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Research Article

Common Fixed Point Theorem in Partially Ordered $\mathcal{L}$-Fuzzy Metric Spaces

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We introduce partially ordered $\mathcal{L}$-fuzzy metric spaces and prove a common fixed point theorem in these spaces.

1. Introduction

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions [1–43]. Recently Nieto and Rodríguez-López [27–29] and Ran and Reurings [33] presented some new results for contractions in partially ordered metric spaces. The main idea in [27–33] involves combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

Recall that if $(X, \leq)$ is a partially ordered set and $F : X \to X$ is such that for $x, y \in X$, $x \leq y$ implies $F(x) \leq F(y)$, then a mapping F is said to be nondecreasing. The main result of Nieto and Rodríguez-López [27–33] and Ran and Reurings [33] is the following fixed point theorem.

Theorem 1.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Suppose that F is a nondecreasing mapping with*

$$d(F(x), F(y)) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$, $x \leq y$, where $0 < k < 1$. Also suppose the following.

(a) *F is continuous.*

(b) *If* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to x$ *in* X,

then $x_n \leq x$ *for all* n *hold.*

If there exists an $x_0 \in X$ *with* $x_0 \leq F(x_0)$, *then* F *has a fixed point.*

The works of Nieto and Rodríguez-López [27, 28] and Ran and Reurings [33] have motivated Agarwal et al. [1], Bhaskar and Lakshmikantham [3], and Lakshmikantham and Ćirić [23] to undertake further investigation of fixed points in the area of ordered metric spaces. We prove the existence and approximation results for a wide class of contractive mappings in intuitionistic metric space. Our results are an extension and improvement of the results of Nieto and Rodríguez-López [27, 28] and Ran and Reurings [33] to more general class of contractive type mappings and include several recent developments.

2. Preliminaries

The notion of fuzzy sets was introduced by Zadeh [44]. Various concepts of fuzzy metric spaces were considered in [15, 16, 22, 45]. Many authors have studied fixed point theory in fuzzy metric spaces; see, for example, [7, 8, 25, 26, 39, 46–48]. In the sequel, we will adopt the usual terminology, notation, and conventions of $\mathcal{L}$-fuzzy metric spaces introduced by Saadati et al. [36] which are a generalization of fuzzy metric sapces [49] and intuitionistic fuzzy metric spaces [32, 37].

Definition 2.1 (see [46]). Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a nonempty set called a universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.

Lemma 2.2 (see [13, 14]). *Consider the set* L^* *and the operation* $\leq_{L^*}$ *defined by*

$$L^* = \left\{(x_1, x_2) : (x_1, x_2) \in [0,1]^2,\ x_1 + x_2 \leq 1\right\}, \tag{2.1}$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$, *and* $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Classically, a triangular norm T on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, x) = x$, for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 2.3. A negation on $\mathcal{L}$ is any strictly decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an involutive negation.

In this paper the negation $\mathcal{N} : L \to L$ is fixed.

Definition 2.4. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) (for all $x \in L$)$(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (boundary condition);

(ii) (for all $(x, y) \in L^2$)$(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);

(iii) (for all $(x, y, z) \in L^3$)$(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);

(iv) (for all $(x, x', y, y') \in L^4$)$(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$ (monotonicity).

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be continuous if for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have

$$\lim_n \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y). \tag{2.2}$$

For example, $\mathcal{T}(x, y) = \min(x, y)$ and $\mathcal{T}(x, y) = xy$ are two continuous t-norms on $[0, 1]$. A t-norm can also be defined recursively as an $(n+1)$-ary operation $(n \in \mathbb{N})$ by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1, \ldots, x_{n+1}) = \mathcal{T}\left(\mathcal{T}^{n-1}(x_1, \ldots, x_n), x_{n+1}\right) \tag{2.3}$$

for $n \geq 2$ and $x_i \in L$.

