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Research Article

Nielsen Type Numbers of Self-Maps on the Real Projective Plane

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Employing the induced endomorphism of the fundamental group and using the homotopy classification of self-maps of real projective plane RP^2, we compute completely two Nielsen type numbers, $\mathrm{NP}_n(f)$ and $\mathrm{NF}_n(f)$, which estimate the number of periodic points of f and the number of fixed points of the iterates of map f.

1. Introduction

Topological fixed point theory deals with the estimation of the number of fixed points of maps. Readers are referred to [1] for a detailed treatment of this subject. The number of essential fixed point classes of self-maps f of a compact polyhedron is called the Nielsen number of f, denoted $N(f)$. It is a lower bound for the number of fixed points of f. The Nielsen periodic point theory provides two homotopy invariants $\mathrm{NP}_n(f)$ and $\mathrm{NF}_n(f)$ called the prime and full Nielsen-Jiang periodic numbers, respectively. A Nielsen type number $\mathrm{NP}_n(f)$ was introduced in [1], which is a lower bound for the number of periodic points of least period n. Another Nielsen type number $\mathrm{NF}_n(f)$ can be found in [1, 2], which is a lower bound for the number of fixed points of f^n.

The computation of these two Nielsen type numbers $\mathrm{NP}_n(f)$ and $\mathrm{NF}_n(f)$ is very difficult. There are very few results. Hart and Keppelmann calculated these two numbers for the periodic homeomorphisms on orientable surfaces of positive genus [3]. In [4], Marzantowicz and Zhao extend these computations to the periodic homeomorphisms on arbitrary closed surfaces. In [5], Kim et al. provide an explicit algorithm for the computation of maps on the Klein bottle. Jezierski gave a formula for $H\,\mathrm{Per}(f)$ for all self-maps of real projective spaces of dimension at least 3 in [6], where $H\,\mathrm{Per}(f)$ is the set of homotopy periods of f which consists of the set of natural numbers n such that every map homotopic to f has periodic points of minimal period n. Actually, $H\,\mathrm{Per}(f)$ is just the set $\{n \in N \mid \mathrm{NP}_n(f) \neq 0\}$.

The purpose of this paper is to give a complete computation of the two Nielsen type numbers $\mathrm{NP}_n(f)$ and $\mathrm{NF}_n(f)$ for all maps on the real projective plane RP^2.

2. Preliminaries

We list some definitions and properties we need for our discussion. For the details see [1, 2, 7]. We consider a topological space X with universal covering $p : \widetilde{X} \to X$. Assume f is a self-map of X and let f^n be its nth iterate. The nth iterate $\widetilde{f}^n$ of $\widetilde{f}$ is a lifting of f^n. We write $D(\widetilde{X})$ for the covering transformation group and identify $D(\widetilde{X}) = \pi_1(X)$. We denote the set of all fixed points of f by $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$.

Definition 2.1. Given a lifting $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ of f, then every lifting of f can be uniquely written as $\alpha \circ \widetilde{f}$, with $\alpha \in D(\widetilde{X})$. For every $\alpha \in D(\widetilde{X})$, $\widetilde{f} \circ \alpha$ is also a lifting of f, so there is a unique element α' such that $\alpha' \circ \widetilde{f} = \widetilde{f} \circ \alpha$. This gives a map

$$
\begin{gathered}
\widetilde{f}_\pi : D\left(\widetilde{X}\right) \longrightarrow D\left(\widetilde{X}\right), \\
\alpha \longmapsto \widetilde{f}_\pi(\alpha) = \alpha',
\end{gathered}
\tag{2.1}
$$

that is, $\widetilde{f} \circ \alpha = \widetilde{f}_\pi(\alpha) \circ \widetilde{f}$. This map may depend on the choice of the lift $\widetilde{f}$.

