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Research Article

Stable Iteration Procedures in Metric Spaces which Generalize a Picard-Type Iteration

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This paper investigates the stability of iteration procedures defined by continuous functions acting on self-maps in continuous metric spaces. Some of the obtained results extend the contraction principle to the use of altering-distance functions and extended altering-distance functions, the last ones being piecewise continuous. The conditions for the maps to be contractive for the achievement of stability of the iteration process can be relaxed to the fulfilment of being large contractions or to be subject to altering-distance functions or extended altering functions.

1. Introduction

Banach contraction principle is a very basic and useful result of Mathematical Analysis [1–7]. Basic applications of this principle are related to stability of both continuous-time and discrete-time dynamic systems [4, 8], including the case of high-complexity models for dynamic systems consisting of functional differential equations by the presence of delays [4, 9]. Several generalizations of the contraction principle are investigated in [2] by proving that the result still holds if altering-distance functions [1] are replaced with a difference of two continuous monotone nondecreasing real functions which take zero values only at the origin. The so-called n-times reasonable expansive mappings and the associated existence of unique fixed points are investigated in [7]. The so-called Halpern's iteration [10] and several of its extensions in the context of fixed-point theory have been investigated in [11–13]. Further extended viscosity iteration schemes with nonexpansive mappings based on the above one have been investigated in [9, 10, 12–18], while proving the common existence of unique fixed points for the related schemes and the strong convergence of the iterations to those points for any arbitrary initial conditions. The stability of Picard iteration has been investigated exhaustively (see, e.g., [5, 19–22]). The Picard and approximate Picard methods have been also used in classical papers for proving the existence and uniqueness of solutions in many differential equations including those of Sobolev type (see, e.g., [23]).

This paper presents some generalizations of results concerning the stability of iterations in the sense that the iteration scheme subject to error sequences converges asymptotically to its nominal fixed point provided that the iteration error converges asymptotically to zero. Several generalizations are discussed in the framework of stability of iteration schemes in complete metric spaces including:

(a) the use of altering-distance functions (Definition 1.1) [1, 2], and the so-called then defined extended altering functions (Definition 2.1 in Section 2) where the continuous altering functions are allowed to be piecewise continuous;

(b) the use of iteration schemes which are based on continuous functions which modify the Picard iteration scheme [5, 6];

(c) the removal of the common hypothesis in the context of T-stability that the set of fixed points of the iteration scheme is nonempty by guaranteeing that this is in fact true under contractive mappings, large contractions, or altering- and extended altering-distance functions, [1–4, 6].

Definition 1.1 (see [1] (*altering-distance function*)). A monotone nondecreasing function $\varphi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$, with $\varphi(x) = 0$, if and only if $x = 0$, is said to be an altering-distance function.

If (X, d) is a complete metric space, $T : X \to X$ is a self-mapping on X, and $\varphi(d(Tx, Ty)) \leq c\varphi(d(x, y))$, for all $x, y \in X$ and some real constant $c \in (0, 1)$, then T has a unique fixed point [1, 2]. This result is extendable to the use of monotone nondecreasing functions $\varphi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ satisfying

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \phi(d(x, y)), \quad \forall x, y \in X, \tag{1.1}$$

for some monotone nondecreasing function $\varphi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ satisfying $\phi(t) = \varphi(t) = 0 \Leftrightarrow t = 0$. Those results are directly extended to monotone nondecreasing piecewise continuous functions being continuous at "0" after a preliminary "ad hoc" definition in the subsequent section.

2. Fixed Point Properties Related to Altering- and Extended Altering-Distance Functions

Since $\varphi \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ but continuous at $t = 0$, it can possess bounded isolated discontinuities on $\mathbf{R}_+$ and it is necessary to reflect this fact in the notation as follows. The left (resp., right) limit of φ at $t = d(x, y)$ is simply denoted by $\varphi(d(x, y))$, instead of using the more cumbersome classical notation $\varphi((d(x, y))^-)$ (resp., by $\varphi^+(d(x, y))$ instead of using the more cumbersome $\varphi((d(x, y))^+)$). Since φ is an extended altering-distance function, then continuous at $t = 0$, $\varphi(0) = \varphi(0^+) = 0$. If φ is continuous at a given $t = d(x, y) > 0$, then $\varphi^+(t) = \varphi(t^+) = \varphi(t)$. If φ is has a discontinuity point (of second class), then $\varphi(t) \neq \varphi^+(t)$, with $|\varphi^+(t) - \varphi(t)| < \infty$.

Definition 2.1 (extended altering-distance function). A monotone nondecreasing function $\varphi \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ being continuous at "0", with $\varphi(x) = 0$, if and only if $x = 0$, is said to be an extended altering-distance function.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping on X. Then, the following properties hold.*

(i) *Assume that $\varphi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ is an altering-distance function such that $\varphi(d(Tx, Ty)) \leq c\varphi(d(x, y))$, for all $x, y \in X$ for some real constant $c \in (0, 1)$. Then T has a unique fixed point [1].*

(ii) *Assume that $\varphi \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ is an extended altering-distance function such that $\varphi(d(Tx, Ty)) \leq c(d(x, y))\varphi(d(x, y))$ and $\varphi^+(d(Tx, Ty)) \leq c^+(d(x, y))\varphi^+(d(x, y))$, for all $x, y \in X$ for some real function*

$$c \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+} \cap [0, 1)) \tag{2.1}$$

defined by

$$c(d(x,y)) = \begin{cases} 1 - \dfrac{\phi(d(x,y))}{\varphi(d(x,y))}, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$
$$c^+(d(x,y)) = \begin{cases} 1 - \dfrac{\phi(d(x,y))}{\varphi^+(d(x,y))}, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases} \tag{2.2}$$

for all $x, y \in X$ for some monotone nondecreasing function $\phi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ satisfying $\phi(t) < \varphi(t)$, for all $t \in \mathbf{R}_+$ and $\phi(t) = \varphi(t) = 0$, if and only if $t = 0$. Then $c : \mathbf{R}_{0+} \to \mathbf{R}_{0+} \cap [0, 1)$ is monotone nondecreasing and T has a unique fixed point. In particular, if $\varphi(d(x, y)) = d(x, y)$ so that

$$c(d(x,y)) = \begin{cases} 1 - \dfrac{\phi(d(x,y))}{d(x,y)}, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases} \tag{2.3}$$

for all $x, y \in X$ for some monotone nondecreasing function $\phi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ satisfying $\phi(d(x, y)) < d(x, y)$, for all $x, y \neq x \in X$ and $\phi(d(x, y)) = 0$, if and only if $y = x \in X$, then T has a unique fixed point.

