Q_G z_n\|^2 \le \|p_n - z_n\|^2\\
&\le \|y_n - z_n\|^2 + 2\langle p_n - y_n, J(p_n - z_n)\rangle\\
&= \|y_n - z_n\|^2 + 2\langle p_n - y_n, J(y_n - z_n)\rangle\\
&\quad + 2\langle p_n - y_n, J(p_n - z_n) - J(y_n - z_n)\rangle\\
&= \|y_n - z_n\|^2 - 2\epsilon_n\langle A_n y_n + \alpha_n y_n - f_n, J(y_n - z_n)\rangle\\
&\quad + 2\langle p_n - y_n, J(p_n - z_n) - J(y_n - z_n)\rangle\\
&\le \|y_n - z_n\|^2 - 2\epsilon_n\langle A z_n + \alpha_n z_n - f, J(y_n - z_n)\rangle\\
&\quad + 2\epsilon_n(\|A_n y_n - A y_n\| + \|f_n - f\|)\|y_n - z_n\| - 2\epsilon_n\alpha_n\|y_n - z_n\|^2\\
&\quad + 2\langle p_n - y_n, J(p_n - z_n) - J(y_n - z_n)\rangle.
\end{aligned} \tag{5.56}$$

Now (5.39), (5.46), and (5.56) yield

$$\begin{aligned}
\|y_{n+1}-z_n\|^2 &\le \|y_n - z_n\|^2 - 2\epsilon_n\alpha_n\|y_n - z_n\|^2 + 2\epsilon_n\|y_n - z_n\|(\omega_n\zeta(\|y_n\|) + \mu_n + \delta_n)\\
&\quad + 2\langle p_n - y_n, J(p_n - z_n) - J(y_n - z_n)\rangle\\
&\le \|y_n - z_n\|^2 - 2\epsilon_n\alpha_n\|y_n - z_n\|^2 + 2\epsilon_n\|y_n - z_n\|(\omega_n\zeta(\|y_n\|) + \mu_n + \delta_n)\\
&\quad + 16\|p_n - y_n\|^2 + C_4(n)\rho_E(\|p_n - y_n\|),
\end{aligned} \tag{5.57}$$

where

$$C_4(n) = 8\max\{2L, \|z_n - y_n\| + \|z_n - p_n\|\}. \tag{5.58}$$

Since $\{y_n\}$ and $\{z_n\}$ are bounded and A is a bounded operator, there exists a constant $d_1 > 0$ such that

$$\begin{gathered}
\|p_n - y_n\| = \epsilon_n\|A_n y_n + \alpha_n y_n - f_n\| \le \epsilon_n d_1,\\
C_4(n) \le 8\max\{2L, \widetilde{C} + M + 2\|x^*\|\} = C_4.
\end{gathered} \tag{5.59}$$

Then

$$16\epsilon_n^2 d_1^2 + C_4(n)\rho_E(\epsilon_n d_1) \le C_5\rho_E(\epsilon_n d_1), \tag{5.60}$$

where

$$C_5 = (16\bar{d}^{-1} + C_4), \qquad \bar{d} = \left(\sqrt{1+d_1^2} + 1\right)^{-1}. \tag{5.61}$$

Substituting (5.57) for (5.55), we deduce that

$$\begin{aligned}||y_{n+1}-z_{n+1}||^2 \le (1-2\epsilon_n\alpha_n)||y_n-z_n||^2 &+ 2\epsilon_n||y_n-z_n||(\omega_n\zeta(||y_n||)+\delta_n+\mu_n)\\ &+C_5\rho_E(\epsilon_n d_1)+C_6\rho_E(||z_{n+1}-z_n||)\\ &+2||z_n-y_{n+1}||\cdot||z_{n+1}-z_n||,\end{aligned} \tag{5.62}$$

where $C_6 = 16\bar{c}^{-1}+C_1$. Finally, there exists $C>0$ such that $\|y_n-z_n\|\le C$ and then

$$\begin{aligned}||y_{n+1}-z_{n+1}||^2 \le ||y_n-z_n||^2 - 2\epsilon_n\alpha_n||y_n-z_n||^2 + 2\epsilon_n C(\omega_n\zeta(\widetilde{C})+\delta_n+\mu_n)\\ +C_5\rho_E(\epsilon_n d_1)+2||x^*||(C_6+C)\frac{|\alpha_n-\alpha_{n+1}|}{\alpha_n},\end{aligned} \tag{5.63}$$

because $\rho_E(\tau)\le\tau$. Now, the conclusion $\|y_n-z_n\|\to 0$ follows from Lemma 2.3. By Theorem 3.8,

$$||y_n-x^*||\le||y_n-z_n||+||z_n-x^*||\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{5.64}$$

Thus, $\{y_n\}$ converges strongly to x^*. The proof is accomplished. □

5.3. Next we study the iterative regularization method for (3.1) defined by

$$w_{n+1} := Q_{G_{n+1}}(w_n-\epsilon_n(Aw_n+\alpha_n w_n-f)), \quad n=0,1,2,\dots, \tag{5.65}$$

on approximately given sets G_n, where, for each n, Q_{G_n} is a sunny nonexpansive retraction of E onto G_n.