We obtain $\widetilde{f}_\pi = f_\pi$, where f_π is the homomorphism of the fundamental group induced by map f (see [1, Lemma 1.3]). Two liftings $\widetilde{f}$ and $\widetilde{f}'$ of $f : X \to X$ are said to be conjugate if there exists $\gamma \in D(\widetilde{X})$ such that $\widetilde{f}' = \gamma \circ \widetilde{f} \circ \gamma^{-1}$. Lifting classes are equivalence classes by conjugacy, denoted by $[\widetilde{f}] = \{\gamma \circ \widetilde{f} \circ \gamma^{-1} \mid \gamma \in D(\widetilde{X})\}$, we will also call them fixed point classes and denote their set by $\mathrm{FPC}(f)$. We will call about these classes referring either to the fixed point class $[\widetilde{f}]$ or to the set $p\,\mathrm{Fix}(\widetilde{f})$(Nielsen class).

The restriction $f : \mathrm{Fix}(f^n) \to \mathrm{Fix}(f^n)$ permutes Nielsen classes. We denote the corresponding self-map of $\mathrm{FPC}(f^n)$ by f_{FPC}. This map can be described as follows. For a given $[\alpha \widetilde{f}^n] \in \mathrm{FPC}(f^n)$, there is a unique $\beta \in D(\widetilde{X})$ such that the diagram

$$
\begin{tikzcd}
\widetilde{X} \arrow[r, "\alpha\widetilde{f}^n"] \arrow[d, "\widetilde{f}"'] & \widetilde{X} \arrow[d, "\widetilde{f}"] \\
\widetilde{X} \arrow[r, "\beta\widetilde{f}^n"'] & \widetilde{X}
\end{tikzcd}
\tag{2.2}
$$

commutes. We put $f_{\mathrm{FPC}}[\alpha \widetilde{f}^n] = [\beta \widetilde{f}^n]$.

Let $\widetilde{f}$ be a given lifting of f. Obviously, we have $p\,\mathrm{Fix}(\widetilde{f}) \subset p\,\mathrm{Fix}(\widetilde{f}^n)$.

Definition 2.2. Let $[\tilde{f}]$ be a lifting class of $f : X \to X$. Then the lifting class $[\tilde{f}^n]$ of f^n is evidently independent of the choice of representative $\tilde{f}$, so we have a well-defined correspondence

$$\begin{gathered}\iota : \mathrm{FPC}(f) \longrightarrow \mathrm{FPC}(f^n),\\ \left[\tilde{f}\right] \longrightarrow \left[\tilde{f}^n\right].\end{gathered} \tag{2.3}$$

Thus, for $m \mid n$, we also have

$$\iota : \mathrm{FPC}(f^m) \longrightarrow \mathrm{FPC}(f^n). \tag{2.4}$$

The next proposition shows that $f_{\mathrm{FPC}} : \mathrm{FPC}(f^n) \to \mathrm{FPC}(f^n)$ is a built-in automorphism. And the correspondence can help us to study the relations and properties between the fixed point classes of f^n.

Proposition 2.3 (see [1, Proposition 3.3]). (i)*Let $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$ be liftings of f, then $f_{\mathrm{FPC}} : [\tilde{f}_n \circ \cdots \circ \tilde{f}_2 \circ \tilde{f}_1] \mapsto [\tilde{f}_1 \circ \tilde{f}_n \cdots \circ \tilde{f}_2]$.*

(ii)*$f(p\,\mathrm{Fix}(\tilde{f}_n \circ \cdots \circ \tilde{f}_2 \circ \tilde{f}_1) = p\,\mathrm{Fix}(\tilde{f}_1 \circ \tilde{f}_n \cdots \circ \tilde{f}_2)$, thus the f-image of a fixed point class of f^n is again a fixed point class of f^n.*