Proof of Property (ii). Note that $c(d(x, x)) = 1 - \lim_{y \to x}\phi(d(x, y))/(\varphi(d(x, y))) = 1 - \phi'(0)/\varphi'(0) = 0$ from l'Hopital rule and the fact that both functions φ and ϕ are continuous at

"0" with $\phi(0) = \varphi(0) = 0$. Note that after taking left and right limits at each nonnegative real argument

$$\begin{aligned} 1 > c\left(d\left(x^{'}, y^{'}\right)\right) &= 1 - \frac{\phi\left(d\left(x^{'}, y^{'}\right)\right)}{\varphi(d(x^{'}, y^{'}))} \geq c^{+}(d(x, y)) = 1 - \frac{\phi(d(x, y))}{\varphi^{+}(d(x, y))} \\ &\geq c(d(x, y)) = 1 - \frac{\phi(d(x, y))}{\varphi(d(x, y))}, \\ c^{+}(d(z, z)) = 1 - \frac{\phi(d(z, z))}{\varphi^{+}(d(z, z))} &= c(d(z, z)) = 1 - \frac{\phi(d(z, z))}{\varphi(d(z, z))} = 0, \end{aligned} \tag{2.4}$$

for all $x, x^{'}(\neq x), y, y^{'}(\neq y), z \in X$, such that $d(x^{'}, y^{'}) \geq d(x, y)$, since

$$\begin{gathered} 0 < \phi(d(x, y)) < \varphi(d(x, y)) \leq \varphi^{+}(d(x, y)) \leq \varphi\left(d\left(x^{'}, y^{'}\right)\right) > \phi\left(d\left(x^{'}, y^{'}\right)\right) \\ \varphi^{+}(d(z, z)) = \varphi(d(z, z)) = \phi(d(z, z)) = 0, \quad \forall x, x^{'}(\neq x), y, y^{'}(\neq y) \in X. \end{gathered} \tag{2.5}$$

Then, $c : \mathbf{R}_{0+} \rightarrow \mathbf{R}_{0+} \cap [0, 1)$ is monotone nondecreasing from simple inspection of the above properties. Thus,

$$\varphi(d(Tx, Ty)) \leq c\varphi(d(x, y)) \leq c^{+}(d(x, y))\varphi(d(x, y)) \leq c^{+}(d(x, y))\varphi^{+}(d(x, y)) \tag{2.6}$$

so that

$$\varphi(d(Tx, Ty)) \leq \varphi^{+}(d(Tx, Ty)) \leq c^{+}(d(x, y))\varphi^{+}(d(x, y)) < \varphi^{+}(d(x, y)), \quad \forall x(\neq y) \in X. \tag{2.7}$$

Now, it is proven by contradiction that there is no $\varepsilon \in \mathbf{R}_{+}$ such that $d(x, y) \geq \varepsilon$, for any given $x(\neq y) \in X$. Take two arbitrary $x_0(\neq y_0) \in X$. Assume that $\varphi^{+}(d(T^j x_0, T^j y_0)) \geq \varepsilon$, for all $j \in \overline{k} \cup \{0\}$, and some given $k \in \mathbf{Z}_{+}$, so that if $\varphi^{+}(d(T^j x_0, T^j y_0)) \geq \varepsilon$ also for all $j \in \mathbf{Z}_{0+}$, then for some $\varepsilon_0 \in \mathbf{R}_{0+}$,

$$\begin{aligned} \varepsilon \leq \varphi^{+}\left(d\left(T^{k+N}x_0, T^{K+N}y_0\right)\right) &\leq \prod_{j=1}^{N}\left[c^{+}\left(d\left(T^{k+j}x_0, T^{k+j}y_0\right)\right)\right]\varphi^{+}(d(x, y)) \\ &= (\varepsilon + \varepsilon_0)\prod_{j=1}^{N}\left[c^{+}\left(d\left(T^{k+j}x_0, T^{k+j}y_0\right)\right)\right], \end{aligned} \tag{2.8}$$

for all $N \in \mathbf{Z}_{0+}$. But, it always exist a finite $N_0 \in \mathbf{Z}_{0+}$ such that $\prod_{j=1}^{N}[c^{+}(d(T^{k+j}x_0, T^{k+j}y_0))] < \varepsilon/(\varepsilon + \varepsilon_0) \leq 1$, for all $N(\geq N_0) \in \mathbf{Z}_{0+}$ since $0 < c^{+}(d(x_j, y_j)) < 1$; $x_j = T^j x_0(\neq y_j = T^j y_0) \in X$, what leads to a contradiction. Thus, there is no $\varepsilon \in \mathbf{R}_+$ such that $\varphi^+(d(T^k x_0, T^k y_0)) \geq \varepsilon$, for all $k \in \mathbf{Z}_{0+}$ for any given $x_0 (\neq y_0) \in X$. As a result, the subsequent relations are true:

$$\lim_{k\to\infty} \prod_{j=1}^{k} \left[c^+\left(d\left(T^j x_0, T^j y_0 \right) \right) \right] = 0$$
$$\Longrightarrow \lim_{k\to\infty} \varphi\left(d\left(T^k x_0, T^k y_0 \right) \right) = \lim_{k\to\infty} \varphi^+\left(d\left(T^k x_0, T^k y_0 \right) \right) = 0 \Longleftrightarrow \lim_{k\to\infty} d\left(T^k x_0, T^k y_0 \right) = 0, \tag{2.9}$$

