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Research Article

Existence and Uniqueness of Solutions for Coupled Systems of Higher-Order Nonlinear Fractional Differential Equations

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We study an initial value problem for a coupled Caputo type nonlinear fractional differential system of higher order. As a first problem, the nonhomogeneous terms in the coupled fractional differential system depend on the fractional derivatives of lower orders only. Then the nonhomogeneous terms in the fractional differential system are allowed to depend on the unknown functions together with the fractional derivative of lower orders. Our method of analysis is based on the reduction of the given system to an equivalent system of integral equations. Applying the nonlinear alternative of Leray-Schauder, we prove the existence of solutions of the fractional differential system. The uniqueness of solutions of the fractional differential system is established by using the Banach contraction principle. An illustrative example is also presented.

1. Introduction

In recent years, the applications of fractional calculus in physics, chemistry, electrochemistry, bioengineering, biophysics, electrodynamics of complex medium, polymer rheology, aerodynamics, continuum mechanics, signal processing, electromagnetics, and so forth are highlighted in the literature. The methods of fractional calculus, when defined as a Laplace, Sumudu, or Fourier convolution product, are suitable for solving many problems in emerging biomedical research. The electrical properties of nerve cell membranes and the propagation of electrical signals are well characterized by differential equations of fractional order. The fractional derivative accurately describes natural phenomena that occur in common engineering problems such as heat transfer, electrode/electrolyte behavior, and subthreshold nerve propagation. Application of fractional derivatives to viscoelastic materials establishes, in a natural way, hereditary integrals and the power law stress-strain relationship for modeling biomaterials. A systematic presentation of the applications of fractional differential equations can be found in the book of Oldham and Spanier [1]. For more details, see the monographs of Miller and Ross [2], Samko et al. [3], Podlubny [4], and Kilbas et al. [5]. In consequence, the subject of fractional differential equations is gaining much importance and attention; see [6–31] and the references therein. There has also been a surge in the study of the theory of fractional differential systems. The study of coupled systems involving fractional differential equations is quite important as such systems occur in various problems of applied nature; for instance, see [32–35] and the references therein. Recently, Su [36] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Ahmad and Nieto [37] studied a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Ahmad and Graef [38] proved the existence of solutions for nonlocal coupled systems of nonlinear fractional differential equations. For applications and examples of fractional order systems, we refer the reader to the papers in [39–47]. Motivated by the recent work on coupled systems of fractional order, we consider an initial value problem for a coupled differential system of fractional order given by

$$ {}^cD^{\rho}u(t) = f\left(t, {}^cD^{\beta} v(t)\right), \quad u^{(k)}(0) = \eta_k, \quad 0 < t \le 1, \tag{1.1} $$

$$ {}^cD^{\sigma}v(t) = g\left(t, {}^cD^{\alpha} u(t)\right), \quad v^{(k)}(0) = \xi_k, \quad 0 < t \le 1, \tag{1.2} $$

where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are given functions, ${}^cD$ denotes the Caputo fractional derivative, $\rho,\ \sigma \in (m-1, m)$, $\alpha,\ \beta \in (n-1, n)$, $m,\ n \in \mathbf{N}$, $\rho > \beta$, $\sigma > \alpha$, $k = 0, 1, 2, \ldots, m-1, \rho,\ \sigma,\ \beta,\ \alpha \notin \mathbf{N}$, and η_k, ξ_k are suitable real constants. We also discuss the case when the nonlinearities f and g in (1.1) are of the form $f(t, v(t), {}^cD^{\beta} v(t))$ and $g(t, u(t), {}^cD^{\alpha} u(t))$, that is, f and g depend on v and u in addition to ${}^cD^{\beta}v(t)$ and ${}^cD^{\alpha}u(t)$, respectively.

2. Preliminaries

First of all, we recall some basic definitions [3–5].

Definition 2.1. For a function $f \in C^m[0,1], m \in \mathbf{N}$, the Caputo derivative of fractional order $\alpha \in (m-1, m)$ is defined as

$$ {}^cD^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds. \tag{2.1} $$

Definition 2.2. The Riemann-Liouville fractional integral of order α, inversion of D^{α}, is the expression given by

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{2.2} $$

Definition 2.3. The Riemann-Liouville fractional derivative of order α for a function $f(t)$ is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^{m} \int_0^t (t-s)^{m-\alpha-1} f(s) ds. \tag{2.3}$$

Now we state a known result [48] which provides a relationship between (2.1) and (2.2).