(iii) *$\mathrm{index}(f^n, p\,\mathrm{Fix}(\tilde{f}_n \circ \cdots \circ \tilde{f}_2 \circ \tilde{f}_1)) = \mathrm{index}(f^n, p\,\mathrm{Fix}(\tilde{f}_1 \circ \tilde{f}_n \cdots \circ \tilde{f}_2))$, f induces an index-preserving permutation among the fixed point classes of f^n.*

(iv) *$(f_{\mathrm{FPC}})^n = id : \mathrm{FPC}(f^n) \to \mathrm{FPC}(f^n)$.*

Proposition 2.4. *Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lifting of f. Then $\iota[\alpha \circ \tilde{f}] = [\alpha^{(n)} \circ \tilde{f}^n]$, where $\alpha^{(n)} = \alpha f_\pi(\alpha) \cdots f_\pi^{n-1}(\alpha)$, and $f_{\mathrm{FPC}}[\alpha \circ \tilde{f}^n] = [f_\pi(\alpha) \circ \tilde{f}^n]$.*

As usual a periodic point class of f with period n is synonymous with a fixed point class of f^n. The quotient set of $\mathrm{FPC}(f^n)$ under the action of the automorphism f_{FPC} is denoted by $\mathrm{Orb}_n(f)$. Every element in $\mathrm{Orb}_n(f)$ is called a periodic point class orbit of f with period n.

Definition 2.5. A periodic point class $[\sigma \tilde{f}^n]$ of period n is reducible to period m if it contains some periodic point class $[\xi \tilde{f}^m]$ of period m, that is $\sigma \tilde{f}^n = (\xi \tilde{f}^m)^{n/m}$, with $\sigma, \xi \in D(\tilde{X})$. It is irreducible if it is not reducible to any lower period.

We say that an orbit $\langle\alpha\rangle \in \mathrm{Orb}_n(f)$ is reducible to m, with $m \mid n$, if there exists a $\langle\beta\rangle \in \mathrm{Orb}_m(f)$ for some $m \mid n$, such that $\iota(\langle\beta\rangle) = \langle\alpha\rangle$. We define the depth of $\langle\alpha\rangle$ as the smallest positive integer to which $\langle\alpha\rangle$ is reducible, denoted by $d = d(\langle\alpha\rangle)$. If $\langle\alpha\rangle$ is not reducible to any $m \mid n$ with $m \neq n$, then that element is said to be irreducible.

From Proposition 2.4, we have a correspondence $f_{\mathrm{FPC}} : [\beta] \to [f_\pi(\beta)]$, Thus we consider the following corollary.

Corollary 2.6. *The fixed point class represented by $[\beta]$ is reducible if and only if the fixed point class represented by $[f_\pi(\beta)]$ is reducible.*

Suppose that X is a connected compact polyhedron and f is a self-map of X.

Definition 2.7. The prime Nielsen-Jiang periodic number $\mathrm{NP}_n(f)$ is defined by

$$\mathrm{NP}_n(f) = n \times \#\{\langle\alpha\rangle \in \mathrm{Orb}_n(f) \mid \langle\alpha\rangle \text{ is essential and irreducible}\}. \tag{2.5}$$

Definition 2.8. A periodic orbit set S is said to be a representative of T if every orbit of T reduces to an orbit of S. A finite set of orbits S is said to be a set of n-representatives if every essential m-orbit $\langle\beta\rangle$ with $m \mid n$ is reducible to some $\langle\alpha\rangle \in S$.

Definition 2.9. The full Nielsen-Jiang periodic number $\mathrm{NF}_n(f)$ is defined as

$$\mathrm{NF}_n(f) = \min\left\{ \sum_{\langle\alpha\rangle\in S} d(\langle\alpha\rangle) \mid S \text{ is a set of } n\text{-representatives} \right\}. \tag{2.6}$$

3. Nielsen Numbers of Self-Maps on the Real Projective Plane

Let $p : S^2 \to \mathrm{RP}^2$ be the universal covering. Let $f : \mathrm{RP}^2 \to \mathrm{RP}^2$ be a self-map, then f has a lifting $\tilde{f} : S^2 \to S^2$, that is, the diagram

$$\begin{tikzcd} S^2 \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & S^2 \arrow[d, "p"] \\ \mathrm{RP}^2 \arrow[r, "f"'] & \mathrm{RP}^2 \end{tikzcd} \tag{3.1}$$

commutes. Assume $\tilde{f}$ is a lifting of f, then the other lifting of f is $\tau\tilde{f}^n$, where τ is the nontrivial element of $\pi_1(\mathrm{RP}^2)$. Here we give the definition of the absolute degree (see also [8]).