for all $x_0 (\neq y_0) \in X$, with the above limits since $\varphi(t)$ is continuous at $t = 0$ and $\varphi(t) = 0$, if and only if $t = 0$. Furthermore, any sequence $\{x_k\}$ with $x_k = T^k x$, for all $x \in X$, is a Cauchy sequence since for any arbitrarily small prefixed constant $\varepsilon \in \mathbf{R}_+$, there exist sequences $\{N_k\}$, $\{n_k\}$, and $\{m_k\}$ of nonnegative integers satisfying $\mathbf{Z}_{0+} \ni N_k \to \infty$; $n_k > N_k, m_k > n_k$, such that

$$\varphi^+\left(d\left(T^{m_k+n_k+1} x, T^{n_k+1} x \right) \right) \leq \prod_{j=1}^{n_k} \left[c^+\left(d\left(T^{m_k+j} x, T^j x \right) \right) \right] \leq \varepsilon. \tag{2.10}$$

Thus, there is a unique $z \in \operatorname{cl} X$ which is in $F(T)$, the set of fixed points of T, that is, $z = Tz = \lim_{k\to\infty} T^k x$, for all $x \in X$. Since (X, d) is a complete metric space and the sequence $\{x_k\}$ with $x_k = T^k x$ is a Cauchy sequence, for all $x \in X$, then $X \supset F(T) = \{z\}$. It holds trivially that all the above proof also holds for special case $\varphi(d(x, y)) = d(x, y)$ and some monotone nondecreasing function $\phi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ satisfying $\phi(d(x, y)) < d(x, y)$ (i.e., $T : X \to X$ is a weak contraction) as may be proven [3] (see also [24]). Property (ii) has been fully proven. □

Theorem 2.2 might be linked to the concept of large contraction which is less restrictive than that of contraction. The related discussion follows.

Definition 2.3 (see [4] (*large contraction*)). Let (X, d) be a complete metric space. Then, the self-mapping $T : X \to X$ on X is said to be a large contraction, if $d(Tx, Ty) < d(x, y)$, for all $x (\neq y) \in X$, and if for any given $\varepsilon \in \mathbf{R}_+$, such that $d(Tx, Ty) \geq \varepsilon$, then there exist $\delta = \delta(\varepsilon) \in [0, 1)$, such that $d(Tx, Ty) \leq \delta d(x, y)$.

It turns out that a contraction is also a large contraction with $\delta \in [0, 1)$ being independent of ε in Definition 2.3. The following result proves that the self-mapping T on X satisfying Theorem 2.2(ii) is a large contraction.

Proposition 2.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping on X. If $\varphi \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ is a modified altering-distance function which satisfies the conditions of Theorem 2.2(ii), then T is a large contraction.*

Proof. Given $\varphi(d(Tx, Ty)) \leq c(d(x, y))\varphi(d(x, y)) < \varphi(d(x, y))$, for all $x (\neq y) \in X$, since $c(d(x, y)) < 1$, if $d(x, y) > 0$. Since $\varphi \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ is an extended altering-distance function it is monotone nondecreasing of nonnegative values and taking the zero value only at "0". Thus, $[\varphi(d(Tx, Ty)) < \varphi(d(x, y))$, for all $x (\neq y) \in X] \Rightarrow d(Tx, Ty) < d(x, y)$. Furthermore, it is proven by contradiction that for any given $\varepsilon \in \mathbf{R}_+$, such that $d(Tx, Ty) \geq \varepsilon$,

$\exists\delta = \delta(\varepsilon) < 1$, such that $d(Tx,Ty) \le \delta d(x,y)$. Take $x(\neq y) \in X$, such that $d(x,y) > 0$, and assume that $d(Tx,Ty) \ge d(x,y)$. Since $\varphi \in PC^{(0)}(\mathbf{R}_{0+},\mathbf{R}_{0+})$ is monotone nondecreasing, then $\varphi(d(Tx,Ty)) \ge \varphi(d(x,y)) > 0$, for all $x(\neq y) \in X$, and one also gets that

$$\varphi^{+}(d(Tx,Ty)) \ge \max(\varphi(d(Tx,Ty)),\varphi^{+}(d(x,y))) \ge \varphi(d(x,y)), \quad \forall x(\neq y) \in X. \tag{2.11}$$

Then,

$$\begin{gathered}\varphi(d(x,y)) > \varphi(d(x,y)) - \phi(d(x,y)) \ge \varphi(d(Tx,Ty)) \ge \varphi(d(x,y)) > 0,\\ \varphi^{+}(d(x,y)) > \varphi^{+}(d(x,y)) - \phi(d(x,y)) \ge \varphi^{+}(d(Tx,Ty)) \ge \varphi^{+}(d(x,y)) \ge \varphi(d(x,y)) > 0,\end{gathered} \tag{2.12}$$

which are two contradictions. Thus, $\rho = d(x,y) > 0 \Rightarrow d(Tx,Ty) \le \delta(\rho)d(x,y) < \rho$, for some $\delta(\rho) < 1$ and T is a large contraction. □

It is now proven that the sequence $\{d(x,T^k x)\}$ is uniformly bounded if Theorem 2.2(ii) holds.