A t-norm $\mathcal{T}$ is said to be of *Hadžić type* if the family $\{\mathcal{T}^n\}_{n \in \mathbb{N}}$ is equicontinuous at $x = 1_{\mathcal{L}}$, that is,

$$\forall \varepsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\} \exists \delta \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\} : a >_L \mathcal{N}(\delta) \Longrightarrow \mathcal{T}^n(a) >_L \mathcal{N}(\varepsilon) \quad (n \geq 1)\ . \tag{2.4}$$

$\mathcal{T}_M$ is a trivial example of a t-norm of Hadžić type, but there exist t-norms of Hadžić type weaker than $\mathcal{T}_M$ [50] where

$$\mathcal{T}_M(x, y) = \begin{cases} x, & \text{if } x \leq_L y, \\ y, & \text{if } y \leq_L x. \end{cases} \tag{2.5}$$

Definition 2.5. The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy metric space* if X is an arbitrary (nonempty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2 \times\ ]0, +\infty[$ satisfying the following conditions for every x, y, z in X and t, s in $]0, +\infty[$:

(a) $\mathcal{M}(x, y, t) >_L 0_{\mathcal{L}}$;

(b) $\mathcal{M}(x, y, t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$;

(c) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;

(d) $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)$;

(e) $\mathcal{M}(x, y, \cdot) :]0, \infty[\ \to L$ is continuous.

If the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ satisfies the condition:

$$(f) \lim_{t\to\infty} \mathcal{M}(x, y, t) = 1_{\mathcal{L}}, \tag{2.6}$$

then $(X, \mathcal{M}, \mathcal{T})$ is said to be *Menger $\mathcal{L}$-fuzzy metric space* or for short a **M**$\mathcal{L}$-fuzzy metric space.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in]0, +\infty[$, we define the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) >_L \mathcal{N}(r)\}. \tag{2.7}$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is called the *topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$*.

Example 2.6 (see [38]). Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{t}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right). \tag{2.8}$$

Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 2.7. Let $X = \mathbb{N}$. Define $\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2b_2)$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^*, and let $\mathcal{M}(x, y, t)$ on $X^2 \times (0, \infty)$ be defined as follows:

$$\mathcal{M}(x, y, t) = \begin{cases} \left(\dfrac{x}{y}, \dfrac{y - x}{y}\right) & \text{if } x \leq y, \\ \left(\dfrac{y}{x}, \dfrac{x - y}{x}\right) & \text{if } y \leq x \end{cases} \tag{2.9}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}, \mathcal{T})$ is an $\mathcal{L}$-fuzzy metric space.

Lemma 2.8 (see [49]). *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x, y, t)$ is nondecreasing with respect to t, for all x, y in X.*

Definition 2.9. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a *Cauchy sequence*, if for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ ($n \geq m \geq n_0$),

$$\mathcal{M}(x_m, x_n, t) >_L \mathcal{N}(\varepsilon). \tag{2.10}$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be *convergent* to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_{\mathcal{L}}$ whenever $n \to +\infty$ for every $t > 0$. A $\mathcal{L}$-fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent.

Definition 2.10. Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times]0, \infty[$ if

$$\lim_{n \to \infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t) \tag{2.11}$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times]0, \infty[$ converges to a point $(x, y, t) \in X \times X \times]0, \infty[$, that is, $\lim_n \mathcal{M}(x_n, x, t) = \lim_n \mathcal{M}(y_n, y, t) = 1_{\mathcal{L}}$ and $\lim_n \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)$.

Lemma 2.11. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X \times X \times]0, \infty[$.*

Proof. The proof is the same as that for fuzzy spaces (see [35, Proposition 1]). □

Lemma 2.12. *If an $\mathbf{M}\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ satisfies the following condition:*

$$\mathcal{M}(x, y, t) = C, \quad \forall t > 0, \tag{2.12}$$

then one has $C = 1_{\mathcal{L}}$ and $x = y$.

Proof. Let $\mathcal{M}(x, y, t) = C$ for all $t > 0$. Then by (f) of Definition 2.5, we have $C = 1_{\mathcal{L}}$ and by (b) of Definition 2.5, we conclude that $x = y$. □

Lemma 2.13 (see [50]). *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathbf{M}\mathcal{L}$-fuzzy metric space in which $\mathcal{T}$ is Hadžic′ type. Suppose*

$$\mathcal{M}(x_n, x_{n+1}, t) \geq_L \mathcal{M}\left(x_0, x_1, \frac{t}{k^n}\right) \tag{2.13}$$

for some $0 < k < 1$ and $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

3. Main Results

Definition 3.1. Suppose that $(X, \leq)$ is a partially ordered set and $F, h : X \to X$ are mappings of X into itself. We say that F is h-nondecreasing if for $x, y \in X$,

$$h(x) \leq h(y) \quad \text{implies } F(x) \leq F(y). \tag{3.1}$$

Now we present the main result in this paper.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there is an $\mathcal{L}$-fuzzy metric $\mathcal{M}$ on X such that $(X, \mathcal{M}, \mathcal{T})$ is a complete $\mathbf{M}\mathcal{L}$-fuzzy metric space in which $\mathcal{T}$ is Hadžic′ type. Let $F, h : X \to X$ be two self-mappings of X such that there exist $k \in (0, 1)$ and $q \in (0, 1)$ such that*