Definition 3.1. Let $f : \mathrm{RP}^2 \to \mathrm{RP}^2$ be a self-map, and let $\tilde{f} : S^2 \to S^2$ be a lifting of f. The lifting degree of f is defined to be the absolute value of the degree of $\tilde{f}$, denoted $\widetilde{\deg}(f)$.

Obviously, this definition is independent of the choice of representative $\tilde{f}$ in $[\tilde{f}]$, moreover homotopic maps have the same lifting degree.

The endomorphism on the fundamental group induced by f is f_π. Since $\pi_1(\mathrm{RP}^2) = Z_2$, either f_π is the identity or it is trivial. If f_π is trivial, then f has a lifting $f' : \mathrm{RP}^2 \to S^2$. We define the mod 2 degree $\widetilde{\deg}_2(f) \in Z_2$ as $\widetilde{\deg}_2(f) = \deg(f') \bmod 2$. The homotopy classification of self-maps on real projective plane is as follows.

Proposition 3.2 (see [9, Theorems III and II]). *Let $f, g : RP^2 \to RP^2$ be self-maps, they are homotopic if and only if one of the cases is satisfied:*

(1) *the endomorphism $f_\pi = g_\pi$ is the identity and $\widetilde{\deg}(f) = \widetilde{\deg}(g)$;*

(2) *the endomorphism $f_\pi = g_\pi$ is trivial and $\widetilde{\deg}_2(f) = \widetilde{\deg}_2(g)$.*

In the first case, in which the degree of f is nonzero, the homotopy classification is completely determined by the lifting degree. Since f_π is the identity, every lifting $\tilde{f}$ commutes with the antipodal map of S^2, thus $\widetilde{\deg}(f)$ is odd. In the second case, we note that the lifting degree is zero. Then we get two classes: $\widetilde{\deg}_2(f) = 0$ or 1.

The Nielsen numbers of all self-maps on RP^2 were computed in [8], we give the proposition here.

Proposition 3.3. *Let f be a self-map of RP^2 with lifting degree $\widetilde{\deg}(f)$.Then*

$$N(f) = \begin{cases} 1, & \text{if } \widetilde{\deg}(f) = 0 \text{ or } 1, \\ 2, & \text{if } \widetilde{\deg}(f) > 1. \end{cases} \tag{3.2}$$

4. Nielsen Type Numbers of Self-Maps on RP^2

4.1. The Reducibility of Periodic Point Classes

Let $f : \mathrm{RP}^2 \to \mathrm{RP}^2$ be a self-map and let $\tilde{f}$ be a lifting of f. We will use the following proposition to examine the reducibility of the periodic point classes of f.

Proposition 4.1. *The two periodic point classes $p\,\mathrm{Fix}(\tilde{f}^n)$ and $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ of f with period n are the same periodic point class if and only if the homomorphism $f_\pi : \pi_1(RP^2) \to \pi_1(RP^2)$ induced by f is trivial.*

Proof. Sufficiency is obvious. It remains to prove necessity.