Proposition 2.5. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping on X. If $\varphi \in PC^{(0)}(\mathbf{R}_{0+},\mathbf{R}_{0+})$ is a modified altering-distance function which satisfies the conditions of Theorem 2.2(ii), then $d(x,T^k x) \le L < \infty$, for all $x \in X$ and $k \in \mathbf{Z}_{0+}$.*

Proof. Proceed by contradiction by assuming that $d(x,T^k x) \le L < \infty$, for all $x \in X$ and $k \in \mathbf{Z}_{0+}$, is false so that $\{d(x,T^k x)\}$, $k \in \mathbf{Z}_{0+}$ is unbounded. Thus, there is a subsequence $\{T^{j_k}\}_{j_k \in \mathbf{Z}_\alpha \subset \mathbf{Z}_{0+}}$ of self-mappings on X, with $\mathbf{Z}_\alpha \ni j_k \to \infty$, as $\mathbf{Z}_{0+} \ni k \to \infty$, such that the real subsequence $\{d(x,T^{j_k}x)\}_{j_k \in \mathbf{Z}_\alpha}$ is strictly monotone increasing so that it diverges to $+\infty$, so that $L_{k+1} = d(x,T^{k+1}x) > L_k$ and $L_k \to \infty$, as $Z_\alpha \ni k \to \infty$. Since $\varphi \in PC^{(0)}(\mathbf{R}_{0+},\mathbf{R}_{0+})$ is monotone nondecreasing, one gets

$$\varphi^{+}(L_{k+1}) \ge \max(\varphi(L_{k+1}),\varphi^{+}(L_k)) \ge \varphi(L_k), \tag{2.13}$$

since $\varphi \in PC^{(0)}(\mathbf{R}_{0+},\mathbf{R}_{0+})$ is monotone nondecreasing. Since the inequalities are nonstrict, the above subsequences might either converge to nonnegative real limits $\varphi_\infty(x,z)$ and $\varphi^{+}_\infty(x,z)$, or diverge to $+\infty$. The event that $\varphi^{+}_\infty(x,z)$ and $\varphi_\infty(x,z)$ are one finite and the other infinity is not possible since $\varphi \in PC^{(0)}(\mathbf{R}_{0+},\mathbf{R}_{0+})$ so that any existing discontinuity is a finite-jump type discontinuity. Thus, both limits are either finite, although eventually distinct or both are $+\infty$ so that $\varphi(L_k) \to \varphi_\infty(x,z) \le \infty$ and $\varphi^{+}(L_k) \to \varphi^{+}_\infty(x,z) \le \infty$ (and simultaneously finite or infinity) as $\mathbf{Z}_\alpha \ni k \to \infty$, where $z = z(x) = Tx \in X$ for the given $x \in X$. Such a z always exists in X for each given $x \in X$ since T is a self-mapping on X. Then,

$$+\infty \ge \varphi_\infty(x,z) \longleftarrow \varphi(L_{k+1}) \le \varphi(L_k) - \phi(L_k) \longrightarrow \varphi_\infty(x,z) - \phi(L_k), \quad \text{as } \mathbf{Z}_\alpha \ni k \longrightarrow \infty, \tag{2.14}$$

which is a contradiction unless $\phi(L_k) \to 0$, as $\mathbf{Z}_\alpha \ni k \to \infty \Rightarrow L_k \to 0$, as $\mathbf{Z}_\alpha \ni k \to \infty$, since ϕ is continuous at $t = 0$ and $\phi(t) = 0$, if and only if $t = 0$. But, if the subsequence $\{L_k\}_{k \in \mathbf{Z}_\alpha}$ has a zero limit as $\mathbf{Z}_\alpha \ni k \to \infty$, then it is a bounded sequence. Thus, $L_k \to \infty$ as $\mathbf{Z}_\alpha \ni k \to \infty$ is false and then $\{d(x, T^k x)\}$, $k \in \mathbf{Z}_{0+}$ being unbounded fails so that the contradiction follows. The right-limit convergence $\varphi^+(L_k) \rightarrow \varphi^+_\infty(x, z) \le \infty$ leads to the same conclusion. As a result, there is no $x \in x$ such that $\{d(x, T^k x)\}$, $k \in \mathbf{Z}_{0+}$ is unbounded and the result is fully proven. □

An alternative proof to that of Theorem 2.2(ii) related to the existence of a unique fixed point in X, follows directly by using Theorem 1.2.4 in [4] since T is a large contraction and the sequence $\{d(x, T^k x)\}$ is uniformly bounded (Propositions 2.4 and 2.5).

Proposition 2.6. *Let (x, d) be a complete metric space and $T : X \rightarrow X$ be a large contraction. If $\varphi \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ is a modified altering-distance function which satisfies the conditions of Theorem 2.2(ii), then T has a unique fixed point in X.*

Proof. T is a large contraction from Proposition 2.4, since it fulfils Theorem 2.2(ii). Also, $d(x, T^k x) \le L < \infty$, for all $x \in X$ and $k \in \mathbf{Z}_{0+}$ from Proposition 2.5. Thus, from [4, Theorem 1.2.4], T has a unique fixed point in X. □

The following result is a direct consequence of Theorem 2.2, Propositions 2.4 and 2.5.

Proposition 2.7. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a weak contraction on X. Then, $d(x, T^k x) \le L < \infty$, for all $x \in X$ and $k \in \mathbf{Z}_{0+}$ and T has a unique fixed point on X.*

3. (f, T)-Stability Related to a Class of Nonlinear Iterations Related to Distance and Altering-Distance Functions

Assume that (X, d) is a complete metric space, $T : X \rightarrow X$ is a self-mapping on X. The iteration process $x_{k+1} = f(Tx_k)$ has a fixed point if $F(f, T) := \{z \in x : z = f(Tz)\} \neq \emptyset$. A necessary condition for $f : X \rightarrow X$ to have a fixed point is that it to be injective. The T-stability of the Picard iteration has been investigated in a set of papers (see, e.g., [5, 19, 20]). The Picard iteration is said to be T-stable if $\lim_{k\to\infty} d(x_{k+1}, Tx_k) = 0 \Rightarrow \lim_{k\to\infty} x_k = z \in X$, for all $x_0 \in X$. The subsequent result is an extension of a previous one in [5] for the so-called (f, T)-stability of the iteration $x_{k+1} = f(Tx_k)$, for all $k \in \mathbf{Z}_{0+}$ if the pair (f, T) satisfies the so-called (L, h) property defined by

$$d\big(f(Tx), q\big) \le Ld\big(f(Tx), x\big) + hd(x, q) \tag{3.1}$$

for all $x \in X$; $q \in F(f, T) := \{z \in X : z = f(Tz)\}$, with $0 \le h < 1$ and $L \ge 0$, provided that the set of fixed points $F(f, T)$ is nonempty. If $f : X \rightarrow X$ is identity, then the above property is stated as T satisfying the (L, h) property.