Lemma 2.4. *For $\alpha \in (m-1, m), m \in \mathbf{N}$, let $f \in C^m[0,1]$ and $g \in C^1[0,1]$. Then*

(i) ${}^cD^{\alpha} I^{\alpha} g(t) = g(t)$;

(ii) $I^{\alpha}\ {}^cD^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \dfrac{t^k}{k!} f^{(k)}(0)$;

(iii) $\lim_{t\to 0^+} {}^cD^{\alpha} f(t) = \lim_{t \to 0^+} I^{\alpha} f(t) = 0$;

(iv) *if there exist $i_k < n$ such that $\sum_{j=1}^{i_k} \alpha_j = k$ for each $k = 1, \ldots, m-1$ with $\alpha_i \in (0,1]$, $i = 1, 2, \ldots, n$, and $\alpha = \sum_{i=1}^{n} \alpha_i$, then ${}^cD^{\alpha} f(t) = {}^cD^{\alpha_n} \cdots {}^cD^{\alpha_2}\ {}^cD^{\alpha_1} f(t)$.*

Remark 2.5. In the sequel, ${}^cD^{\alpha} f(0)$ will be understood in the sense of the limit, that is, $\lim_{t\to 0} {}^cD^{\alpha} f(t) = {}^cD^{\alpha} f(0)$. We also point out that the fractional order derivatives do not satisfy the relation of the form ${}^cD^{\alpha_1}\ {}^cD^{\alpha_2} f = {}^cD^{\alpha_1+\alpha_2} f$ (in general).

For the sequel, we need the following results [26].

Lemma 2.6. *Assume that $m-1 < \alpha < \beta < m$ and $f \in C^m[0,1]$. Then, for all $k \in \{1,2,\ldots,m-1\}$ and for all $t \in [0,1]$, the following relations hold:*

$${}^cD^{\beta-m+k} f^{(m-k)}(t) = {}^cD^{\beta} f(t), \tag{2.4}$$

$${}^cD^{\beta-\alpha}\ {}^cD^{\alpha} f(t) = {}^cD^{\beta} f(t). \tag{2.5}$$

Lemma 2.7. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $f(0,0) = 0$ and $f(t,0) \neq 0$ on a compact subinterval of $(0,1]$. Then, for $\rho \in (m-1, m)$, $\beta \in (n-1, n)$, $m, n \in \mathbf{N}$ with $\rho > \beta$ and $n-1 < \beta < n \leq m-1 < \rho < m$, a function $u \in C^m[0,1]$ is a solution of the initial value problem*

$${}^cD^{\rho} u(t) = f\left(t, {}^cD^{\beta} u(t)\right), \quad 0 < t \leq 1, \tag{2.6}$$

$$u^{(k)}(0) = \eta_k, \quad k = 1, 2, \ldots, m-1, \tag{2.7}$$

if and only if

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} \eta_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} v(s) ds, \quad 0 < t \leq 1, \tag{2.8}$$

where $v \in C[0,1]$ is a solution of the integral equation

$$v(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i} + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} v(\tau)d\tau\right)ds. \tag{2.9}$$

Proof. For the sake of completeness and later use, we outline the proof. Using (2.4) with $k = m-n$, $\beta = \rho$ yields

$${}^cD^{\rho-n}u^{(n)}(t) = {}^cD^{\rho}u(t) = f\left(t, {}^cD^{\beta}u(t)\right). \tag{2.10}$$

On the other hand, in view of (2.1), we have

$${}^cD^{\rho-n}u^{(n)}(t) = f\left(t, \int_0^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)} u^{(n)}(s)ds\right). \tag{2.11}$$

Using (2.2) and Lemma 2.4 (ii) together with the substitution $u^{(n)}(t) = v(t)$, we obtain

$$v(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!}v^{(i)}(0) + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} v(\tau)d\tau\right)ds. \tag{2.12}$$

Applying the initial conditions (2.7) and the fact that $u^{(n+i)}(t) = v^{(i)}(t)$, (2.12) transforms to (2.9).

Conversely, suppose that $v \in C[0,1]$ is a solution of (2.9). Then

$$\begin{aligned} u^{(n)}(t) = v(t) &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i} + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} v(\tau)d\tau\right)ds \\ &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i} + I^{\rho-n} f\left(t, {}^cD^{\beta}u(t)\right). \end{aligned} \tag{2.13}$$

As $\rho - n \in (m-n-1, m-n)$, it follows by Lemma 2.4 (i) and Lemma 2.6 that

$$\begin{aligned} {}^cD^{\rho}u(t) = {}^cD^{\rho-n}u^{(n)}(t) &= {}^cD^{\rho-n}\left(\sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i}\right) + {}^cD^{\rho-n}I^{\rho-n}f\left(t, {}^cD^{\beta}u(t)\right) \\ &= f\left(t, {}^cD^{\beta}u(t)\right), \quad 0 < t \leq 1. \end{aligned} \tag{2.14}$$