For each n, if $p\,\mathrm{Fix}(\tilde{f}^n) = p\,\mathrm{Fix}(\tau\tilde{f}^n)$, then we have $\tau^{-1}(\tau\tilde{f}^n)\tau = \tilde{f}^n$, that is $\tilde{f}^n\tau = \tilde{f}^n$. By applying Definition 2.1 we get $f_\pi^n(\tau)\tilde{f}^n = \tilde{f}^n$, thus $f_\pi^n(\tau) = \mathrm{id}$. This shows that f_π^n is trivial. □

From this proposition we conclude that if f_π is trivial, then there is a unique periodic point class $p\,\mathrm{Fix}(\tilde{f}^n)$ of f with any period n; if f_π is the identity, then there are two distinct periodic point classes $p\,\mathrm{Fix}(\tilde{f}^n)$ and $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ of f for any period n.

Theorem 4.2. *Let $f : RP^2 \to RP^2$ be a self-map, and let $f_\pi : \pi_1(RP^2) \to \pi_1(RP^2)$ be the homomorphism induced by f. Let $\tilde{f}$ be a lifting of f. Then, for each $n = 2^s \cdot t$ with $s \geq 0$ and odd t,*

(1) *if f_π is trivial, the unique periodic point class $p\,\mathrm{Fix}(\tilde{f}^n)$ of f is reducible to the periodic point class of period 1.*

(2) *if f_π is the identity, the two distinct periodic point classes $p\,\mathrm{Fix}(\tilde{f}^n)$ and $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ of f lie in different periodic orbits. Moreover, the periodic point class $p\,\mathrm{Fix}(\tilde{f}^n)$ is reducible to $p\,\mathrm{Fix}(\tilde{f})$ and the orbit containing $p\,\mathrm{Fix}(\tilde{f}^n)$ has depth 1. The periodic point class $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ is reducible to $p\,\mathrm{Fix}(\tau\tilde{f})$ and the orbit containing $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ has depth 1 if n is odd; is reducible to $p\,\mathrm{Fix}(\tau\tilde{f}^{2^s})$ and the orbit containing $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ has depth 2^s if $n = 2^s \cdot t$ with odd $t > 1$ and $s > 0$; and is irreducible if $n = 2^s$ with $s > 0$.*

Proof. We analyze the reducibility as follows.

Case 1 (f_π is trivial). Now, the unique point class in $\mathrm{FPC}(f^n)$ reduces to the unique point class in $\mathrm{FPC}(f)$, hence its depth equals 1.

Case 2 (f_π is the identity). There are two periodic point classes $p\,\mathrm{Fix}(\tilde{f}^n)$ and $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ of f for each n. By Proposition 2.4, we have $f_{\mathrm{FPC}}[\tau\tilde{f}^n] = [f_\pi(\tau)\tilde{f}^n] = [\tau\tilde{f}^n]$, hence, these two periodic point classes lie in different orbits. It is easy to see that the class $p\,\mathrm{Fix}(\tilde{f}^n)$ is reducible to $p\,\mathrm{Fix}(\tilde{f})$. So the depth of this periodic point class orbit of f is 1. Determining whether the periodic point class $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ is reducible or not is a little complicated because it depends on the value of n.

Notice that $(\tau\tilde{f})^n = \underbrace{\tau\tilde{f}\circ\tau\tilde{f}\cdots\circ\tau\tilde{f}}_{n} = \tau\cdot f_\pi(\tau)\cdot f_\pi^2(\tau)\cdot\cdots\cdot f_\pi^{n-1}(\tau)\tilde{f}^n = \tau^n\tilde{f}^n$.

We discuss the cases for $n = 2^s\cdot t$ with $s \ge 0$ and odd t as follows. Let us recall that $\tau^n = \tau$ for n odd and $\tau^n = 1$ for n even.

Subcase 2.1. If $s = 0$, that is, n is odd, then we have $(\tau\tilde{f})^n = \tau\tilde{f}^n$. The periodic point class $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ is reducible to $p\,\mathrm{Fix}(\tau\tilde{f})$. We conclude that the depth of the periodic point class orbit of f with period odd n is 1.