Theorem 3.1. *Assume that*

(1) *(X, d) is a complete metric space, $f : X \rightarrow X$ is a continuous mapping, and $T : X \rightarrow X$ is a self-mapping on X such that the set of fixed points $F(f, T)$ of the iteration procedure $x_{k+1} = f(Tx_k)$, for all $k \in \mathbf{Z}_{0+}$, is nonempty;*

(2) *the pair (f, T) satisfies the (L, h) property; that is, $d(f(Tx), q) \le Ld(f(T, x), x) + hd(x, q)$; for all $q \in F(f, T) := \{z \in X : z = f(Tz)\}$ with $0 \le h < 1$ and $L \ge 0$;*

(3) *$\lim_{k\to\infty} d(f(Tx_k), x_k) = 0$, for all $x_0 \in X$.*

Then, the iteration procedure $x_{k+1} = f(Tx_k)$*, for all* $k \in \mathbf{Z}_{0+}$*, is* (f,T)*-stable and it possesses a unique fixed point.*

Proof. For any given $q \in F(f,T)$, which exists since $F(f,T) \neq \emptyset$, and for all $x_k \in X$ such that $x \ni x_{k+1} = f(Tx_k) + \varepsilon_k$ with $\{\varepsilon_k\}$ being the computation error sequence, one has

$$\begin{aligned} d(f(Tx_k),q) &\leq Ld(f(Tx_k),x_k) + hd(x_k,q) \\ &\leq (L+h)d(f(Tx_k),x_k) + hd(q,f(Tx_k)) \\ &\Longrightarrow (1-h)d(f(Tx_k),q) \leq (L+h)d(f(Tx_k),x_k). \end{aligned} \tag{3.2}$$

Since $0 \leq h < 1$, $L \geq 0$, and $d(f(Tx_k),x_k) \to 0$, as $k \to \infty$, then,

$$d(f(Tx_k),q) \longrightarrow d(f(Tx_k),x_k) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty \tag{3.3}$$

from (3.2), [5, 6]. Also, $d(q,x_k) \leq d(x_k,f(Tx_k)) + d(q,f(Tx_k)) \to 0$, as $k \to \infty \Rightarrow d(q,x_k) \to 0$, as $k \to \infty$. Also,

$$\begin{aligned} d(f(Tx_k),x_{k+1}) &\leq d(f(Tx_k),q) + d(q,x_{k+1}) \\ &\leq (L+h)d(f(Tx_k),x_k) + hd(q,f(Tx_k)) + d(q,x_{k+1}) \\ &\leq (L+h)d(f(Tx_k),x_k) + hd(q,f(Tx_k)) + d(q,x_k) + d(x_k,x_{k+1}) \\ &\longrightarrow d(x_k,x_{k+1}), \end{aligned} \tag{3.4}$$

as $k \to \infty$, since $d(q,x_k) \to d(f(Tx_k),q) \to d(f(Tx_k),x_k) \to 0$, as $k \to \infty$. From the above inequalities either $d(f(Tx_k),x_{k+1}) \to d(x_k,x_{k+1}) \to 0$, as $k \to \infty$, or $\liminf_{k\to\infty} d(x_k,x_{k+1}) > 0$, with $d(f(Tx_k),x_k) \to 0$, as $k \to \infty$, but in this second case, $d(q,x_k) \to 0$, as $k \to \infty$, is false so that $q \notin F(f,T)$. Then, $d(f(Tx_k),x_{k+1}) \to d(x_k,x_{k+1}) \to 0$, as $k \to \infty$. Thus, $d(f(Tx_k),x_{k+1}) \to d(f(Tx_k),x_k) \to 0$, as $k \to \infty$ from (3.4). Then, $x_{k+1} \to f(Tx_k) \to x_k$ and $f(T,x_k) \to q$, as $k \to \infty$. Since $f : X \to X$ is injective, $x_k \to q$, as $k \to \infty$ so that f is T-stable. It is proven by contradiction that the fixed point of the iteration procedure $x_{k+1} = f(Tx_k)$, for all $k \in \mathbf{Z}_{0+}$ is unique. Assume that there exists $p \neq q \in F(f,T)$. Then,

$$0 \neq d(p,q) = d(f(Tp),q) \leq Ld(p,f(Tp)) + hd(p,q) = hd(p,q) < d(p,q) = 0, \tag{3.5}$$

what is impossible if $p \neq q$. Then, $F(f,T) = \{q\}$. □

Theorem 3.1 is now extended by extending the (L,h) property of the pair (f,T) to that of the triple (φ,f,T), where $\varphi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ is an appropriate continuous function.

Theorem 3.2. *Assume that*

(1) (X,d) *is a complete metric space,* $f : X \to X$ *is a continuous mapping, and* $T : X \to X$ *is a self-mapping on* X *such that the set of fixed points* $F(f,T)$ *of the iteration procedure* $x_{k+1} = f(Tx_k)$*, for all* $k \in \mathbf{Z}_{0+}$*, is nonempty;*

(2) $\varphi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *is continuous, satisfies* $\varphi(x) = 0 \Leftrightarrow x = 0$, *possesses the subadditive property, and, furthermore, the triple* (φ, f, T) *satisfies the* (L, h) *property defined by* $\varphi(d(f(Tx), q)) \leq L\varphi(d(f(T, x), x)) + h\varphi(d(x, q))$; *for all* $q \in F(f, T) := \{z \in X : z = f(Tz)\}$ *with* $0 \leq h < 1$ *and* $L \geq 0$;

(3) $\lim_{k \to \infty} d(f(Tx_k), x_k) = 0$, *for all* $x_0 \in X$.

