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Research Article

Best Proximity Points of Cyclic φ-Contractions on Reflexive Banach Spaces

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We provide a positive answer to a question raised by Al-Thagafi and Shahzad (Nonlinear Analysis, 70 (2009), 3665-3671) about best proximity points of cyclic φ-contractions on reflexive Banach spaces.

1. Introduction

As a generalization of Banach contraction principle, Kirk et al. proved, in 2003, the following fixed point result; see [1].

Theorem 1.1. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and there exists $k \in (0, 1)$ such that $d(Tx, Ty) \le kd(x, y)$ for all $x \in A$ and $y \in B$. Then, T has a unique fixed point in $A \cap B$.*

Let A and B be nonempty closed subsets of a metric space (X, d) and $\varphi : [0, \infty) \to [0, \infty)$ a strictly increasing map. We say that $T : A \cup B \to A \cup B$ is a cyclic φ-contraction map [2] whenever $T(A) \subseteq B$, $T(B) \subseteq A$ and

$$d(Tx, Ty) \le d(x, y) - \varphi\big(d(x, y)\big) + \varphi(d(A, B)) \tag{1.1}$$

for all $x \in A$ and $y \in B$, where $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. Also, $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$. As a special case, when $\varphi(t) = (1 - \alpha)t$, in which $\alpha \in (0, 1)$ is a constant, T is called cyclic contraction.

In 2005, Petrusel proved some periodic point results for cyclic contraction maps [3]. Then, Eldered and Veeramani proved some results on best proximity points of cyclic contraction maps in 2006 [4]. They raised a question about the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space. In 2009, Al-Thagafi and Shahzad gave a positive answer to the question [2]. More precisely, they proved some results on the existence and convergence of best proximity points of cyclic contraction maps defined on reflexive (and strictly convex) Banach spaces [2, Theorems 9, 10, 11, and 12]. They also introduced cyclic φ-contraction maps and raised the following question in [2].

Question 1. It is interesting to ask whether Theorems 9 and 10 (resp., Theorems 11 and 12) hold for cyclic φ-contraction maps where the space is only reflexive (resp., reflexive and strictly convex) Banach space.

In this paper, we provide a positive answer to the above question. For obtaining the answer, we use some results of [2].

2. Main Results

First, we give the following extension of [4, Proposition 3.3] for cyclic φ-contraction maps, where φ is unbounded.

Theorem 2.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing unbounded map. Also, let A and B be nonempty subsets of a metric space (X,d), $T : A\cup B \to A\cup B$ a cyclic φ-contraction map, $x_0 \in A\cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose that $x_0 \in A$ (the proof when $x_0 \in B$ is similar). By [2, Theorem 3], $d(x_{2n}, x_{2n+1}) \to d(A,B)$. Hence, it is sufficient to prove that $\{x_{2n+1}\}$ is bounded. Since φ is unbounded, there exists $M > 0$ such that

$$\varphi(M) > d(x_0, Tx_0) + \varphi(d(A,B)). \tag{2.1}$$

If $\{x_{2n+1}\}$ is not bounded, then there exists a natural number n_0 such that

$$d\left(T^2x_0, T^{2n_0+1}x_0\right) > M, \qquad d\left(T^2x_0, T^{2n_0-1}x_0\right) \leq M. \tag{2.2}$$

Then, we have

$$\begin{aligned}
M &< d\left(T^2x_0, T^{2n_0+1}x_0\right) \leq d\left(Tx_0, T^{2n_0}x_0\right) - \varphi\left(d\left(Tx_0, T^{2n_0}x_0\right)\right) + \varphi(d(A,B)) \\
&\leq d\left(x_0, T^{2n_0-1}x_0\right) - \left[\varphi\left(d\left(Tx_0, T^{2n_0}x_0\right)\right) + \varphi\left(d\left(x_0, T^{2n_0-1}x_0\right)\right)\right] + 2\varphi(d(A,B)) \\
&\leq d\left(x_0, T^2x_0\right) + d\left(T^2x_0, T^{2n_0-1}x_0\right) - \left[\varphi\left(d\left(Tx_0, T^{2n_0}x_0\right)\right) + \varphi\left(d\left(x_0, T^{2n_0-1}x_0\right)\right)\right] \\
&\quad + 2\varphi(d(A,B)) \\
&\leq d(x_0, Tx_0) + d\left(Tx_0, T^2x_0\right) + M - \left[\varphi\left(d\left(Tx_0, T^{2n_0}x_0\right)\right) + \varphi\left(d\left(x_0, T^{2n_0-1}x_0\right)\right)\right] \\
&\quad + 2\varphi(d(A,B)).
\end{aligned} \tag{2.3}$$

