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Research Article

Fixed Point in Topological Vector Space-Valued Cone Metric Spaces

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We obtain common fixed points of a pair of mappings satisfying a generalized contractive type condition in TVS-valued cone metric spaces. Our results generalize some well-known recent results in the literature.

1. Introduction and Preliminaries

Many authors [1–16] studied fixed points results of mappings satisfying contractive type condition in Banach space-valued cone metric spaces. In a recent paper [17] the authors obtained common fixed points of a pair of mapping satisfying generalized contractive type conditions without the assumption of normality in a class of topological vector space-valued cone metric spaces which is bigger than that of studied in [1–16]. In this paper we continue to study fixed point results in topological vector space valued cone metric spaces.

Let (E, τ) be always a topological vector space (TVS) and P a subset of E. Then, P is called a cone whenever

(i) P is closed, nonempty, and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b,

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

Definition 1.1. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a topological vector space-valued cone metric on X, and (X, d) is called a topological vector space-valued cone metric space.

If E is a real Banach space then (X, d) is called (Banach space-valued) cone metric space [9].

Definition 1.2. Let (X, d) be a TVS-valued cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1.3. *Let (X, d) be a TVS-valued cone metric space, P be a cone. Let $\{x_n\}$ be a sequence in X,and $\{a_n\}$ be a sequence in P converging to $\mathbf{0}$. If $d(x_n, x_m) \leq a_n$ for every $n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Fix $\mathbf{0} \ll c$ take a symmetric neighborhood V of 0 such that $c + V \subseteq \operatorname{int} P$. Also, choose a natural number n_0 such that $a_n \in V$, for all $n \geq n_0$. Then $d(x_n, x_m) \leq a_n \ll c$ for every $m, n \geq n_0$. Therefore, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. □

Remark 1.4. Let A, B, C, D, E be nonnegative real numbers with $A+B+C+D+E < 1$, $B = C$, or $D = E$. If $F = (A + B + D)(1 - C - D)^{-1}$ and $G = (A + C + E)(1 - B - E)^{-1}$, then $FG < 1$. In fact, if $B = C$ then

$$FG = \frac{A + B + D}{1 - C - D} \cdot \frac{A + C + E}{1 - B - E} = \frac{A + C + D}{1 - B - E} \cdot \frac{A + B + E}{1 - C - D} < 1, \tag{1.1}$$

and if $D = E$,

$$FG = \frac{A + B + D}{1 - C - D} \cdot \frac{A + C + E}{1 - B - E} = \frac{A + B + E}{1 - C - D} \cdot \frac{A + C + D}{1 - B - E} < 1. \tag{1.2}$$

2. Main Results

The following theorem improves/generalizes the results of [5, Theorems 1, 3, and 4] and [4, Theorems 2.3, 2.6, 2.7, and 2.8].

Theorem 2.1. *Let (X, d) be a complete topological vector space-valued cone metric space, P be a cone and m, n be positive integers. If a mapping $T : X \to X$ satisfies*

$$d(T^m x, T^n y) \leq Ad(x, y) + Bd(x, T^m x) + Cd(y, T^n y) + Dd(x, T^n y) + Ed(y, T^m x) \tag{2.1}$$

for all $x, y \in X$, where A, B, C, D, E are non negative real numbers with $A+B+C+D+E < 1$, $B = C$, or $D = E$. Then T has a unique fixed point.

Proof. For $x_0 \in X$ and $k \geq 0$, define

$$\begin{aligned} x_{2k+1} &= T^m x_{2k}, \\ x_{2k+2} &= T^n x_{2k+1}. \end{aligned} \tag{2.2}$$

Then

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(T^m x_{2k}, T^n x_{2k+1}) \\ &\leq Ad(x_{2k}, x_{2k+1}) + Bd(x_{2k}, T^m x_{2k}) + Cd(x_{2k+1}, T^n x_{2k+1}) \\ &\quad + Dd(x_{2k}, T^n x_{2k+1}) + Ed(x_{2k+1}, T^m x_{2k}) \\ &\leq [A + B]d(x_{2k}, x_{2k+1}) + Cd(x_{2k+1}, x_{2k+2}) + Dd(x_{2k}, x_{2k+2}) \\ &\leq [A + B + D]d(x_{2k}, x_{2k+1}) + [C + D]d(x_{2k+1}, x_{2k+2}). \end{aligned} \tag{2.3}$$

It implies that

$$[1 - C - D]d(x_{2k+1}, x_{2k+2}) \leq [A + B + D]d(x_{2k}, x_{2k+1}). \tag{2.4}$$

That is,

$$d(x_{2k+1}, x_{2k+2}) \leq Fd(x_{2k}, x_{2k+1}), \tag{2.5}$$

where $F = (A + B + D)/(1 - C - D)$.