Thus, u is a solution of (2.6). Now, differentiating (2.9), we obtain

$$v^{(k)}(t)=\sum_{i=0}^{m-n-k-1}\frac{t^i}{i!}\eta_{n+i+k}+\prod_{l=1}^{k}(\rho-n-l)\int_0^t\frac{(t-s)^{\rho-n-1-k}}{\Gamma(\rho-n)}f\left(s,\int_0^s\frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)}u^{(n)}(\tau)d\tau\right)ds, \tag{2.15}$$

for each $k=0,\ 1,\dots,\ m-n-1$. Since $\rho-n-1-k\in(-1,m-n-1)$, the second term in the above expression becomes zero as $t\to 0$. Thus, we have

$$u^{(n+k)}(0)=v^{(k)}(0)=\eta_{n+k},\quad k=0,1,\dots,m-n-1, \tag{2.16}$$

which implies that $u^{(k)}(0)=\eta_k, k=0,1,\dots,m-1$. Also, it is easy to infer that $v^{(m-n)}=u^{(m)}\in C[0,1]$. Hence we conclude that $u\in C^m[0,1]$ is a solution of (2.6) and (2.7). □

3. Existence Result

For the forthcoming analysis, we introduce the following assumptions:

(A_1) let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function with $f(0,0)=0$ and $f(t,0)\neq 0$ on a compact subinterval of $(0,1]$;

(A_2) let $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function with $g(0,0)=0$ and $g(t,0)\neq 0$ on a compact subinterval of $(0,1]$;

(A_3) there exist nonnegative functions $a_1,a_2,b_1,b_2\in C[0,1]$ such that

$$\begin{gathered}|f(t,x)|\le a_1(t)+a_2(t)|x|,\quad t\in[0,1],\\ |g(t,x)|\le b_1(t)+b_2(t)|x|,\quad t\in[0,1].\end{gathered} \tag{3.1}$$

Now we state a result which describes the nonlinear alternative of Leray and Schauder [49].

Theorem 3.1. *Let $\mathcal{X}$ be a normed linear space, $\mathcal{M}\subset\mathcal{X}$ be a convex set, and $\mathcal{N}$ be open in $\mathcal{M}$ with $0\in\mathcal{N}$. Let $\mathcal{T}:\overline{\mathcal{N}}\to\mathcal{M}$ be a continuous and compact mapping. Then either the mapping $\mathcal{T}$ has a fixed point in $\overline{\mathcal{N}}$ or there exist $n\in\partial\mathcal{N}$ and $\lambda\in(0,1)$ with $n=\lambda\mathcal{T}n$.*

Lemma 3.2. *Suppose that the assumption (A_1) holds and $n-1<\alpha,\ \beta<n\le m-1<\rho,\ \sigma<m$. Then, a function $u\in C^m[0,1]$ is a solution of the initial value problem* (1.1) *if and only if*

$$u(t)=\sum_{k=0}^{n-1}\frac{t^k}{k!}\eta_k+\int_0^t\frac{(t-s)^{n-1}}{\Gamma(n)}w_1(s)ds,\quad 0<t\le 1, \tag{3.2}$$

where $w_1=u^{(n)}(t)\in C[0,1]$ with $u^{(n+i)}(t)=w_1^{(i)}(t)$ is a solution of the integral equation

$$w_1(t)=\sum_{i=0}^{m-n-1}\frac{t^i}{i!}\eta_{n+i}+\int_0^t\frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}f\left(s,\int_0^s\frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)}w_2(\tau)d\tau\right)ds, \tag{3.3}$$

and a function $v \in C^m[0,1]$ *is a solution of the initial value problem* (1.2) *if and only if*

$$v(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\xi_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_2(s)ds, \quad 0 < t \le 1, \tag{3.4}$$

where $w_2 = v^{(n)}(t) \in C[0,1]$ *with* $v^{(n+i)}(t) = w_2^{(i)}(t)$ *is a solution of the integral equation*

$$w_2(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\xi_{n+i} + \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} g\left(s, \int_0^s \frac{(s-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_1(\tau)d\tau\right)ds. \tag{3.5}$$

We do not provide the proof as it is similar to that of Lemma 2.7. Consider the coupled system of integral equations

$$\begin{aligned} u(t) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\eta_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_1(s)ds, \\ v(t) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\xi_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_2(s)ds, \end{aligned} \tag{3.6}$$

where w_1 and w_2 are given by (3.3) and (3.5), respectively.

Let $C([0,1])$ denote the space of all continuous functions defined on $[0,1]$. Let $X = \{u \mid u \in C([0,1])\}$ and $Y = \{v \mid v \in C([0,1])\}$ be normed spaces with the sup-norm $\|u\|_X$ and $\|v\|_Y$, respectively. Then, $(X \times Y, \|\cdots\|_{X\times Y})$ is a normed space endowed with the sup-norm defined by $\|(u,v)\|_{X\times Y}$.

Lemma 3.3. *Assume that* $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ *are continuous functions. Then* $(u,v) \in X \times Y$ *is a solution of* (1.1)-(1.2) *if and only if* $(u,v) \in X \times Y$ *is a solution of* (3.6).