Since $d(Tx,Ty) \leq d(x,y)$ for all $x \in A$ and $y \in B$, we obtain

$$M < 2d(x_0,Tx_0) + M - \left[\varphi\left(d\left(Tx_0,T^{2n_0}x_0\right)\right) + \varphi\left(d\left(x_0,T^{2n_0-1}x_0\right)\right)\right] + 2\varphi(d(A,B)). \tag{2.4}$$

Since $d(Tx_0,T^{2n_0}x_0) \leq d(x_0,T^{2n_0-1}x_0)$, we have

$$\varphi\left(d\left(Tx_0,T^{2n_0}x_0\right)\right) \leq \varphi\left(d\left(x_0,T^{2n_0-1}x_0\right)\right). \tag{2.5}$$

Thus, we obtain $\varphi(d(Tx_0,T^{2n_0}x_0)) < d(x_0,Tx_0) + \varphi(d(A,B))$. Since

$$M < d\left(T^2x_0,T^{2n_0+1}x_0\right) \leq d\left(Tx_0,T^{2n_0}x_0\right), \tag{2.6}$$

$\varphi(M) < \varphi(d(Tx_0,T^{2n_0}x_0))$. Hence, $\varphi(M) < d(x_0,Tx_0) + \varphi(d(A,B))$. This contradiction completes the proof. □

Since the proof of last result was classic, we presented it separately. Here, we provide our key result via a special proof which is a general case of Theorem 2.1.

Theorem 2.2. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. Also, let A and B be nonempty subsets of a metric space (X,d), $T : A \cup B \to A \cup B$ a cyclic φ-contraction map, $x_0 \in A \cup B$, and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose that $x_0 \in A$ (the proof when $x_0 \in B$ is similar). By [2, Theorem 3], $d(x_{2n},x_{2n+1}) \to d(A,B)$. Hence, either $\{x_{2n+1}\}$ and $\{x_{2n}\}$ are bounded or both are unbounded. Suppose that both sequences are unbounded. Fix $n_1 \in \mathbb{N}$ and define

$$e_{n,k} = d\left(T^{2n}x_0,T^{2(n_1+k)+1}x_0\right) \tag{2.7}$$

for all $n,k \geq 1$. Since $\{x_{2n+1}\}$ is unbounded, $\limsup_{k\to\infty} e_{n,k} = \infty$ for all $n \geq 1$. Thus, we can choose a strictly increasing subsequence $\{e_{1,k_i^1}\}_{i\geq1}$ of the sequence $\{e_{1,k}\}_{k\geq1}$. Since $d(T^2x_0,T^{2(n_1+k_i^1)+1}x_0) \leq d(T^2x_0,T^4x_0) + d(T^4x_0,T^{2(n_1+k_i^1)+1}x_0)$, we have

$$\limsup_{i\to\infty} e_{2,k_i^1} = \infty. \tag{2.8}$$

Again, we can choose a strictly increasing subsequence $\{e_{2,k_i^2}\}_{i\geq1}$ of the sequence $\{e_{2,k_i^1}\}_{i\geq1}$ such that $\limsup_{i\to\infty} e_{2,k_i^2} = \infty$. By continuing this process, for each natural number n, we can choose a strictly increasing subsequence $\{e_{n,k_i^n}\}_{i\geq1}$ of the sequence $\{e_{n,k_i^{n-1}}\}_{i\geq1}$ such that $\limsup_{i\to\infty} e_{n,k_i^n} = \infty$. By the construction, if we consider the sequence $\{k_i^i\}_{i\geq1}$, then $\lim_{i\to\infty} k_i^i = \infty$, $\{e_{n,k_i^i}\}_{i\geq1}$ is a strictly increasing subsequence of $\{e_{n,k_i^n}\}_{i\geq1}$ and $\limsup_{i\to\infty} e_{n,k_i^i} = \infty$ for all $n \geq 1$. Now, define $n_2 = n_1 + k_2^2 - k_1^1$. Also, by induction define the sequence $\{n_m\}_{m\geq1}$ by $n_m = n_1 + k_m^m - k_1^1$. Note that, the sequence $\{n_m\}_{m\geq1}$ is strictly increasing and $\limsup_{m\to\infty} n_m = \infty$. Since T is a cyclic φ-contraction map, $\{d(T^{2n_m}x_0,T^{2(n_m+k_1^1)+1}x_0)\}_{m\geq1}$ is a decreasing sequence. Hence by the construction of the sequence $\{n_m\}_{m\geq 1}$, $\{d(T^{2n_m}x_0, T^{2(n_1+k_m^m)+1}x_0)\}_{m\geq 1}$ is a decreasing sequence. Let $m \geq 1$ be given. Since $e_{n_m,k_1^1} \leq e_{n_m,k_m^m}$, we have

