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Research Article

A Continuation Method for Weakly Contractive Mappings under the Interior Condition

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Recently, Frigon proved that, for weakly contractive maps, the property of having a fixed point is invariant by a certain class of homotopies, obtaining as a consequence a Leray-Schauder alternative for this class of maps in a Banach space. We prove here that the Leray-Schauder condition in the aforementioned result can be replaced by a modification of it, the interior condition. We also show that our arguments work for a certain class of generalized contractions, thus complementing a result of Agarwal and O'Regan.

1. Introduction

Suppose that X is a Banach space, that $U \subset X$ is an open bounded subset of X, containing the origin, and that $f : \overline{U} \to X$ is a mapping. It is well known that if f satisfies the Leray-Schauder condition defined as

$$f(x) \neq \lambda x, \quad \text{for } x \in \partial U, \ \lambda > 1 \tag{L-S}$$

and f is a strict set-contraction or, more generally, condensing, then f has a fixed point in $\overline{U}$ (see, e.g., [1] or [2]). The first continuation method in the setting of a complete metric space for contractive maps comes from the hands of Granas [3], in 1994, who gave a homotopy result for contractive maps (for more information on this topic see, e.g., [4, 5] or [6]).

On the other hand, it has been recently shown in [7] that, for condensing mappings, the condition (L-S) can be replaced by a modification of it which we call the interior condition, and is defined as follows: a mapping $f : \overline{U} \to X$ satisfies the *Interior Condition* (I-C), if there exists $\delta > 0$ such that

$$f(x) \neq \lambda x, \quad \text{for } x \in U_\delta,\ \lambda > 1,\ f(x) \notin \overline{U}, \tag{I-C}$$

where $U_\delta = \{x \in U : \operatorname{dist}(x, \partial U) < \delta\}$ (some generalizations of this result can be found in [8, 9]).

We remark that the condition (I-C) by itself cannot be a substitute for the condition (L-S), and an additional assumption on the domain of f needs to be made in order to guarantee the existence of a fixed point for f. The class of sets that we need is defined as follows: suppose that $U \subset X$ is an open neighborhood of the origin. We say that U is strictly star shaped if for any $x \in \partial U$ we have that $\{\lambda x : \lambda > 0\} \cap \partial U = \{x\}$. It was shown in [7] that if U is bounded and strictly star shaped and $f : \overline{U} \to X$ is a condensing mapping satisfying the condition (I-C), then f has a fixed point. Of course, this result includes the case of a contractive map (i.e., a map f for which there exists $k \in [0, 1)$ such that $d(f(x), f(y)) \leq k d(x, y)$ for all $x, y \in \overline{U}$), but our aim in this note is, following the pattern of Granas [3] and Frigon et al. [10], to give a continuation method for weakly contractive mappings, in the setting of a complete metric space, under some conditions on the homotopy which are the counterpart of the condition (I-C) and the notion of a strictly star shaped set in a space without a vector structure. Finally, in the last section we show that our arguments also work for a class of generalized contractions, thus complementing a result of Agarwal and O'Regan [11].

2. Weakly Contractive Maps

In this chapter we deal with the concept of weakly contractive maps, as it was introduced by Dugundji and Granas in [12].

Definition 2.1. Let (X, d) be a complete metric space and U an open subset of X. A function $f : \overline{U} \to X$ is said to be weakly contractive if there exists $\psi : X \times X \to (0, \infty)$ compactly positive (i.e., $\inf\{\psi(x, y) : a \leq d(x, y) \leq b\} = \theta(a, b) > 0$ for every $0 < a \leq b$) such that

$$d(f(x), f(y)) \leq d(x, y) - \psi(x, y). \tag{2.1}$$

If ψ is a compactly positive function, we define for $0 < a \leq b$

$$\gamma(a, b) = \min\{a, \theta(a, b)\}. \tag{2.2}$$

It was shown in [12] that any weakly contractive map $f : X \to X$ defined on a complete metric space X has a unique fixed point. Some years later, Frigon [5] proved that, for weakly contractive maps, the property of having a fixed point is invariant by a certain class of homotopies, obtaining as a consequence a Leray-Schauder alternative for weakly contractive maps in the setting of a Banach space. We prove here that the Leray-Schauder condition in the aforementioned result can be replaced by the condition (I-C), and it will also be obtained as a consequence of a continuation method. The definition of homotopy that we need for our purposes is the following.