$F(X) \subseteq h(X)$, *F is a h-nondecreasing mapping and*

$$\begin{aligned}\mathcal{M}(F(x),F(y),kt)\geq_L\mathcal{T}_M\{&\mathcal{M}(h(x),h(y),t),\mathcal{M}(h(x),F(x),t),\mathcal{M}(h(y),F(y),t),\\&\mathcal{M}(h(x),F(y),(1+q)t),\mathcal{M}(h(y),F(x),(1-q)t)\}\end{aligned} \tag{3.2}$$

for all $x, y \in X$ *for which* $h(x) \leq h(y)$ *and all* $t > 0$.

Also suppose that

$$\begin{aligned}&\text{if } \{h(x_n)\} \subset X \text{ is a nondecreasing sequence with } h(x_n) \longrightarrow h(z) \text{ in } h(X),\\&\text{then } h(z) \leq h(h(z)) \quad \text{and } h(x_n) \leq h(z) \quad \forall n \text{ hold.}\end{aligned} \tag{3.3}$$

Also suppose that $h(X)$ *is closed. If there exists an* $x_0 \in X$ *with* $h(x_0) \leq F(x_0)$, *then F and h have a coincidence. Further, if F and h commute at their coincidence points, then F and h have a common fixed point.*

Proof. Let $x_0 \in X$ be such that $h(x_0) \leq F(x_0)$. Since $F(X) \subseteq h(X)$, we can choose $x_1 \in X$ such that $h(x_1) = F(x_0)$. Again from $F(X) \subseteq h(X)$ we can choose $x_2 \in X$ such that $h(x_2) = F(x_1)$. Continuing this process we can choose a sequence $\{x_n\}$ in X such that

$$h(x_{n+1}) = F(x_n) \quad \forall n \geq 0. \tag{3.4}$$

Since $h(x_0) \leq F(x_0)$ and $h(x_1) = F(x_0)$, we have $h(x_0) \leq h(x_1)$. Then from (3.1),

$$F(x_0) \leq F(x_1), \tag{3.5}$$

that is, by (3.4), $h(x_1) \leq h(x_2)$. Again from (3.1),

$$F(x_1) \leq F(x_2), \tag{3.6}$$

that is, $h(x_2) \leq h(x_3)$. Continuing we obtain

$$F(x_0) \leq F(x_1) \leq F(x_2) \leq F(x_3) \leq \cdots \leq F(x_n) \leq F(x_{n+1}) \leq \cdots. \tag{3.7}$$

Now we will show that a sequence $\{\mathcal{M}(F(x_n),F(x_{n+1}),t)\}$ converges to $1_{\mathcal{L}}$ for each $t > 0$. If $\mathcal{M}(F(x_n),F(x_{n+1}),t) = 1_{\mathcal{L}}$ for some n and for each $t > 0$, then it is easily to show that $\mathcal{M}(F(x_{n+k}),F(x_{n+k+1}),t) = 1_{\mathcal{L}}$ for all $k \geq 0$. So we suppose that $\mathcal{M}(F(x_n),F(x_{n+1}),t) <_L 1_{\mathcal{L}}$ for all n. We show that for each $t > 0$,

$$\mathcal{M}(F(x_n),F(x_{n+1}),kt)\geq_L\mathcal{M}(F(x_{n-1}),F(x_n),t) \quad \forall n \geq 1. \tag{3.8}$$

Since from (3.4) and (3.7) we have $h(x_{n-1}) \leq h(x_n)$, from (3.1) with $x = x_n$ and $y = x_{n+1}$,

$$\begin{aligned}\mathcal{M}(F(x_n), F(x_{n+1}), kt) \geq_L \mathcal{T}_M \{&\mathcal{M}(h(x_n), h(x_{n+1}), t), \mathcal{M}(h(x_n), F(x_n), t), \mathcal{M}(h(x_{n+1}), F(x_{n+1}), t),\\ &\mathcal{M}(h(x_n), F(x_{n+1}), (1+q)t), \mathcal{M}(h(x_{n+1}), F(x_n), (1-q)t)\}.\end{aligned} \tag{3.9}$$