$$d\left(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0\right) \leq d\left(T^{2n_m}x_0, T^{2(n_1+k_m^m)+1}x_0\right). \tag{2.9}$$

Thus,

$$d\left(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0\right) \leq d\left(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) \tag{2.10}$$

for all $m \geq 1$. Hence, we have

$$\begin{aligned} d\left(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) &\leq d\left(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(T^{2n_m}x_0, T^{2(n_1+k_m^m)+1}x_0\right) \\ &= d\left(T^{2n_m}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(T^{2n_m}x_0, T^{2(n_m+k_1^1)+1}x_0\right) \\ &\leq d\left(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(T^{2n_m}x_0, T^{2(n_m+k_1^1)+1}x_0\right) \end{aligned} \tag{2.11}$$

for all $m \geq 1$. Since $d(Tx, Ty) \leq d(x, y)$ for all $x \in A$ and $y \in B$, we obtain

$$\begin{aligned} d\left(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) &\leq d\left(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(T^{2n_m-1}x_0, T^{2(n_m+k_1^1)}x_0\right) \\ &\leq d\left(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(x_0, T^{2k_1^1+1}x_0\right) \end{aligned} \tag{2.12}$$

for all $m \geq 1$. Consequently

$$\begin{aligned} d\left(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)}x_0\right) &\leq d\left(T^{2(n_1+k_m^m)+1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(T^{2(n_1+k_1^1)+1}x_0, T^{2(n_1+k_1^1)}x_0\right) \\ &\leq d\left(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(x_0, T^{2k_1^1+1}x_0\right) \\ &\quad + d\left(T^{2(n_1+k_1^1)+1}x_0, T^{2(n_1+k_1^1)}x_0\right) \end{aligned} \tag{2.13}$$

for all $m \geq 1$. This implies that

$$e_{(n_1+k_1^1),k_m^m} \leq \mu \tag{2.14}$$

for all $m \geq 1$, where

$$\mu = d\left(T^{2n_1}x_0, T^{2(n_1+k_1^1)+1}x_0\right) + d\left(x_0, T^{2k_1^1+1}x_0\right) + d\left(T^{2(n_1+k_1^1)+1}x_0, T^{2(n_1+k_1^1)}x_0\right) \tag{2.15}$$

is a constant. But, $\limsup_{i\to\infty} e_{n,k_i^i} = \infty$ for all $n \geq 1$. This contradiction completes the proof. □

Now by using this key result, we provide our main results which give positive answer to the question. Their proofs are basically due to Al-Thagafi and Shahzad [2]. However, the crucial role is played by our key result. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \xrightarrow{w} x$.

Theorem 2.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. Also, let A and B be nonempty weakly closed subsets of a reflexive Banach space and $T : A \cup B \to A \cup B$ a cyclic φ-contraction map. Then there exists $(x, y) \in A \times B$ such that*

$$\|x - y\| = d(A, B). \tag{2.16}$$

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for all $n \geq 1$. By Theorem 2.2, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded. Since X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ such that $x_{2n_k} \xrightarrow{w} x \in A$. As $\{x_{2n_k+1}\}$ is bounded and B is weakly closed, we can say, without loss of generality, that $x_{2n_k+1} \xrightarrow{w} y \in B$ as $k \to \infty$. Since $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - y \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0,\infty)$ such that $\|f\| = 1$ and $f(x - y) = \|x - y\|$. For each $k \geq 1$, we have

$$\left|f(x_{2n_k} - x_{2n_k+1})\right| \leq \|f\| \|x_{2n_k} - x_{2n_k+1}\| = \|x_{2n_k} - x_{2n_k+1}\|. \tag{2.17}$$