Proof. For $k = m-n,\ \beta = \rho$ in (2.4), we have

$${}^cD^{\rho-n}u^{(n)}(t) = {}^cD^{\rho}u(t) = f\left(t, {}^cD^{\beta}\, v(t)\right) = f\left(t, \int_0^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)} v^{(n)}(s)ds\right). \tag{3.7}$$

Using the fact

$$I^{\rho}\ {}^cD^{\rho}u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0), \tag{3.8}$$

and making the substitutions $u^{(n)}(t) = w_1(t)$, $v^{(n)}(t) = w_2(t)$, we obtain

$$w_1(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!} w_1^{(i)}(0) + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} v^{(n)}(\tau)d\tau\right)ds. \tag{3.9}$$

Using the initial conditions of (1.1) together with $u^{(n+i)}(t) = w_1^{(i)}(t)$ and $v^{(n)}(t) = w_2(t)$, (3.9) becomes (3.3), and an application of Cauchy function yields the first equation of (3.6). The converse of the theorem follows by applying the arguments used to prove the converse of Lemma 2.7. Similarly, it can be shown that v satisfying the second equation of (3.6) together with (3.5) is a solution of (1.2) and vice versa. Thus, $(u, v) \in X \times Y$ satisfying (3.6) is a solution of the system (1.1)-(1.2) and vice versa. □

Theorem 3.4. *Let the assumptions* (A_1)*–*(A_3) *hold. Then there exists a solution for the coupled integral equations* (3.3) *and* (3.5) *if*

$$\begin{aligned}
B_1 &= \frac{1}{\Gamma(\rho-n)\Gamma(n-\beta+1)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\rho-n-1} s^{n-\beta} a_2(s)ds < 1, \\
B_2 &= \frac{1}{\Gamma(\sigma-n)\Gamma(n-\alpha+1)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\sigma-n-1} s^{n-\alpha} b_2(s)ds < 1, \\
0 < C_1 &= \sup_{t\in[0,1]} \left(|\nu_1(t)| + \frac{1}{\Gamma(\rho-n)} \int_0^t (t-s)^{\rho-n-1} a_1(s)ds \right) < \infty, \\
0 < C_2 &= \sup_{t\in[0,1]} \left(|\nu_2(t)| + \frac{1}{\Gamma(\sigma-n)} \int_0^t (t-s)^{\sigma-n-1} b_1(s)ds \right) < \infty,
\end{aligned} \tag{3.10}$$

where

$$\nu_1(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!} \eta_{n+i}, \qquad \nu_2(t) = \sum_{i=0}^{m-n-1} \frac{t^i}{i!} \xi_{n+i}. \tag{3.11}$$

Proof. Let us define an operator $\digamma : X \times Y \to X \times Y$ by

$$\digamma(w_1, w_2)(t) = (\digamma_1 w_2(t), \digamma_2 w_1(t)), \tag{3.12}$$

where

$$\begin{aligned}
\digamma_1 w_2(t) &= \nu_1(t) + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\right)ds, \\
\digamma_2 w_1(t) &= \nu_2(t) + \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} g\left(s, \int_0^s \frac{(s-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_1(\tau)d\tau\right)ds,
\end{aligned} \tag{3.13}$$

and $\nu_1(t), \nu_2(t)$ are given by (3.11). In view of (A_1)-(A_2), it follows that $\digamma$ is well defined and continuous.

Define a ball U in the normed space $X \times Y$ as

$$U = \{(w_1(t), w_2(t)) \mid (w_1(t), w_2(t)) \in X \times Y, \|(w_1(t), w_2(t))\|_{X\times Y} < R, t \in [0,1]\}, \tag{3.14}$$

where $R = C/(1-B)$, $B = \max\{B_1, B_2\}, C = \max\{C_1, C_2\}$, and let $\mathcal{D} \subset X \times Y$ be such that $\mathcal{D} = \overline{U}$.

Let $(w_1, w_2) \in \overline{U}$. Then $\|(w_1, w_2)\|_{X\times Y} \le R$, and

$$\begin{aligned}
\|\mathcal{F}_1 w_2\|_X &= \sup_{t\in[0,1]}\left| v_1(t) + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\right)ds\right| \\
&\le \sup_{t\in[0,1]}\left(|v_1(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\left|f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\right)\right|ds\right) \\
&\le \sup_{t\in[0,1]}\left(|v_1(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\left(a_1(s) + a_2(s)\int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)}|w_2(\tau)|d\tau\right)ds\right) \\
&\le \sup_{t\in[0,1]}\left(|v_1(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} a_1(s)ds\right) \\
&\quad + \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\left(\int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)}d\tau\right)a_2(s)ds\right)\|w_2\|_Y \\
&\le \sup_{t\in[0,1]}\left(|v_1(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} a_1(s)ds\right) \\
&\quad + \frac{1}{\Gamma(\rho-n)\Gamma(n-\beta+1)}\sup_{t\in[0,1]}\int_0^t (t-s)^{\rho-n-1}s^{n-\beta}a_2(s)ds\|w_2\|_Y \\
&= C_1 + B_1\|w_2\|_Y \le C + BR = R.
\end{aligned} \tag{3.15}$$