Then, the iteration procedure $x_{k+1} = f(Tx_k)$, *for all* $k \in \mathbf{Z}_{0+}$ *is* (f, T)*-stable and it possesses a unique fixed point.*

Proof. For any given $q \in F(f, T)$, which exists since $F(f, T) \neq \emptyset$, and for all $x_k \in X$ such that $x \ni x_{k+1} = f(Tx_k) + \varepsilon_k$, with $\{\varepsilon_k\}$ being the computation error sequence and, since $\varphi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ possesses the sub-additive property, one has

$$\begin{aligned}
\varphi\big(d\big(f(Tx_k), q\big)\big) &\leq L\varphi\big(d\big(f(T, x_k), x_k\big)\big) + h\varphi\big(d(x_k, q)\big) \\
&\leq L\varphi\big(d\big(f(T, x_k), x_k\big)\big) + h\varphi\big(d\big(x_k, f(T, x_k)\big) + d\big(f(T, x_k), q\big)\big) \\
&\leq L\varphi\big(d\big(f(T, x_k), x_k\big)\big) + h\big(\varphi\big(d\big(x_k, f(T, x_k)\big)\big) + \varphi\big(d\big(f(T, x_k), q\big)\big)\big) \\
&\Longrightarrow (1-h)\varphi\big(d\big(f(Tx_k), q\big)\big) \leq (L+h)\varphi\big(d\big(f(Tx_k), x_k\big)\big) \\
&\Longrightarrow \ 0 \leftarrow \varphi\big(d\big(f(Tx_k), q\big)\big) \leq \frac{L+h}{1-h}\varphi\big(d\big(f(Tx_k), x_k\big)\big) \longrightarrow 0 \ \text{ as } k \longrightarrow \infty
\end{aligned} \tag{3.6}$$

according to Hypothesis (3) since $0 \leq h < 1$ and $L + h \geq 0$ from Hypothesis (2). Since $\varphi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ is everywhere continuous and satisfies $\varphi(x) = 0 \Leftrightarrow x = 0$, then $d(f(Tx_k), q) \to d(f(Tx_k), x_k) \to 0$ as $k \to \infty$. Also, $d(q, x_k) \leq d(x_k, f(Tx_k)) + d(q, f(Tx_k)) \to 0$ as $k \to \infty \Rightarrow d(q, x_k) \to 0$ as $k \to \infty$. The remaining of the proof follows with the same arguments as in that of Theorem 3.1. □

Theorem 3.3. *Assume that*

(1) (X, d) *is a complete metric space,* $f : X \to X$ *is a continuous mapping, and* $T : X \to X$ *is a self-mapping on* X *such that the set of fixed points* $F(f, T)$ *of the iteration procedure* $x_{k+1} = f(Tx_k)$, *for all* $k \in \mathbf{Z}_{0+}$, *is nonempty;*

(2) $\varphi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *and* $\phi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *are both continuous and monotone nondecreasing while satisfying* $\varphi(x) = \phi(x) = 0 \Leftrightarrow x = 0$, *and, furthermore, the quadruple* (φ, ϕ, f, T) *satisfies the* (L, h) *property:* $\varphi(d(f(Tx_k), q)) \leq \varphi(Ld(f(Tx_k), x_k) + hd(x_k, q)) - \phi(Ld(f(Tx_k), x_k) + hd(x_k, q))$, *for all* $x \in X$; *and for all* $q \in F(f, T) := \{z \in X : z = f(Tz)\}$ *with* $0 \leq h < 1$ *and* $L \geq 0$;

(3) $\lim_{k \to \infty} \varphi(d(f(Tx_k), q)) = 0$, *for all* $x_0 \in X$.

Then, the iteration procedure $x_{k+1} = f(Tx_k)$, *for all* $k \in \mathbf{Z}_{0+}$ *is* (f, T)*-stable and it possesses a unique fixed point.*

Proof. Since $\varphi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ and $\psi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ are both continuous and monotone nondecreasing then Hypothesis (2) implies that $d(f(Tx),q) \le Ld(f(T,x),x) + hd(x,q)$, for all $k \in \mathbf{Z}_{0+}$ and $x \in X$; for all $q \in F(f,T) := \{z \in X : z = f(Tz)\}$ with $0 \le h < 1$ and $L \ge 0$ which is Hypothesis (2) of Theorem 3.1. Furthermore, $\lim_{k\to\infty}\varphi(d(f(Tx_k),q)) = \varphi(\lim_{k\to\infty}d(f(Tx_k),q)) = 0 \Rightarrow \lim_{k\to\infty}d(f(Tx_k),q) = 0$ from the continuity of $\varphi : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ everywhere within its definition domain $\mathbf{R}_{0+}$ and its property $\varphi(x) = \psi(x) = 0 \Leftrightarrow x = 0$. Thus, the proof follows as in Theorem 3.1 since Hypothesis (1) to (3) of this theorem hold. □

The following direct particular result of Theorems 3.1 to 3.3 follows.

Corollary 3.4. *Theorems 3.1, 3.2, and 3.3 hold "mutatis-mutandis" stated for the function $f : X \to X$ being the identity mapping on X and $F(I,T) = F(T) \neq \emptyset$.*

Corollary 3.4 referred to Theorem 3.1 was first proven in [5]. It is now of interest the removal of the condition of the set of fixed points to be nonempty by guaranteeing that is in fact nonempty consisting of a unique element under extra contractive properties of the pair (f,T). The following result holds.