Since $\lim_{k\to\infty} f(x_{2n_k} - x_{2n_k+1}) = f(x - y) = \|x - y\|$, by using [2, Theorem 3] we obtain

$$\|x - y\| = \lim_{k\to\infty} \left|f(x_{2n_k} - x_{2n_k+1})\right| \leq \lim_{k\to\infty} \|x_{2n_k} - x_{2n_k+1}\| = \|x_{2n_k} - x_{2n_k+1}\| = d(A, B). \tag{2.18}$$

Hence, $\|x - y\| = d(A, B)$. □

Definition 2.4. (see [2]) Let A and B be nonempty subsets of a normed space X, $T : A \cup B \to A \cup B$, $T(A) \subseteq B$, and $T(B) \subseteq A$. We say that T satisfies the proximal property if

$$x_n \xrightarrow{w} x \in A \cup B, \quad \|x_n - Tx_n\| \longrightarrow d(A, B) \Longrightarrow \|x - Tx\| = d(A, B). \tag{2.19}$$

Theorem 2.5. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. Also, let A and B be nonempty subsets of a reflexive Banach space X such that A is weakly closed and $T : A \cup B \to A \cup B$ a cyclic φ-contraction map. Then, there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*

(a) *T is weakly continuous on A.*

(b) *T satisfies the proximal property.*

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for all $n \geq 1$. By Theorem 2.2, the sequence $\{x_{2n}\}$ is bounded. Since X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ such that $x_{2n_k} \xrightarrow{w} x \in A$ as $k \to \infty$.

(a) Since T is weakly continuous on A and $T(A) \subseteq B$, we have $x_{2n_k+1} \xrightarrow{w} Tx \in B$ as $k \to \infty$. So $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - Tx \neq 0$ as $k \to \infty$. The rest of the proof is similar to that of Theorem 2.3.

(b) By [2, Theorem 3], we have

$$\|x_{2n_k} - Tx_{2n_k}\| = \|x_{2n_k} - x_{2n_k+1}\| \longrightarrow d(A, B) \tag{2.20}$$

as $k \to \infty$. Since T satisfies the proximal property, we have $\|x - Tx\| = d(A, B)$. □

Theorem 2.6. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. Also, let A and B be nonempty closed and convex subsets of a reflexive and strictly convex Banach space and $T : A \cup B \to A \cup B$ a cyclic φ-contraction map. If $(A - A) \cap (B - B) = \{0\}$, then there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A, B)$.*

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. By Theorem 2.3, there exists $(x, y) \in A \times B$ with $\|x - y\| = d(A, B)$. To show the uniqueness of (x, y), suppose that there exists another $(x', y') \in A \times B$ with $\|x' - y'\| = d(A, B)$. Since $(A - A) \cap (B - B) = \{0\}$, we conclude that $x - y \neq x' - y'$. As both A and B are convex, by the strict convexity of X, we have

$$\left\| \frac{x + x'}{2} - \frac{y + y'}{2} \right\| = \left\| \frac{x - y}{2} + \frac{x' - y'}{2} \right\| < d(A, B), \tag{2.21}$$

which is a contradiction. Since $\|Ty - Tx\| = \|Tx - Ty\| = \|x - y\| = d(A, B)$, we obtain, from the uniqueness of (x, y), that $(Ty, Tx) = (x, y)$. Hence $Tx = y$, $Ty = x$ and $T^2x = x$. □

Theorem 2.7. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. Also, let A and B be nonempty subsets of a reflexive and strictly convex Banach space X such that A is closed and convex and $T : A \cup B \to A \cup B$ a cyclic φ-contraction map. Then, there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*

(a) *T is weakly continuous on A.*

(b) *T satisfies the proximal property.*

Proof. If $d(A, B) = 0$, the result follows from [2, Theorem 1]. So, we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. By Theorem 2.5 that there exists $x \in A$ with $\|x - Tx\| = d(A, B)$. Thus, $T^2x = x$. Indeed, if we assume that $T^2x - Tx \neq x - Tx$. Then from the convexity of A and the strict convexity of X, we have

$$\left\| \frac{T^2x + x}{2} - Tx \right\| = \left\| \frac{T^2x - Tx}{2} + \frac{x - Tx}{2} \right\| < d(A, B), \tag{2.22}$$

which is a contradiction. The uniqueness of x follows as in the proof of [2, Theorem 8]. □

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