Similarly, it can be shown that

$$\|\mathcal{F}_2 w_1\|_Y \le C_2 + B_2\|w_1\|_X \le C + BR = R. \tag{3.16}$$

Hence we conclude that $\|\mathcal{F}(w_1, w_2)\|_{X\times Y} \le R$. This implies that $\mathcal{F}(w_1, w_2) \in \mathcal{D}$. Now we show that $\mathcal{F}$ is a completely continuous operator (continuous and compact). To do this, we first set

$$\begin{aligned}
M_f &= \max_{t\in[0,1]}\left|f\left(t, \int_0^t \frac{(t-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\right)\right|, \\
M_g &= \max_{t\in[0,1]}\left|g\left(t, \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_1(\tau)d\tau\right)\right|.
\end{aligned} \tag{3.17}$$

For $(w_1, w_2) \in U$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}|\mathcal{F}_1 w_2(t_1) - \mathcal{F}_1 w_2(t_2)| &= \left| v_1(t_1) - v_1(t_2) + \int_0^{t_1} \frac{(t_1 - s)^{\rho-n-1}}{\Gamma(\rho - n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau) d\tau\right) ds \right. \\ &\quad \left. - \int_0^{t_2} \frac{(t_2 - s)^{\rho-n-1}}{\Gamma(\rho - n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau) d\tau\right) ds \right| \\ &\leq |v_1(t_1) - v_1(t_2)| + M_f \left| \int_0^{t_1} \frac{(t_1 - s)^{\rho-n-1}}{\Gamma(\rho - n)} ds - \int_0^{t_2} \frac{(t_2 - s)^{\rho-n-1}}{\Gamma(\rho - n)} ds \right| \\ &= |v_1(t_1) - v_1(t_2)| + \frac{M_f}{\Gamma(\rho - n + 1)} \left| t_1^{\rho-n} - t_2^{\rho-n} \right|.\end{aligned} \tag{3.18}$$

Similarly,

$$|\mathcal{F}_2 w_1(t_1) - \mathcal{F}_2 w_1(t_2)| \leq |v_2(t_1) - v_2(t_2)| + \frac{M_g}{\Gamma(\sigma - n + 1)} \left| t_1^{\sigma-n} - t_2^{\sigma-n} \right|. \tag{3.19}$$

Since the functions $t^k, t^{\rho-n}, t^{\sigma-n}$ are uniformly continuous on $[0,1]$, it follows from the above estimates that $\mathcal{F}U$ is an equicontinuous set. Also, it is uniformly bounded as $\mathcal{F}U \subset U$. Thus, we conclude that $\mathcal{F}$ is a completely continuous operator.

Now, let us consider the eigenvalue problem

$$(w_1, w_2) = \lambda \mathcal{F}(w_1, w_2) = (\lambda \mathcal{F}_1 w_2, \lambda \mathcal{F}_2 w_1), \quad \lambda \in (0,1). \tag{3.20}$$

Assuming that (w_1, w_2) is a solution of (3.20) for $\lambda \in (0,1)$, we find that

$$\begin{aligned}\|w_1\|_X &= \sup_{t\in[0,1]} |\lambda \mathcal{F}_1 w_2| \\ &= \sup_{t\in[0,1]} \left| \lambda v_1(t) + \lambda \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho - n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau) d\tau\right) ds \right| \\ &\leq \sup_{t\in[0,1]} \left(|v_1(t)| + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho - n)} \left| f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau) d\tau\right) \right| ds \right) \\ &\leq C + B\|w_2\|_Y,\end{aligned} \tag{3.21}$$

and, in a similar manner,

$$\|w_2\|_Y = \sup_{t\in[0,1]} |\lambda \mathcal{F}_2 w_1| \le C + B\|w_1\|_X, \tag{3.22}$$

which imply that $(w_1, w_2) \notin \partial U$. Hence, by Theorem 3.1, $\mathcal{F}$ has a fixed point (w_{10}, w_{20}) in $\overline{U}$ such that $\|(w_{10}, w_{20})\|_{X\times Y} \le R$. This completes the proof. □

Thus, by Lemma 3.2 and Theorem 3.4, the solution (u_0, v_0) of (1.1)-(1.2) is given by

$$\begin{aligned} u_0(t) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\eta_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_{10}(s)ds, \\ v_0(t) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\xi_k + \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} w_{20}(s)ds, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} w_{10}(t) &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i} + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_{20}(\tau)d\tau\right)ds, \\ w_{20}(t) &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\xi_{n+i} + \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} g\left(s, \int_0^s \frac{(s-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_{10}(\tau)d\tau\right)ds. \end{aligned} \tag{3.24}$$

Now we allow the nonlinear f in (1.1) to depend on v in addition to ${}^cD^\beta v(t)$ and g in (1.2) to depend on $u(t)$ together with ${}^cD^\alpha u(t)$. Precisely, for $\rho, \sigma \in (m-1, m), \alpha, \beta \in (n-1, n), m, n \in \mathbf{N}$, we consider the following fractional differential system:

$$\begin{aligned} {}^cD^\rho u(t) &= f\left(t, v(t), {}^cD^\beta v(t)\right), \quad 0 < t \le 1, \\ {}^cD^\sigma v(t) &= g\left(t, u(t), {}^cD^\alpha u(t)\right), \quad 0 < t \le 1, \end{aligned} \tag{3.25}$$

subject to the initial conditions given by (1.1)-(1.2), where $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions.