Similarly,

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &= d(T^m x_{2k+2}, T^n x_{2k+1}) \\ &\leq Ad(x_{2k+2}, x_{2k+1}) + Bd(x_{2k+2}, T^m x_{2k+2}) + Cd(x_{2k+1}, T^n x_{2k+1}) \\ &\quad + Dd(x_{2k+2}, T^n x_{2k+1}) + Ed(x_{2k+1}, T^m x_{2k+2}) \\ &\leq Ad(x_{2k+2}, x_{2k+1}) + Bd(x_{2k+2}, x_{2k+3}) + Cd(x_{2k+1}, x_{2k+2}) \\ &\quad + D \quad d(x_{2k+2}, x_{2k+2}) + Ed(x_{2k+1}, x_{2k+3}) \\ &\leq [A + C + E]d(x_{2k+1}, x_{2k+2}) + [B + E]d(x_{2k+2}, x_{2k+3}), \end{aligned} \tag{2.6}$$

which implies

$$d(x_{2k+2}, x_{2k+3}) \leq G d(x_{2k+1}, x_{2k+2}), \tag{2.7}$$

with $G = (A + C + E)/(1 - B - E)$.

Now by induction, we obtain for each $k = 0, 1, 2, \ldots$

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\leq F\ d(x_{2k}, x_{2k+1}) \\ &\leq (FG) d(x_{2k-1}, x_{2k}) \\ &\leq F(FG) d(x_{2k-2}, x_{2k-1}) \\ &\leq \cdots \leq F(FG)^k d(x_0, x_1), \\ d(x_{2k+2}, x_{2k+3}) &\leq G d(x_{2k+1}, x_{2k+2}) \\ &\leq \cdots \leq (FG)^{k+1} d(x_0, x_1). \end{aligned} \tag{2.8}$$

By Remark 1.4, for $p < q$ we have

$$\begin{aligned} d\big(x_{2p+1}, x_{2q+1}\big) &\leq d\big(x_{2p+1}, x_{2p+2}\big) + d\big(x_{2p+2}, x_{2p+3}\big) + d\big(x_{2p+3}, x_{2p+4}\big) + \cdots + d\big(x_{2q}, x_{2q+1}\big) \\ &\leq \left[F \sum_{i=p}^{q-1} (FG)^i + \sum_{i=p+1}^{q} (FG)^i \right] d(x_0, x_1) \\ &\leq \left[\frac{F(FG)^p}{1 - FG} + \frac{(FG)^{p+1}}{1 - FG} \right] d(x_0, x_1) \\ &\leq (1 + F) \left[\frac{(FG)^p}{1 - FG} \right] d(x_0, x_1). \end{aligned} \tag{2.9}$$

In analogous way, we deduced

$$\begin{aligned} d\big(x_{2p}, x_{2q+1}\big) &\leq (1 + F) \left[\frac{(FG)^p}{1 - FG} \right] d(x_0, x_1), \\ d\big(x_{2p}, x_{2q}\big) &\leq (1 + F) \left[\frac{(FG)^p}{1 - FG} \right] d(x_0, x_1), \\ d\big(x_{2p+1}, x_{2q}\big) &\leq (1 + F) \left[\frac{(FG)^p}{1 - FG} \right] d(x_0, x_1). \end{aligned} \tag{2.10}$$

Hence, for $0 < n < m$

$$d(x_n, x_m) \leq a_n, \tag{2.11}$$

where $a_n = (1 + F)[(FG)^p/(1 - FG)] d(x_0, x_1)$ with p the integer part of $n/2$.