Definition 2.2. Let (X,d) be a complete metric space, and U an open subset of X. Let $f,g:\overline{U}\to X$ be two weakly contractive maps. We say that f is (I-C)-homotopic to g if there exists $H:\overline{U}\times[0,1]\to X$ with the following properties:

(P1) $H(x,1)=f(x)$ and $H(x,0)=g(x)$ for every $x\in\overline{U}$;

(P2) there exists $\delta>0$ such that $x\neq H(x,t)$ for every $x\in U_\delta$, with $f(x)\notin\overline{U}$, and $t\in[0,1]$, where $U_\delta=\{x\in U:\operatorname{dist}(x,\partial U)<\delta\}$;

(P3) there exists a compactly positive function $\psi: X\times X\to(0,\infty)$ such that $d(H(x,t),H(y,t))\le d(x,y)-\psi(x,y)$ for every $x,y\in\overline{U}$, and $t\in[0,1]$;

(P4) there exists a continuous function $\phi:[0,1]\to\mathbb{R}$ such that, for every $x\in\overline{U}$ and $t,s\in[0,1]$, $d(H(x,t),H(x,s))\le|\phi(t)-\phi(s)|$;

(P5) if $x\in\partial U$ and $0\le\lambda<1$, with $H(x,\lambda)\in\partial U$, then $H(x,1)\notin\overline{U}$.

In the proof of the main result of this chapter we shall make use of the following lemma (see Frigon [5]).

Lemma 2.3. *Let $x_0\in X$, $r>0$, and $h:\overline{B(x_0,r)}\to X$ weakly contractive. If $d(x_0,h(x_0))<\gamma(r/2,r)$, then h has a fixed point.*

Theorem 2.4. *Let $f,g:\overline{U}\to X$ be two weakly contractive maps. Suppose that f is homotopic to g and $g(\overline{U})$ is bounded. If g has a fixed point in U, then f has a fixed point in $\overline{U}$.*

Proof. We argue by contradiction. Suppose that f does not have any fixed point in $\overline{U}$, and let H be a homotopy between f and g, in the sense of Definition 2.1. Consider the set

$$A=\{\lambda\in[0,1]:x=H(x,\lambda)\text{ for some }x\in U\},\tag{2.3}$$

and notice that A is nonempty since g has a fixed point in U, that is, $0\in A$. We will show that A is both open and closed in $[0,1]$, and hence, by connectedness, we will have that $A=[0,1]$. As a result, f will have a fixed point in U, which establishes a contradiction.

To show that A is closed, suppose that $\{\lambda_n\}$ is a sequence in A converging to $\lambda\in[0,1]$ and let us show that $\lambda\in A$. Since $\lambda_n\in A$, there exists $x_n\in U$ with $x_n=H(x_n,\lambda_n)$. Fix $\varepsilon>0$. Using that $g(\overline{U})$ is bounded and that ϕ is continuous on the compact interval $[0,1]$, it is easy to show that there exists $M>\varepsilon$ such that $\operatorname{diam} H(\overline{U}\times[0,1])\le M$, and hence $d(x_n,x_m)\le M$ for all $n,m\in\mathbb{N}$. Define $\mu=\theta(\varepsilon,M)$ and let $n_0\in\mathbb{N}$ be such that for all $n,m\ge n_0$, $|\phi(\lambda_n)-\phi(\lambda_m)|<\mu$. Then $d(x_n,x_m)<\varepsilon$ for all $n,m\ge n_0$ because, otherwise, we would have $d(x_n,x_m)\ge\varepsilon$ for some $n,m\ge n_0$, and then

$$\begin{aligned}
d(x_n,x_m)&=d(H(x_n,\lambda_n),H(x_m,\lambda_m))\\
&\le d(H(x_n,\lambda_n),H(x_n,\lambda_m))+d(H(x_n,\lambda_m),H(x_m,\lambda_m))\\
&\le\left|\phi(\lambda_n)-\phi(\lambda_m)\right|+d(x_n,x_m)-\psi(x_n,x_m)\\
&<\mu+d(x_n,x_m)-\psi(x_n,x_m)\\
&\le d(x_n,x_m),
\end{aligned}\tag{2.4}$$