In order to prove the existence of solution for the system (3.25), we need the following assumptions:

($\overline{\mathrm{A}}_1$) let $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $f(0,0,0) = 0$ and $f(t,0,0) \ne 0$ on a compact subinterval of $(0,1]$;

($\overline{\mathrm{A}}_2$) let $g : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $g(0,0,0) = 0$ and $g(t,0,0) \ne 0$ on a compact subinterval of $(0,1]$;

$(\overline{\mathrm{A}}_3)$ there exist nonnegative functions $a_1, a_2, a_3, b_1, b_2, b_3 \in C[0,1]$ such that

$$\begin{aligned} |f(t,x,y)| &\le a_1(t) + a_2(t)|x| + a_3(t)|y|, \quad t \in [0,1], \\ |g(t,x,y)| &\le b_1(t) + b_2(t)|x| + b_3(t)|y|, \quad t \in [0,1]. \end{aligned} \tag{3.26}$$

In this case, w_1 and w_2 involved in the coupled system of integral equations (3.6) modify to the following form:

$$\begin{aligned} w_1(t) &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\eta_{n+i} + \int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} f\left(s, w_2(s), \int_0^s \frac{(s-\tau)^{n-\beta-1}}{\Gamma(n-\beta)} w_2(\tau)d\tau\right) ds, \\ w_2(t) &= \sum_{i=0}^{m-n-1} \frac{t^i}{i!}\xi_{n+i} + \int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} g\left(s, w_1(s), \int_0^s \frac{(s-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} w_1(\tau)d\tau\right) ds. \end{aligned} \tag{3.27}$$

The following corollary presents the analogue form of Theorem 3.4 for the fractional differential system (3.25).

Corollary 3.5. *Suppose that the assumptions $(\overline{A}_1)$–$(\overline{A}_3)$ hold. Then there exists a solution for the coupled integral equation* (3.27) *if*

$$\begin{aligned} \overline{B}_1 &= \frac{1}{\Gamma(\rho-n)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\rho-n-1}\left(a_2(s) + \frac{s^{n-\beta}}{\Gamma(n-\beta+1)} a_3(s)\right) ds < 1, \\ \overline{B}_2 &= \frac{1}{\Gamma(\sigma-n)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\sigma-n-1}\left(b_2(s) + \frac{s^{n-\alpha}}{\Gamma(n-\alpha+1)} b_3(s)\right) ds < 1, \\ 0 < C_1 &= \sup_{t\in[0,1]} \left(|\nu_1(t)| + \frac{1}{\Gamma(\rho-n)} \int_0^t (t-s)^{\rho-n-1} a_1(s) ds\right) < \infty, \\ 0 < C_2 &= \sup_{t\in[0,1]} \left(|\nu_2(t)| + \frac{1}{\Gamma(\sigma-n)} \int_0^t (t-s)^{\sigma-n-1} b_1(s) ds\right) < \infty, \end{aligned} \tag{3.28}$$

where $\nu_1(t)$ and $\nu_2(t)$ are given by (3.11).

The method of proof is similar to that of Theorem 3.3.

4. Uniqueness Result

To prove the uniqueness of solutions of (1.1)-(1.2), we need the following assumptions.

(A_4) For each $R > 0$, there exist nonnegative functions $\delta_1(t)$ and $\delta_2(t)$ such that

$$\begin{aligned} |f(t,x_1) - f(t,x_2)| &\le \delta_1(t)|x_1 - x_2|, \quad t \in [0,1],\ x_1, x_2 \in \mathbb{R}, \\ |g(t,x_1) - g(t,x_2)| &\le \delta_2(t)|x_1 - x_2|, \quad t \in [0,1],\ x_1, x_2 \in \mathbb{R}. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Assume that* (A_1), (A_2), *and* (A_4) *hold. Furthermore,*

$$\zeta_1 = \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\frac{s^{n-\beta}}{\Gamma(n-\beta+1)}\delta_1(s)ds\right) < 1, \tag{4.2}$$

$$\zeta_2 = \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)}\frac{s^{n-\alpha}}{\Gamma(n-\alpha+1)}\delta_2(s)ds\right) < 1, \tag{4.3}$$

$$0 < \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\left|f(s,0)\right|ds\right) < \infty, \tag{4.4}$$

$$0 < \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)}\left|g(s,0)\right|ds\right) < \infty. \tag{4.5}$$

Then there exists a unique solution for the coupled integral equations (3.3) *and* (3.5).