So by (3.4),

$$\begin{aligned}\mathcal{M}(F(x_n), F(x_{n+1}), kt) \geq_L \mathcal{T}_M \{&\mathcal{M}(F(x_{n-1}), F(x_n), t), \mathcal{M}(F(x_{n-1}), F(x_n), t), \mathcal{M}(F(x_n), F(x_{n+1}), t),\\ &\mathcal{M}(F(x_{n-1}), F(x_{n+1}), (1+q)t), 1_{\mathcal{L}}\}.\end{aligned} \tag{3.10}$$

Since by (d) of Definition 2.5

$$\mathcal{M}(F(x_{n-1}), F(x_{n+1}), (1+q)t) \geq_L \mathcal{T}_M \{\mathcal{M}(F(x_{n-1}), F(x_n), t), \mathcal{M}(F(x_n), F(x_{n+1}), qt)\}, \tag{3.11}$$

we have

$$\begin{aligned}\mathcal{M}(F(x_n), F(x_{n+1}), kt) \geq_L \mathcal{T}_M \{&\mathcal{M}(F(x_{n-1}), F(x_n), t), \mathcal{M}(F(x_n), F(x_{n+1}), t),\\ &\mathcal{M}(F(x_n), F(x_{n+1}), qt)\}.\end{aligned} \tag{3.12}$$

As t-norm is continuous, letting $q \to 1_{\mathcal{L}}$ we get

$$\mathcal{M}(F(x_n), F(x_{n+1}), kt) \geq_L \mathcal{T}_M \{\mathcal{M}(F(x_{n-1}), F(x_n), t), \mathcal{M}(F(x_n), F(x_{n+1}), t)\}. \tag{3.13}$$

Consequently,

$$\mathcal{M}(F(x_n), F(x_{n+1}), t) \geq_L \mathcal{T}_M \left\{\mathcal{M}\left(F(x_{n-1}), F(x_n), \frac{1}{k}t\right), \mathcal{M}\left(F(x_n), F(x_{n+1}), \frac{1}{k}t\right)\right\}. \tag{3.14}$$

By repeating the above inequality, we obtain

$$\mathcal{M}(F(x_n), F(x_{n+1}), t) \geq_L \mathcal{T}_M \left\{\mathcal{M}\left(F(x_{n-1}), F(x_n), \frac{1}{k}t\right), \mathcal{M}\left(F(x_n), F(x_{n+1}), \frac{1}{k^p}t\right)\right\}. \tag{3.15}$$

Since $\mathcal{M}(F(x_n), F(x_{n+1}), (1/k^p)t) \to 1_{\mathcal{L}}$ as $p \to \infty$, it follows that

$$\mathcal{M}(F(x_n), F(x_{n+1}), t) \geq_L \mathcal{M}\left(F(x_{n-1}), F(x_n), \frac{1}{k}t\right). \tag{3.16}$$

Thus we proved (3.7). By repeating the above inequality (3.7), we get

$$\mathcal{M}(F(x_n), F(x_{n+1}), t) \geq_L \mathcal{M}\left(F(x_0), F(x_1), \frac{1}{k^n}t\right). \tag{3.17}$$

Since $\mathcal{M}(x,y,t) \to 1_{\mathcal{L}}$ as $t \to +\infty$ and $k<1$, letting $n \to \infty$ in (3.17) we get

$$\lim_{n\to\infty} \mathcal{M}(F(x_n), F(x_{n+1}), t) = 1_{\mathcal{L}} \quad \text{for each } t>0. \tag{3.18}$$

Now we will prove that $\{F(x_n)\}$ is a Cauchy sequence which means that for every $\delta>0$ and $\epsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $n(\delta,\epsilon) \in \mathbb{N}$ such that

$$M(F(x_n), F(x_{n+p}), \delta) >_L \mathcal{N}(\epsilon) \quad \text{for every } n \geq n(\delta,\epsilon) \text{ and every } p \in \mathbb{N}. \tag{3.19}$$

Let $\epsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $\delta>0$ be arbitrary. For any $p \geq 1$ we have

$$\delta = \delta(1-k)(1+k+\cdots+k^p+\cdots) > \delta(1-k)\left(1+k+\cdots+k^{p-1}\right). \tag{3.20}$$