Subcase 2.2. If $s > 0$ and $t = 1$, that is $n = 2^s$, then we have $(\tau\tilde{f})^n \neq \tau\tilde{f}^n$. The periodic point class $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ is irreducible.

Subcase 2.3. If $s > 0$ and $t > 1$, then we have $\tau\tilde{f}^n = (\tau\tilde{f}^{2^s})^t$. The periodic point class $p\,\mathrm{Fix}(\tau\tilde{f}^n)$ is reducible to $p\,\mathrm{Fix}(\tau\tilde{f}^{2^s})$. Therefore, the depth of the periodic point class orbit of f with period $2^s\cdot t$ with $s > 0$, $t > 1$ is 2^s.

□

For any k, we set $F_0^{(k)} = p\,\mathrm{Fix}(\tilde{f}^k)$ and $F_\tau^{(k)} = p\,\mathrm{Fix}(\tau\tilde{f}^k)$. Thus, if the homomorphism f_π induced by f is trivial, we find that the periodic point class orbit with period k is $\{\langle F_0^{(k)}\rangle\}$; whereas if f_π is the identity, the two periodic point class orbits with period k are $\{\langle F_0^{(k)}\rangle\}$ and $\{\langle F_\tau^{(k)}\rangle\}$. Moreover, for each k, whether f_π is trivial or the identity, we have $\mathrm{FPC}(f^k) = \mathrm{Orb}_k(f)$ and each periodic point class orbit with period k of f has a unique k-periodic point class of f. We discuss the k-periodic point class in the following result.

Lemma 4.3. *Let $f : RP^2 \to RP^2$ be a self-map and let $\tilde{f}$ be a lifting of f. Then*

$$\mathrm{index}\left(f, p\,\mathrm{Fix}\left(\tilde{f}\right)\right) = \begin{cases} \dfrac{1+\deg\left(\tilde{f}\right)}{2}, & \text{if } \deg\left(\tilde{f}\right) \text{ is odd,} \\ 1, & \text{if } \deg\left(\tilde{f}\right) \text{ is even.} \end{cases} \tag{4.1}$$

Corollary 4.4. *Let $f : RP^2 \to RP^2$ be a self-map, and let $f_\pi : \pi_1(RP^2) \to \pi_1(RP^2)$ be the homomorphism induced by f. Then, for any k,*

(1) *If f_π is trivial, then the periodic point class $p\,\mathrm{Fix}(\tilde{f}^k)$ is essential.*

(2) *If f_π is the identity, then the periodic point class $p\,\mathrm{Fix}(\tilde{f}^k)$ is essential; the fixed point class $p\,\mathrm{Fix}(\tau\tilde{f}^k)$ is inessential if $\widetilde{\deg}(f) = 1$ and is essential if $\widetilde{\deg}(f) > 1$, where $\tilde{f}$ is the lifting of f with $\deg(\tilde{f}) > 0$.*

The above corollary is crucial to our theorem in the next two subsections.

Table 1

	$n=1$	$n>1$ and n is odd	$n=2^s$, $s>0$	$n=2^s\cdot t$, $s>0$ and $t\neq 1$
$\widetilde{\deg}(f)\leq 1$	1	0	0	0
$\widetilde{\deg}(f)>1$	2	0	n	0

4.2. The Prime Nielsen-Jiang Periodic Number $NP_n(f)$ of RP^2

The number $\mathrm{NP}_n(f)$ is a lower bound for the number of periodic points with least period n. The computation of $\mathrm{NP}_n(f)$ is somewhat difficult. We give a detailed computation of $\mathrm{NP}_n(f)$ of RP^2 in this subsection as follows.

Theorem 4.5. *Assume $f : RP^2 \to RP^2$ is a self-map. Then $NP_n(f)$ is given by Table 1.*

Proof. The equality $\mathrm{NP}_1(f) = N(f)$ is true in general, since all Nielsen classes in $\mathrm{Fix}(f)$ are irreducible. Now we assume that $n \geq 2$. For the computation of $\mathrm{NP}_n(f)$, the important thing is to compute the number of essential and irreducible orbits of f.