Proof. For $t \in [0,1]$, we define

$$\mathcal{D} = \{(w_1(t), w_2(t)) \in X \times Y : \|(w_1(t), w_2(t))\|_{X\times Y} \le R\}, \tag{4.6}$$

where

$$R = \frac{1}{1-\zeta_1}\sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\left|f(s,0)\right|ds\right). \tag{4.7}$$

As before, we define the operator $\mathcal{F} : X\times Y \to X\times Y$ by $\mathcal{F}(w_1, w_2)(t) = (\mathcal{F}_1 w_2(t), \mathcal{F}_2 w_1(t))$, which is well defined and continuous. For $(w_1, w_2) \in \mathcal{D}$, using (4.2) and (4.4), we have

$$\begin{aligned}\|\mathcal{F}_1 w_2\|_X &\le \|\mathcal{F}_1 w_2 - \mathcal{F}_1 0\|_X + \|\mathcal{F}_1 0\|_X \\ &\le \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\frac{s^{n-\beta}}{\Gamma(n-\beta+1)}\delta_1(s)ds\right)\|w_2\|_Y \\ &\quad + \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\left|f(s,0)\right|ds\right) \\ &\le \zeta_1 R + (1-\zeta_1)R \le R.\end{aligned} \tag{4.8}$$

Similarly, by using (4.3) and (4.5), it can be shown that $\|\mathcal{F}_2 w_1\|_Y \le R$. Thus, $\mathcal{F} : \mathcal{D} \to \mathcal{D}$.

Now, for $(w_1, w_2), (\overline{w}_1, \overline{w}_2) \in \mathcal{P}$, we obtain

$$
\begin{aligned}
\|\mathcal{F}_1 w_2 - \mathcal{F}_1 \overline{w}_2\|_X &= \sup_{t\in[0,1]} |\mathcal{F}_1 w_2 - \mathcal{F}_1 \overline{w}_2| \\
&\le \sup_{t\in[0,1]} \left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} \frac{s^{n-\beta}}{\Gamma(n-\beta+1)} \delta_1(s) ds \right) \|w_2 - \overline{w}_2\|_Y \\
&= \zeta_1 \|w_2 - \overline{w}_2\|_Y.
\end{aligned}
\tag{4.9}
$$

In a similar manner, we find that

$$
\|\mathcal{F}_2 w_1 - \mathcal{F}_2 \overline{w}_1\|_Y \le \zeta_2 \|w_1 - \overline{w}_1\|_X. \tag{4.10}
$$

Since $\zeta_1 < 1$, $\zeta_2 < 1$, therefore $\mathcal{F}$ is a contraction. Hence, by Banach contraction principle, $\mathcal{F}$ has a unique fixed point $(\overline{w}_{10}, \overline{w}_{20})$ in $\mathcal{P}$ such that $\|(\overline{w}_{10}, \overline{w}_{20})\|_{X\times Y} \le R$, which is a solution of the coupled integral equations (3.3) and (3.5). This completes the proof. □

The following Corollary ensures the uniqueness of the solutions of (3.25). We do not provide the proof as it is similar to that of Theorem 4.1.

Corollary 4.2. *Assume that* $(\overline{A}_1)$, $(\overline{A}_2)$, *and the following condition hold:*

$(\overline{A}_4)$ *For each* $R > 0$, *there exist nonnegative functions* $\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$, *and* $\mu_4(t)$ *such that*

$$
\begin{aligned}
\left| f(t, x_1, \overline{x}_1) - f(t, x_2, \overline{x}_2) \right| &\le \mu_1(t)|x_1 - x_2| + \mu_2(t)|\overline{x}_1 - \overline{x}_2|, \quad t \in [0,1],\ x_1, x_2, \overline{x}_1, \overline{x}_2 \in \mathbb{R}, \\
\left| g(t, x_1, \overline{x}_1) - g(t, x_2, \overline{x}_2) \right| &\le \mu_3(t)|x_1 - x_2| + \mu_4(t)|\overline{x}_1 - \overline{x}_2|, \quad t \in [0,1],\ x_1, x_2, \overline{x}_1, \overline{x}_2 \in \mathbb{R}.
\end{aligned}
\tag{4.11}
$$

Furthermore,

$$
\begin{aligned}
\eta_1 &= \frac{1}{\Gamma(\rho-n)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\rho-n-1} \left(\mu_1(s) + \frac{s^{n-\beta}}{\Gamma(n-\beta+1)} \mu_2(s) \right) ds < 1, \\
\eta_2 &= \frac{1}{\Gamma(\sigma-n)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\sigma-n-1} \left(\mu_3(s) + \frac{s^{n-\alpha}}{\Gamma(n-\alpha+1)} \mu_4(s) \right) ds < 1, \\
0 &< \sup_{t\in[0,1]} \left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)} \left| f(s,0,0) \right| ds \right) < \infty, \\
0 &< \sup_{t\in[0,1]} \left(\int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)} \left| g(s,0,0) \right| ds \right) < \infty.
\end{aligned}
\tag{4.12}
$$

Then there exists a unique solution for the coupled integral equation (3.27).