Since $M(x,y,t)$ is nondecreasing with respect to t, for all x, y in X,

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq_L \mathcal{M}\left(F(x_n), F(x_{n+p}), \delta(1-k)\left(1+k^n+\cdots+k^{p-1}\right)\right) \tag{3.21}$$

and hence, by (d) of Definition 2.5,

$$\begin{aligned}\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq_L \mathcal{T}_M^{p-2}\Big\{&\mathcal{M}(F(x_n), F(x_{n+1}), (1-k)\delta), \mathcal{M}(F(x_{n+1}), F(x_{n+2}), (1-k)\delta k) \\ &, \ldots, \mathcal{M}\left(F(x_{n+p-1}), F(x_{n+p}), (1-k)\delta k^{p-1}\right)\Big\}.\end{aligned} \tag{3.22}$$

From (3.17) it follows that

$$\mathcal{M}(F(x_{n+i}), F(x_{n+i+1}), t) \geq_L \mathcal{M}\left(F(x_n), F(x_{n+1}), \frac{t}{k^i}\right) \quad \text{for each } i \geq_L 1_{\mathcal{L}}. \tag{3.23}$$

From (3.23) with $t=(1-k)\delta k^i$ we get

$$\mathcal{M}\left(F(x_{n+i}), F(x_{n+i+1}), (1-k)\delta k^i\right) \geq_L \mathcal{M}(F(x_n), F(x_{n+1}), (1-k)\delta). \tag{3.24}$$

Thus by (3.22),

$$\begin{aligned}\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq_L \mathcal{T}_M^{n}\{&\mathcal{M}(F(x_n), F(x_{n+1}), (1-k)\delta), \mathcal{M}(F(x_n), F(x_{n+1}), (1-k)\delta) \\ &, \ldots, \mathcal{M}(F(x_n), F(x_{n+1}), (1-k)\delta)\}.\end{aligned} \tag{3.25}$$

Hence we get

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq_L \mathcal{M}(F(x_n), F(x_{n+1}), (1-k)\delta). \tag{3.26}$$

From (3.26) and (3.17),

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq_L \mathcal{M}\left(F(x_0), F(x_1), \frac{(1-k)\delta}{k^n}\right). \tag{3.27}$$

Hence we conclude, as $\mathcal{M}(x, y, t) \to 1_{\mathcal{L}}$ as $t \to +\infty$ and $k < 1$, that there exists $n(\delta, \epsilon) \in \mathbb{N}$ such that

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) >_L \mathcal{N}(\epsilon) \quad \text{for every } n \geq n(\delta, \epsilon) \text{ and every } p \in \mathbb{N}. \tag{3.28}$$

Thus we proved that $\{F(x_n)\}$ is a Cauchy sequence.

Since $h(X)$ is closed and as $F(x_n) = h(x_{n+1})$, there is some $z \in X$ such that

$$\lim_{n \to \infty} h(x_n) = h(z). \tag{3.29}$$

Now we show that z is a coincidence of F and h. Since from (3.3) and (3.29) we have $h(x_n) \leq h(z)$ for all n, then from (3.2) and by (d) of Definition 2.5 we have

$$\begin{aligned} \mathcal{M}(F(x_n), F(z), kt) \geq_L \mathcal{T}_M\{&\mathcal{M}(h(x_n), h(z), t), \mathcal{M}(h(x_n), F(x_n), t), \mathcal{M}(h(z), F(z), t), \\ &\mathcal{M}(h(x_n), F(z), (1+q)t), M(h(z), F(x_n), (1-q)t)\}. \end{aligned} \tag{3.30}$$

Letting $n \to \infty$ we get

$$\begin{aligned} \mathcal{M}(h(z), F(z), kt) \geq_L \mathcal{T}_M\{&\mathcal{M}(h(z), h(z), t), \mathcal{M}(h(z), h(z), t), \mathcal{M}(h(z), F(z), t), \\ &\mathcal{M}(h(z), F(z), (1+q)t), \mathcal{M}(h(z), h(z), (1-q)t)\} \end{aligned} \tag{3.31}$$

for all $t > 0$. Therefore,

$$\mathcal{M}(h(z), F(z), t) \geq_L \mathcal{M}\left(h(z), F(z), \frac{1}{k}t\right). \tag{3.32}$$

Hence we get

$$\mathcal{M}(h(z), F(z), t) \geq_L \mathcal{M}\left(h(z), F(z), \frac{1}{k^n}t\right) \longrightarrow 1_{\mathcal{L}} \quad \text{as } n \longrightarrow \infty \ \forall t > 0. \tag{3.33}$$

Hence we conclude that $\mathcal{M}(h(z), F(z), t) = 1_{\mathcal{L}}$ for all $t > 0$. Then by (b) of Definition 2.5 we have $F(z) = h(z)$. Thus we proved that F and h have a coincidence.