Theorem 3.5. *The following two properties hold.*

(i) *Consider the Picard T-iteration process $x_{k+1} = Tx_k$, for all $x \in X$ and $k \in \mathbf{Z}_{0+}$. If T satisfies the (L,h) property while it is a k-contraction (i.e., a contractive mapping with constant $0 \le k < 1$), then $F(T) = \{q\}$ for some $q \in X$, and, furthermore,*

$$d(Tx,q) \le \frac{L+h}{1-h}d(x,Tx), \quad d\left(T^2x,q\right) \le \frac{k(L+h)}{1-h}d(Tx,x), \tag{3.7}$$

$$\begin{aligned} d\left(T^2x,Tx\right) \le \min\Bigg(& kLd(x,Tx) + k(h+1)d(x,q), L(k+h+1)d(x,Tx) \\ & +h(h+1)d(x,q), \left(L + \frac{k(L+h)}{1-h}\right)d(Tx,x) + hd(x,q)\Bigg). \end{aligned} \tag{3.8}$$

(ii) *Consider the iteration process $x_{k+1} = f(Tx_k)$, for all $x \in X$ and $k \in \mathbf{Z}_{0+}$. If T satisfies the (L,h) property while the pair (f,T) is a k-contraction (i.e., a contractive mapping with constant $0 \le k < 1$), then $F(f,T) = \{q\}$ for some $q \in x$, and, furthermore,*

$$d(f(Tx),q) \le \frac{L+h}{1-h}d(x,f(Tx)), \quad d\left(f\left(T^2x\right),q\right) \le \frac{k(L+h)}{1-h}d(f(Tx),x), \tag{3.9}$$

$$\begin{aligned} d\left(f\left(T^2x\right),f(Tx)\right) \le \min\Bigg(& kLd(x,f(Tx)) + k(h+1)d(x,q), L(k+h+1)d(x,f(Tx)) \\ & +h(h+1)d(x,q), \left(L + \frac{k(L+h)}{1-h}\right)d(Tx,x) + hd(x,q)\Bigg). \end{aligned} \tag{3.10}$$

Proof. (i) Equation (3.7) follows from the (L,h) property leading to

$$\begin{aligned} d(Tx,q) &\le Ld(x,Tx) + hd(x,q) \le Ld(x,Tx) + hd(x,Tx) + hd(Tx,q) \\ &\Longrightarrow d(Tx,q) \le \frac{L+h}{1-h} d(x,Tx); d\left(T^2x,q\right) \\ &\le \frac{L+h}{1-h} d\left(T^2x,Tx\right) \le k\frac{L+h}{1-h} d(Tx,x), \end{aligned} \tag{3.11}$$

since T is k-contractive, so that it possesses a unique fixed point $q \in X$, and it satisfies the (L,h) property with $0 \le h < 1$. Equation (3.8) follows directly from the following three inequalities:

$$\begin{aligned} d\left(T^2x,Tx\right) &\le kd(Tx,x) \le k(d(Tx,q) + d(x,q)) \\ &\le k(Ld(x,Tx) + hd(x,q)) + kd(x,q) \le kLd(x,Tx) + k(h+1)d(x,q) \end{aligned} \tag{3.12}$$

For $x \ni z = Tx$, for all $x \in X$, one gets

$$\begin{aligned} d\left(T^2x,Tx\right) &\le d\left(T^2x,q\right) + d(q,Tx) \\ &= d(Tz,q) + d(q,z) \le Ld(z,Tz) + (h+1)d(z,q) \\ &\le Lkd(x,Tx) + (h+1)(Ld(x,Tx) + hd(x,q)) \\ &\le L(k+h+1)d(x,Tx) + h(h+1)d(x,q) \end{aligned} \tag{3.13}$$

after using the (L,h) property, and

$$\begin{aligned} d\left(T^2x,Tx\right) &\le d\left(T^2x,q\right) + d(Tx,q) \le \frac{k(L+h)}{1-h} d(Tx,x) + d(Tx,q) \\ &\le \frac{k(L+h)}{1-h} d(Tx,x) + Ld(Tx,x) + hd(x,q) \\ &\le \left(L + \frac{k(L+h)}{1-h}\right) d(Tx,x) + hd(x,q) \end{aligned} \tag{3.14}$$

again with the use of the (L,h) property.

(ii) Equation (3.9) follows from the (L,h) property leading to

$$\begin{aligned} d(f(Tx),q) &\le Ld(x,f(Tx)) + hd(x,q) \le Ld(x,f(Tx)) + hd(x,f(Tx)) + hd(f(Tx),q) \\ &\Longrightarrow d(f(Tx),q) \le \frac{L+h}{1-h} d(x,f(Tx)); \\ d\left(f\left(T^2x\right),q\right) &\le \frac{L+h}{1-h} d\left(f\left(T^2x\right),f(Tx)\right) \le k\frac{L+h}{1-h} d(f(Tx),x), \end{aligned} \tag{3.15}$$

since the pair (f,T) is k-contractive and it satisfies the (L,h) property with $0 \le h < 1$. Equation (3.10) follows directly from the following three inequalities:

$$\begin{aligned}
d\left(f\left(T^2x\right), f(Tx)\right) &\le kd(f(Tx), x) \le kd(q, f(Tx)) + d(x,q) \\
&\le k(Ld(f(Tx), x) + hd(x,q)) + khd(x,q) \\
&\le kLd(x, f(Tx)) + k(h+1)d(x,q),
\end{aligned} \tag{3.16}$$

since the pair (f, T) is k-contractive. Then,

$$\lim_{k\to\infty} d\left(f\left(T^{k+1}x\right), f\left(T^kx\right)\right) = d\left(\lim_{k\to\infty} Tf\left(T^kx\right), \lim_{k\to\infty} f\left(T^kx\right)\right) = 0 \Longrightarrow q = Tq = \lim_{k\to\infty} f\left(T^kx\right), \tag{3.17}$$

for all $x \in X$ and some $q \in X$ independent of $x_0 \in X$. $F(f,T) = \{q\}$. Therefore, for any $x \in X$, one gets

$$\begin{aligned}
d\left(f\left(T^2x\right), f(Tx)\right) &\le d\left(f\left(T^2x\right), q\right) + d(q, f(Tx)) \\
&\le Ld\left(f\left(T^2x\right), f(Tx)\right) + (h+1)d(f(Tx), q) \\
&\le Lkd(f(Tx), x) + (h+1)(Ld(f(Tx), x) + hd(x,q)) \\
&\le L(k+h+1)d(f(Tx), x) + h(h+1)d(x,q)
\end{aligned} \tag{3.18}$$

after using the (L,h) property, and

$$\begin{aligned}
d\left(f\left(T^2x\right), f(Tx)\right) &\le d\left(f\left(T^2x\right), q\right) + d(f(Tx), q) \\
&\le \frac{k(L+h)}{1-h} d(f(Tx), x) + d(f(Tx), q) \\
&\le \frac{k(L+h)}{1-h} d(f(Tx), x) + Ld(f(Tx), x) + hd(x,q) \\
&\le \left(L + \frac{k(L+h)}{1-h}\right) d(f(Tx), x) + hd(x,q)
\end{aligned} \tag{3.19}$$

again with the use of the (L,h) property. □

Theorem 3.5(ii) allows directly extending Theorem 3.1 as follows by removing the requirement of the set of fixed points to be nonempty with a unique element since this is guaranteed by the Banach contractive mapping principle.