Fix $\mathbf{0} \ll c$ and choose a symmetric neighborhood V of 0 such that $c + V \subseteq \operatorname{int} P$. Since $a_n \to \mathbf{0}$ as $n \to \infty$, by Lemma 1.3, we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete, there exists $u \in X$ such that $x_n \to u$. Fix $\mathbf{0} \ll c$ and choose $n_0 \in \mathbb{N}$ be such that

$$d(u, x_{2k}) \ll \frac{c}{3K}, \qquad d(x_{2k-1}, x_{2k}) \ll \frac{c}{3K}, \qquad d(u, x_{2k-1}) \ll \frac{c}{3K} \tag{2.12}$$

for all $k \geq n_0$, where

$$K = \max\left\{ \frac{1+D}{1-B-E}, \frac{A+E}{1-B-E}, \frac{C}{1-B-E} \right\}. \tag{2.13}$$

Now,

$$\begin{aligned} d(u, T^m u) &\leq d(u, x_{2k}) + d(x_{2k}, T^m u) \\ &\leq d(u, x_{2k}) + d(T^n x_{2k-1}, T^m u) \\ &\leq d(u, x_{2k}) + Ad(u, x_{2k-1}) + Bd(u, T^m u) + Cd(x_{2k-1}, T^n x_{2k-1}) \\ &\quad + Dd(u, T^n x_{2k-1}) + Ed(x_{2k-1}, T^m u) \\ &\leq d(u, x_{2k}) + Ad(u, x_{2k-1}) + Bd(u, T^m u) + Cd(x_{2k-1}, x_{2k}) \\ &\quad + Dd(u, x_{2k}) + Ed(x_{2k-1}, u) + Ed(u, T^m u)] \\ &\leq (1+D)d(u, x_{2k}) + (A+E)d(u, x_{2k-1}) + Cd(x_{2k-1}, x_{2k}) + (B+E)d(u, T^m u). \end{aligned} \tag{2.14}$$

So,

$$\begin{aligned} d(u, T^m u) &\leq Kd(u, x_{2k}) + Kd(u, x_{2k-1}) + Kd(x_{2k-1}, x_{2k}) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c. \end{aligned} \tag{2.15}$$

Hence

$$d(u, T^m u) \ll \frac{c}{p} \tag{2.16}$$

for every $p \in \mathbb{N}$. From

$$\frac{c}{p} - d(u, T^m u) \in \operatorname{int} P \tag{2.17}$$

being P closed, as $p \to \infty$, we deduce $-d(u, T^m u) \in P$ and so $d(u, T^m u) = \mathbf{0}$. This implies that $u = T^m u$.

Similarly, by using the inequality,

$$d(u,T^n u) \leq d(u,x_{2k+1}) + d(x_{2k+1},T^n u), \tag{2.18}$$

we can show that $u = T^n u$, which in turn implies that u is a common fixed point of T^m, T^n and, that is,

$$u = T^m u = T^n u. \tag{2.19}$$

Now using the fact that

$$\begin{aligned} d(Tu,u) &= d(TT^m u, T^n u) = d(T^m Tu, T^n u) \\ &\leq Ad(Tu,u) + Bd(Tu,T^m Tu) + Cd(u,T^n u) + Dd(Tu,T^n u) + Ed(u,T^m Tu) \\ &\leq Ad(Tu,u) + Bd(Tu,Tu) + Cd(u,u) + Dd(Tu,u) + Ed(u,Tu) \\ &= (A+D+E)d(Tu,u). \end{aligned} \tag{2.20}$$

We obtain u is a fixed point of T. For uniqueness, assume that there exists another point u^* in X such that $u^* = Tu^*$ for some u^* in X. From

$$\begin{aligned} d(u,u^*) &= d(T^m u, T^n u^*) \\ &\leq Ad(u,u^*) + Bd(u,T^m u) + Cd(u^*,T^n u^*) + Dd(u,T^n u^*) + Ed(u^*,T^m u) \\ &\leq Ad(u,u^*) + Bd(u,u) + Cd(u^*,u^*) + Dd(u,u^*) + Ed(u,u^*) \\ &\leq (A+D+E)d(u,u^*), \end{aligned} \tag{2.21}$$

we obtain that $u^* = u$.

Huang and Zhang [9] proved Theorem 2.1 by using the following additional assumptions.

(a) E Banach Space.

(b) P is normal (i.e., there is a number $\kappa \geq 1$ such that for all $x, y, \in E$, $\mathbf{0} \leq x \leq y \Rightarrow \|x\| \leq \kappa\|y\|$).

(c) $m = n = 1$.