Suppose now that F and h commute at z. Set $w = h(z) = F(z)$. Then

$$F(w) = F(h(z)) = h(F(z)) = h(w). \tag{3.34}$$

Since from (3.3) we have $h(z) \le h(h(z)) = h(w)$ and as $h(z) = F(z)$ and $h(w) = F(w)$, from (3.2) we get

$$\begin{aligned} \mathcal{M}(w, F(w), kt) &= \mathcal{M}(F(z), F(w), kt) \\ &\ge_L \mathcal{T}_M\{\mathcal{M}(h(z), h(w), t), \mathcal{M}(h(z), F(z), t), \mathcal{M}(h(w), F(w), t), \\ &\qquad \mathcal{M}(h(w), F(z), (1+q)t), \mathcal{M}(h(z), F(w), (1-q)t)\} \\ &= \mathcal{M}(F(z), F(w), (1-q)t). \end{aligned} \tag{3.35}$$

Letting $q \to 0$ we get

$$\mathcal{M}(F(z), F(w), kt) \ge_L \mathcal{M}(F(z), F(w), t). \tag{3.36}$$

Hence, similarly as above, we get

$$\mathcal{M}(F(z), F(w), t) \ge_L \mathcal{M}\left(F(z), F(w), \frac{1}{k^n}t\right) \longrightarrow 1_{\mathcal{L}} \quad \text{as } n \longrightarrow \infty \ \forall t > 0. \tag{3.37}$$

Hence we conclude that $F(w) = F(z)$. Since $F(z) = h(z) = w$, we have

$$F(w) = h(w) = w. \tag{3.38}$$

Thus we proved that F and h have a common fixed point. □

Remark 3.3. Note that F is h-nondecreasing and can be replaced by F which is h-nonincreasing in Theorem 3.2 provided that $h(x_0) \le F(x_0)$ is replaced by$F(x_0) \ge h(x_0)$ in Theorem 3.2.

Corollary 3.4. *Let $(X, \le)$ be a partially ordered set and suppose that there is an $\mathcal{L}$-fuzzy metric $\mathcal{M}$ on X such that $(X, \mathcal{M}, \mathcal{T})$ is a complete* **M**$\mathcal{L}$*-fuzzy metric space in which $\mathcal{T}$ is Hadžic′ type. Let $F : X \to X$ be a nondecreasing self-mappings of X such that there exist $k \in (0,1)$ and $q \in (0,1)$ such that*

$$\begin{aligned} \mathcal{M}(F(x), F(y), kt) \ge_L \mathcal{T}_M\{&\mathcal{M}(x, y, t), \mathcal{M}(x, F(x), t), \mathcal{M}(y, F(y), t), \\ &\mathcal{M}(x, F(y), (1+q)t), \mathcal{M}(y, F(x), (1-q)t)\} \end{aligned} \tag{3.39}$$

for all $x, y \in X$ for which $x \le y$ and all $t > 0$. Also suppose the following.

(i) *If $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \le z$ for all n hold.*

(ii) *F is continuous.*

If there exists an $x_0 \in X$ with $x_0 \le F(x_0)$, then F has a fixed point.

Proof. Taking $h = I$ (I = the identity mapping) in Theorem 3.2, then (3.3) reduces to the hypothesis (i).

Suppose now that F is continuous. Since from (3.4) we have $x_{n+1} = F(x_n)$ for all $n \geq 0$, and as from (3.29), $x_n \to z$, then

$$F(z) = F\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} F(x_n) = z. \tag{3.40}$$

□

Corollary 3.5. *Let $(X, \leq)$ be a partially ordered set and suppose that there is an $\mathcal{L}$-fuzzy metric $\mathcal{M}$ on X such that $(X, \mathcal{M}, \mathcal{T})$ is a complete $\mathbf{M}\mathcal{L}$-fuzzy metric space in which $\mathcal{T}$ is Hadžic′ type. Let $F : X \to X$ be a nondecreasing self-mappings of X such that there exist $k \in (0,1)$ and $q \in (0,1)$ such that*

$$\mathcal{M}(F(x), F(y), kt) \geq_L \mathcal{T}_M\{\mathcal{M}(x,y,t), \mathcal{M}(x,F(x),t), \mathcal{M}(y,F(y),t)\} \tag{3.41}$$

for all $x, y \in X$ for which $x \leq y$ and all $t > 0$. Also suppose the following.

(i) *If $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \leq z$ for all n hold.*

(ii) *F is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.

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