There are three cases, depending on the lifting degree of f.

Case 1 ($\widetilde{\deg}(f) = 0$). Now f_π is trivial, hence there is a single periodic point class for each n. These classes reduce to $n = 1$, hence $\mathrm{NP}_n(f) = 0$ for $n > 1$.

Case 2 ($\widetilde{\deg}(f) = 1$). We may assume that $f = id_{\mathrm{RP}^2}$. Then we may take $\tilde{f} = id_{S^2}$. Now $[\tilde{f}^n] = [id_{S^2}] \in \mathrm{Orb}_n(f)$ is reducible (for $n \geq 2$), while $[\tau\tilde{f}^n] = [\tau] \in \mathrm{Orb}_n(f)$ is inessential, since $\mathrm{Fix}(\tau)$ is empty. Thus, there is no essential irreducible class.

Case 3 ($\widetilde{\deg}(f) > 1$). We write $F_0^{(k)} = p\,\mathrm{Fix}(\tilde{f}^k)$ and $F_\tau^{(k)} = p\,\mathrm{Fix}(\tau\tilde{f}^k)$ for each k, which are distinct classes. In this case, by Theorem 4.2 (2), the reducibility of periodic point classes of f depends on n. We write $n = 2^s \cdot t$ with $s \geq 0$ and odd t. There are three subcases.

Subcase 3.1 ($s = 0$ and $t > 1$, that is, n is odd and $n > 1$). By Theorem 4.2 (2), both periodic point classes $F_0^{(n)}$ and $F_\tau^{(n)}$ are reducible. Thus, $\mathrm{NP}_n(f) = 0$.

Subcase 3.2 ($s > 0$ and $t = 1$, that is $n = 2^s$). By Theorem 4.2 (2) and Corollary 4.4 (2), the periodic point class $F_0^{(2^s)}$ is reducible and essential; the periodic point class $F_\tau^{(2^s)}$ is irreducible and essential. The number of essential and irreducible periodic point class orbit of f with period 2^s is 1. Thus, $\mathrm{NP}_n(f) = n = 2^s$.

Subcase 3.3 ($s > 0$ and $t > 1$). By Theorem 4.2 (2), the periodic point classes $F_0^{(n)}$ and $F_\tau^{(n)}$ are reducible. Thus, $\mathrm{NP}_n(f) = 0$. □

4.3. The Full Nielsen-Jiang Periodic Number $NF_n(f)$ (See Definition 2.9)

Theorem 4.6. *Let $f : RP^2 \to RP^2$ be a self-map. Then $NF_n(f)$ is given by Table 2.*

Proof. From the definition we have $\mathrm{NF}_1(f) = N(f)$, so we consider the cases for $n \geq 2$. Let S be a set of n-representatives of periodic point class orbits of f and set $h(S) = \{\sum_{<\alpha>\in S} d(\langle\alpha\rangle)\}$.

Table 2

	n is odd	$n = 2^s$, $s > 0$	$n = 2^s \cdot t$, $s > 0$ and $t \neq 1$
$\widetilde{\deg}(f) \leq 1$	1	1	1
$\widetilde{\deg}(f) > 1$	2	$2n$	2^{s+1}

The computation of $\mathrm{NF}_n(f)$ is somewhat different from that of $\mathrm{NP}_n(f)$; we are interested in the reducible orbits of f.

We discuss three cases, depending on the lifting degree of f.

Case 1 ($\widetilde{\deg}(f) = 0$). If f_π is trivial, then there is a single periodic point class for each n. For each $m \mid n$, the periodic point class $F_0^{(m)} = p\,\mathrm{Fix}(\tilde{f}^m)$ is reducible to $F_0^{(1)} = p\,\mathrm{Fix}(\tilde{f})$ and by Corollary 4.4 (1), it is essential. We have that $S = \{\langle F_0^{(1)}\rangle\}$ is a set of n-representatives and $h(S) = 1$. Thus, $\mathrm{NF}_n(f) = 1$.