(d) One of the following is satisfied:

(i) $B = C = D = E = 0$ with $A < 1$ [5, Theorem 1],

(ii) $A = D = E = 0$ with $B = C < 1/2$ [5, Theorem 3],

(iii) $A = B = C = 0$ with $D = E < 1/2$ [5, Theorem 4].

Azam and Arshad [4] improved these results of Huang and Zhang [5] by omitting the assumption (b). □

Theorem 2.2. *Let (X,d) be a complete topological vector space-valued cone metric space, P be a cone and m,n be positive integers. If a mapping $T : X \to X$ satisfies:*

$$d(Tx,Ty) \leq Ad(x,y) + Bd(x,Tx) + Cd(y,Ty) + Dd(x,Ty) + Ed(y,Tx) \tag{2.22}$$

for all $x,y \in X$, where A, B, C, D, E are non negative real numbers with $A+B+C+D+E<1$. Then T has a unique fixed point.

Proof. The symmetric property of d and the above inequality imply that

$$d(Tx,Ty) \leq Ad(x,y) + \frac{B+C}{2}\left[d(x,Tx)+d(y,Ty)\right] + \frac{D+E}{2}\left[d(x,Ty)+d(y,Tx)\right]. \tag{2.23}$$

By substituting $T^m = T^n = T$ in the Theorem 2.1, we obtain the required result. Next we present an example to support Theorem 2.2. □

Example 2.3. $X=[0,1]$, E be the set of all complex-valued functions on X then E is a vector space over $\mathbb{R}$ under the following operations:

$$(f+g)(t) = f(t)+g(t), \qquad (\alpha f)(t) = \alpha f(t) \tag{2.24}$$

for all $f,g \in E$, $\alpha \in \mathbb{R}$. Let τ be the topology on E defined by the the family $\{p_x : x \in X\}$ of seminorms on E, where

$$p_x(f) = |f(x)| \tag{2.25}$$

then (X,τ) is a topological vector space which is not normable and is not even metrizable (see [18, 19]). Define $d : X \times X \to E$ as follows:

$$\begin{gathered}(d(x,y))(t) = (|x-y|, 3|x-y|)3^t,\\ P = \{(x \in E : x(t) \geqslant 0 \ \forall t \in X\}.\end{gathered} \tag{2.26}$$

Then (X,d) is a topological vector space-valued cone metric space. Define $T : X \to X$ as $T(x) = x^2/9$, then all conditions of Theorem 2.2 are satisfied.

Corollary 2.4. *Let (X,d) be a complete Banach space-valued cone metric space, P be a cone, and m,n be positive integers. If a mapping $T : X \to X$ satisfies*

$$d(T^m x, T^n y) \leq Ad(x,y) + Bd(x,T^m x) + Cd(y,T^n y) + Dd(x,T^n y) + Ed(y,T^m x) \tag{2.27}$$

for all $x,y \in X$, where A, B, C, D, E are non negative real numbers with $A+B+C+D+E<1$, $B=C$, or $D=E$. Then T has a unique fixed point.

Next we present an example to show that corollary 2.4 is a generalization of the results [9, Theorems 1, 3, and 4] and [15, Theorems 2.3, 2.6, 2.7, and 2.8].

Example 2.5. Let $X = \{1,2,3\}$, $\mathcal{B} = R^2$, and $P = \{(x,y) \in \mathcal{B} \mid x, y \geq 0\} \subset R^2$. Define $d : X \times X \to R^2$ as follows:

$$d(x,y) = \begin{cases} (0,0), & \text{if } x = y, \\ \left(\dfrac{5}{7}, 5\right), & \text{if } x \neq y,\ x, y \in X - \{2\}, \\ (1,7), & \text{if } x \neq y,\ x, y \in X - \{3\}, \\ \left(\dfrac{4}{7}, 4\right), & \text{if } x \neq y,\ x, y \in X - \{1\}. \end{cases} \tag{2.28}$$

Define the mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} 1, & \text{if } x \neq 2, \\ 3, & \text{if } x = 2. \end{cases} \tag{2.29}$$

Note that the assumptions (d) of results [9, Theorems 1, 3, and 4] and [15, Theorems 2.3, 2.6, 2.7, and 2.8] are not satisfied to find a fixed point of T. In order to apply inequality (2.1) consider mapping $T^2(x) = 1$ for each $x \in X$, then for $A = B = C = D = 0$, $E = 5/7$, T^2, and T satisfy all the conditions of Corollary 2.4 and we obtain $T(1) = 1$.

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