Theorem 3.6. *Assume that: (x, d) is a complete metric space, $f : X \to X$ is a continuous mapping on X, and $T : X \to X$ is a self-mapping on X such that the pair (f, T) is k-contractive and satisfies the (L, h) property (provided that the set of fixed points is nonempty), that is,*

$$d\left(f\left(T^2x\right), f(Ty)\right) \leq kd(f(Tx), y), \quad d(f(Tx), q) \leq Ld(f(Tx), x) + hd(x, q), \quad \forall x, y \in X, \tag{3.20}$$

with q being any fixed point; that is, $q \in F(f, T) \subset X$ and some real constants $0 \leq k < 1$, $0 \leq h < 1$, and $L \geq 0$, and, furthermore,

$$\lim_{k \to \infty} d(f(Tx_k), x_k) = 0, \quad \forall x_0 \in X. \tag{3.21}$$

Then, the iteration procedure $x_{k+1} = f(Tx_k)$, for all $k \in \mathbf{Z}_{0+}$ is (f, T)-stable with q being its unique fixed point, that is, $x \supset F(f, T) = \{q\} \neq \emptyset$.

A direct particular case of Theorem 3.5 applies directly to the case of f being the identity map on X via Theorem 3.5(i).

Corollary 3.7. *Assume that (X, d) is a complete metric space, and $T : X \to X$ is a k-contractive self-mapping on X satisfying the (L, h) property, that is,*

$$d\left(T^2x, Tx\right) \leq kd(Tx, q); \quad d(Tx, q) \leq Ld(Tx, x) + hd(x, q), \quad \forall x \in X, \tag{3.22}$$

with q being any fixed point; that is, $q \in F(f, T) \subset x$ and some real constants $0 \leq k < 1$, $0 \leq h < 1$, and $L \geq 0$, and, furthermore, $\lim_{k \to \infty} d(Tx_k, x_k) = 0$, for all $x_0 \in X$. Then, the Picard iteration procedure $x_{k+1} = Tx_k$, for all $k \in \mathbf{Z}_{0+}$, is T-stable with q being its unique fixed point, that is, $X \supset F(T) = \{q\} \neq \emptyset$.

Theorems 3.5 and 3.6 and Corollary 3.7 are directly extendable to the case that the pair (f, T) is a large contraction. Also, Theorem 3.6 and Corollary 3.7 can be extended directly for the use of distance functions or extended altering-distance functions as follows.

Theorem 3.8. *Assume that*

(1) *(X, d) is a complete metric space, $f : X \to X$ is a continuous mapping on X, and $T : X \to X$ is a self-mapping on X;*

(2) *$\varphi \in PC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ is an extended altering-distance function (Definition 2.1) satisfying the conditions of Theorem 2.2(ii) with some monotone nondecreasing function $\varphi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ satisfying $\phi(t) < \varphi(t)$, for all $t \in \mathbf{R}_{+}$, and $\phi(t) = \varphi(t) = 0$, if and only if $t = 0$;*

(3) *the quadruple* (φ, ϕ, f, T) *satisfies the following* (L, h) *property:*

$$\varphi(d(f(Tx_k), q)) \le \varphi(Ld(f(Tx_k), x_k) + hd(x_k, q)) - \phi(Ld(f(Tx_k), x_k) + hd(x_k, q)), \tag{3.23}$$

for all $x \in X$*, and* $q \in F(f, T) := \{z \in X : z = f(Tz)\}$ *with real constants* $0 \le h < 1$ *and* $L \ge 0$*;*

(4) $\lim_{k\to\infty}\varphi(d(f(Tx_k), q)) = 0$*, for all* $x_0 \in X$*.*

Then, the iteration procedure $x_{k+1} = f(Tx_k)$*, for all* $k \in \mathbf{Z}_{0+}$*, is* (f, T)*-stable with* q *being its unique fixed point, that is,* $X \supset F(f, T) = \{q\} \neq \emptyset$*.*

Corollary 3.9. *Assume that Assumptions (1) and (4) of Theorem 3.8 hold, and, furthermore,*

(1) $\varphi \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ *is an altering-distance function satisfying the conditions of Theorem 2.2(i), for all* $t \in \mathbf{R}_+$*, satisfying* $\phi(0) = 0$*, if and only if* $t = 0$*;*

(2) *the triple* (φ, f, T) *satisfies the following* (L, h) *property:*

$$\varphi(d(f(Tx_k), q)) \le \varphi(Ld(f(Tx_k), x_k) + hd(x_k, q)), \tag{3.24}$$

for all $x \in X$*, and* $q \in F(f, T) := \{z \in X : z = f(Tz)\}$ *with real constants* $0 \le h < 1$ *and* $L \ge 0$*;*

(3) $\lim_{k\to\infty}\varphi(d(f(Tx_k), q)) = 0$*, for all* $x_0 \in X$*.*

Then, the iteration procedure $x_{k+1} = f(Tx_k)$*, for all* $k \in \mathbf{Z}_{0+}$ *is* (f, T)*-stable with* q *being its unique fixed point, that is,* $X \supset F(f, T) = \{q\} \neq \emptyset$*.*

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