5. Example

For $m = 3$ and $n = 2$, we consider the following coupled system of fractional differential equations:

$$
\begin{gathered}
{}^cD^{5/2}u(t) = \frac{t}{6} + \frac{\sqrt{t}(2-t)}{6}\ {}^cD^{3/2}v(t), \quad 0 < t \le 1, \\
u(0) = \eta_0, \quad u'(0) = \eta_1, \quad u''(0) = \eta_2, \\
{}^cD^{11/5}v(t) = \frac{t}{9} + \frac{t^{1/4}(3-t)}{9}\ {}^cD^{5/4}u(t), \quad 0 < t \le 1, \\
v(0) = \xi_0, \quad v'(0) = \xi_1, \quad v''(0) = \xi_2.
\end{gathered}
\tag{5.1}
$$

Here $\rho = 5/2, \sigma = 11/5, \beta = 3/2, \alpha = 5/4$, and $\eta_i, \xi_i \in (0,1), i = 1,2,3$. Clearly, the assumptions (A_1)—(A_3) are satisfied with $a_1(t) = 1/6, a_2(t) = \sqrt{t}/3, b_1(t) = 1/9, b_2(t) = t^{1/4}/3$. In this case

$$
\begin{gathered}
B_1 = \frac{1}{3\Gamma(1/2)\Gamma(3/2)} \sup_{t\in[0,1]}\left(\int_0^t (t-s)^{-1/2} s\,ds\right) = \frac{8}{9\pi} < 1, \\
B_2 = \frac{4}{9\Gamma(1/5)\Gamma(3/4)} \sup_{t\in[0,1]}\left(\int_0^t (t-s)^{-4/5} s\,ds\right) = \frac{50}{27\Gamma(1/5)\Gamma(3/4)} < 1, \\
C_1 = \sup_{t\in[0,1]}\left(\eta_2 + \frac{1}{6\Gamma(1/2)}\int_0^t (t-s)^{-1/2} ds\right) = \eta_2 + \frac{1}{3\Gamma(1/2)} < \infty, \\
C_2 = \sup_{t\in[0,1]}\left(\xi_2 + \frac{1}{9\Gamma(1/5)}\int_0^t (t-s)^{-4/5} ds\right) = \xi_2 + \frac{5}{9\Gamma(1/5)} < \infty,
\end{gathered}
\tag{5.2}
$$

where $\Gamma(1/5) = 4.590843712, \Gamma(3/4) = 1.225416702$. Thus, all the conditions of Theorem 3.4 are satisfied, and hence there exists a solution of (5.1).

To prove the uniqueness of solutions of (5.1), we just need to verify the assumption (A_4). With $\delta_1(t) = \sqrt{t}/3$ and $\delta_2(t) = t^{1/4}/3$, we find that

$$
\begin{gathered}
\zeta_1 = \sup_{t\in[0,1]}\left(\frac{1}{3\Gamma(1/2)\Gamma(3/2)}\int_0^t (t-s)^{-1/2} s\,ds\right) = \frac{8}{9\pi} < 1, \\
\zeta_2 = \frac{4}{9\Gamma(1/5)\Gamma(3/4)} \sup_{t\in[0,1]}\left(\int_0^t (t-s)^{-4/5} s\,ds\right) = \frac{50}{27\Gamma(1/5)\Gamma(3/4)} < 1, \\
\sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\rho-n-1}}{\Gamma(\rho-n)}\left|f(s,0)\right| ds\right) = \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{-1/2}s}{6\Gamma(1/2)} ds\right) = \frac{2}{9\sqrt{\pi}} < \infty, \\
\sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{\sigma-n-1}}{\Gamma(\sigma-n)}\left|g(s,0)\right| ds\right) = \sup_{t\in[0,1]}\left(\int_0^t \frac{(t-s)^{-4/5}s}{9\Gamma(1/5)} ds\right) = \frac{25}{54\Gamma(1/5)} < \infty.
\end{gathered}
\tag{5.3}
$$

As all the conditions of Theorem 4.1 hold, therefore the conclusion of Theorem 4.1 applies, and hence the coupled system of fractional differential equation (5.1) has a unique solution.

6. Conclusions

We have presented some existence and uniqueness results for an initial value problem of coupled fractional differential systems involving the Caputo type fractional derivative. The nonlinearities in the coupled fractional differential system depend on (i) the fractional derivatives of lower orders, (ii) the unknown functions together with the fractional derivative of lower orders. The proof of the existence results is based on the nonlinear alternative of Leray-Schauder, while the uniqueness of the solutions is proved by applying the Banach contraction principle. The present work can be extended to nonlocal coupled systems of nonlinear fractional differential equations.

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