THEOREM 5.4. *Assume that all the conditions of Theorems 3.8 and 5.1 are fulfilled and* $G_n\subseteq \operatorname{int}D(A)$, $n=1,2,\dots$, *are closed convex sets such that the Hausdorff distance* $\mathcal{H}_E(G_n,G)\le\sigma_n\le\bar{\sigma}$, $\sigma_{n+1}\le\sigma_n$. *Denote* $\overline{G}=G\bigcup\widetilde{G}$, *where* $\widetilde{G}=\bigcup G_n$. *Assume that an operator* $A:\overline{G}\to E$ *is accretive and bounded. Assume that conditions (5.22) hold and that*

$$\frac{\sqrt{h_E(\sigma_n)}}{\epsilon_n\alpha_n}\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{5.66}$$

If the iterative sequence $\{w_n\}$ *generated by (5.65) is bounded, then it converges strongly to* $\bar{x}^*$, *where* $\bar{x}^*$ *is the unique solution of inequality (3.9).*

Proof. Denote

$$Zx = x-\epsilon_n(Ax+\alpha_n x-f). \tag{5.67}$$

Since $\{w_n\}$ is bounded and, hence, $\{Aw_n\}$ is also bounded, then there exists a constant $d>0$ such that

$$||Zw_n|| = ||w_n-\epsilon_n(Aw_n+\alpha_n w_n-f)||\le d. \tag{5.68}$$

By analogy,

$$||Zx_n|| = ||x_n-\epsilon_n(Ax_n+\alpha_n x_n-f)||\le d, \tag{5.69}$$

where $\{x_n\}$ is the bounded sequence generated by (5.1) (see Theorem 5.2). From (5.65) and (5.1), we have

$$\begin{aligned}||w_{n+1}-x_{n+1}|| &= ||Q_{G_{n+1}}Zw_n - Q_G Zx_n|| \\ &\le ||Q_G Zw_n - Q_G Zx_n|| + ||Q_{G_{n+1}}Zw_n - Q_G Zw_n||.\end{aligned} \tag{5.70}$$

Estimate the first term of the right-hand side of the previous inequality:

$$\begin{aligned}||Q_G Zw_n - Q_G Zx_n||^2 &\le ||Zw_n - Zx_n||^2 \\ &\le ||w_n - x_n||^2 - 2\epsilon_n\langle Aw_n - Ax_n + \alpha_n(w_n - x_n), J(Zw_n - Zx_n)\rangle \\ &\le ||w_n - x_n||^2 - 2\epsilon_n\alpha_n||w_n - x_n||^2 - 2\epsilon_n\langle Aw_n - Ax_n, J(w_n - x_n)\rangle \\ &\quad - 2\epsilon_n\langle Aw_n - Ax_n + \alpha_n(w_n - x_n), J(Zw_n - Zx_n) - J(w_n - x_n)\rangle \\ &\le ||w_n - x_n||^2 - 2\epsilon_n\alpha_n||w_n - x_n||^2 + 16\epsilon_n^2||Aw_n - Ax_n + \alpha_n(w_n - x_n)||^2 \\ &\quad + C_7(n)\rho_E(\epsilon_n||Aw_n - Ax_n + \alpha_n(w_n - x_n)||),\end{aligned} \tag{5.71}$$

where

$$C_7(n) = 8\max\{2L, ||Zw_n - Zx_n|| + ||w_n - x_n||\}. \tag{5.72}$$

Thus, there exists a constant $C_7 > 0$ such that $C_7(n) \le C_7$.

Using Lemma 2.7, we come to the following inequality:

$$||Q_{G_{n+1}}Zw_n - Q_G Zw_n|| \le 16(R+\bar{\sigma})(2d+q)h_B(8q^{-1}\sigma_{n+1}), \tag{5.73}$$

where $R = 2(2d+q)$, $q = \max\{q_1, q_2\}$, $q_1 = \operatorname{dist}(\theta, G_1)$, $q_2 = \max\{\operatorname{dist}(\theta, G_n)\}$, $n = 0, 1, 2, \dots$, and θ is the origin of the Banach space E. Hence, from (5.70), (5.71), and (5.73) we get

$$\begin{aligned}||w_{n+1}-x_{n+1}||^2 &\le ||w_n - x_n||^2 - 2\epsilon_n\alpha_n||w_n - x_n||^2 + 16\epsilon_n^2 C + C_7\rho_E(\epsilon_n C) \\ &\quad + (\bar{C}h_B(8q^{-1}\sigma_{n+1}))^{1/2},\end{aligned} \tag{5.74}$$

for $\bar{C} = 16(R+\bar{\sigma})(2d+q)$ and some $C > 0$.

Since (5.66) holds, we conclude that $\|w_n - x_n\| \to 0$ as $n \to \infty$. Thus, for all $n \ge 0$, one has

$$||w_{n+1} - \bar{x}^*|| \le ||w_{n+1} - x_{n+1}|| + ||x_{n+1} - \bar{x}^*||, \tag{5.75}$$

and, therefore,

$$||w_n - \bar{x}^*|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.76}$$

The proof is complete. □

Remark 5.5. Obviously, if G_n are bounded, then all w_n are bounded too.

Now we are able to combine Theorems 5.2, 5.3, and 5.4 in order to investigate the iterative regularization method for (3.1) with perturbed data A, f, and G defined by the following algorithm:

$$u_{n+1} := Q_{G_{n+1}}\big(u_n - \epsilon_n(A_n u_n + \alpha_n u_n - f_n)\big), \quad n = 0,1,2,\ldots. \tag{5.77}$$

Theorem 5.6. *Suppose that the conditions of Theorems 5.2, 5.3, and 5.4 are fulfilled. If the iterative sequence $\{w_n\}$ generated by (5.77) is bounded, then it converges strongly to $\bar{x}^*$, where $\bar{x}^*$ is the unique solution of inequality (3.9).*

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