Case 2 ($\widetilde{\deg}(f) = 1$). If $\widetilde{\deg}(f) = 1$, then $\tilde{f}$ is homotopic to the identity or the antipodal map on S^2. From the homotopy classification of self-maps of RP^2, we obtain that f is homotopic to the identity map on RP^2 which has least period 1. Thus, we have $\mathrm{NF}_n(f) = 1$ with $n > 1$.

Case 3 ($\widetilde{\deg}(f) > 1$). In this case, by Corollary 4.4 (2), we know that the periodic point classes $F_0^{(n)}$ and $F_\tau^{(n)}$ are essential. By Theorem 4.2 (2), the reducibility of periodic point classes of f depends on n which we write in the form $n = 2^s \cdot t$ with $s \geq 0$ and odd t.

There are three subcases.

Subcase 3.1 ($s = 0$ and $t > 1$, that is, n is odd and $n > 1$). For each $m \mid n$, by Theorem 4.2 (2), the periodic class $F_0^{(m)}$ reduces to the periodic point class $F_0^{(1)} = p\,\mathrm{Fix}(\tilde{f})$. Also the periodic class $F_\tau^{(m)}$ reduces to $F_\tau^{(1)} = p\,\mathrm{Fix}(\tau\tilde{f})$. Thus, $S = \{\langle F_0^{(1)}\rangle, \langle F_\tau^{(1)}\rangle\}$ is a set of n-representatives with minimal height 2. Thus, $\mathrm{NF}_n(f) = 2$.

Subcase 3.2 ($s > 0$ and $t = 1$, that is $n = 2^s$). For each $m \mid n, m = 2^k (0 \leq k \leq s)$, by Theorem 4.2 (2), the periodic point class $F_0^{(m)}$ reduces to $F_0^{(1)} = p\,\mathrm{Fix}(\tilde{f})$. The set $S = \{\langle F_0^{(1)}\rangle, \langle F_\tau^{(1)}\rangle, \langle F_\tau^{(2^1)}\rangle, \langle F_\tau^{(2^2)}\rangle, \ldots, \langle F_\tau^{(2^s)}\rangle\}$ is a set of n-representatives. By Theorem 4.2 (2), each $F_\tau^{(2^k)}$ $(0 < k \leq s)$ is irreducible, any n-representatives must contain each $F_\tau^{(2^k)}$. Therefore we have $\mathrm{NF}_n(f) = 1 + 1 + 2 + 2^2 + \cdots + 2^s = 2^{s+1} = 2n$.

Subcase 3.3 ($s > 0$ and $t > 1$). For each $m \mid n$, we write $m = 2^k \cdot q$, with $0 \leq k \leq s$ and $q \mid t$. By Theorem 4.2 (2), the periodic point class $F_0^{(m)}$ reduces to $F_0^{(1)} = p\,\mathrm{Fix}(\tilde{f})$. By Theorem 4.2 (2), for $F_\tau^{(m)}$ with $m = 2^k \cdot q$, each $F_\tau^{(m)}$ reduces to $F_\tau^{(2^k)}$ $(0 < k \leq s)$. Thus, the set $S = \{\langle F_0^{(1)}\rangle, \langle F_\tau^{(1)}\rangle, \langle F_\tau^{(2^1)}\rangle, \langle F_\tau^{(2^2)}\rangle, \ldots \langle F_\tau^{(2^s)}\rangle\}$ is a set of n-representatives. Since each $F_\tau^{(2^k)}$ $(0 < k \leq s)$ is irreducible, any n-representatives must contain each $F_\tau^{(2^k)}$. Therefore we have $\mathrm{NF}_n(f) = 1 + 1 + 2 + 2^2 + \cdots + 2^s = 2^{s+1}$. □

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