which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence and, since (X,d) is complete, there exists $x_0 \in \overline{U}$ such that $x_n \to x_0$ as $n \to \infty$. In addition, $x_0 = H(x_0,\lambda)$ since for all $n \in \mathbb{N}$ we have that

$$\begin{aligned} d(x_n, H(x_0,\lambda)) &= d(H(x_n,\lambda_n), H(x_0,\lambda)) \\ &\leq d(H(x_n,\lambda_n), H(x_n,\lambda)) + d(H(x_n,\lambda), H(x_0,\lambda)) \\ &\leq \left|\phi(\lambda_n) - \phi(\lambda)\right| + d(x_n,x_0) - \psi(x_n,x_0) \\ &\leq \left|\phi(\lambda_n) - \phi(\lambda)\right| + d(x_n,x_0). \end{aligned} \tag{2.5}$$

Observe that $0 \leq \lambda < 1$, because if $\lambda = 1$, then $x_0 = H(x_0,1) = f(x_0)$, which contradicts the fact that f does not have any fixed point in $\overline{U}$. Notice that $x_0 \in U$, because, otherwise, we would have $x_0 \in \partial U$, that is, $H(x_0,\lambda) \in \partial U$, and since $0 \leq \lambda < 1$, by (P5), we have that $H(x_0,1) \notin \overline{U}$. However, since $x_0 \in \partial U$, $\{x_n\} \to x_0$ and $x_n \in U$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U_\delta$ for all $n \geq n_0$. Hence, since $x_n = H(x_n,\lambda_n)$ for all $n \geq n_0$, applying (P2), we have that $f(x_n) \in \overline{U}$ for all $n \geq n_0$, that is, $H(x_n,1) \in \overline{U}$ for all $n \geq n_0$. Taking limits, we arrive to the contradiction $H(x_0,1) \in \overline{U}$.

Therefore, $x_0 \in U$ and, consequently, $\lambda \in A$.

Next we show that A is open in $[0,1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0,\lambda_0)$. Let $r > 0$ be such that $B(x_0,r) \subset U$, and let $\delta > 0$ such that $|\phi(\lambda) - \phi(\lambda_0)| < \gamma(r/2,r)$ for every $\lambda \in [0,1]$ with $|\lambda_0 - \lambda| < \delta$. Then, if $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$,

$$\begin{aligned} d(x_0, H(x_0,\lambda)) &= d(H(x_0,\lambda_0), H(x_0,\lambda)) \\ &\leq \left|\phi(\lambda_0) - \phi(\lambda)\right| \\ &< \gamma\left(\frac{r}{2}, r\right). \end{aligned} \tag{2.6}$$

Using Lemma 2.3, we obtain that $H(\cdot,\lambda)$ has a fixed point in U for every $\lambda \in [0,1]$ such that $|\lambda_0 - \lambda| < \delta$. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$, and therefore A is open in $[0,1]$. □

As an immediate consequence of the previous theorem, we obtain the following fixed point result of the Leray-Schauder type for weakly contractive maps under the condition (I-C).

Theorem 2.5. *Suppose that U is an open and strictly star shaped subset of a Banach space $(X, \|\cdot\|)$, with $0 \in U$, and that $f : \overline{U} \to X$ is a weakly contractive map with $f(\overline{U})$ being bounded. If f satisfies the condition* (I-C), *then f has a fixed point in $\overline{U}$.*

Proof. Since f satisfies the condition (I-C), there exists $\delta > 0$ such that $f(x) \neq \lambda x$ for $\lambda > 1$ and $x \in U_\delta$ with $f(x) \notin \overline{U}$. We may assume that $x \neq f(x)$ for every $x \in U_\delta$, because otherwise we are finished. Define $H : \overline{U} \times [0,1] \to X$ as $H(x,t) = tf(x)$, and let g be the zero map. Notice that g has a fixed point in U, that is, $0 = g(0)$ and also that f and g are two weakly contractive mappings. So, the result will follow from Theorem 2.4 once we prove that f is (I-C)-homotopic to g. Let us check it.

(P1) For all $x \in \overline{U}$, $H(x,0) = 0 \cdot f(x) = 0 = g(x)$ and $H(x,1) = 1 \cdot f(x) = f(x)$.

(P2) Since f satisfies the condition (I-C), we have that $f(x) \neq \lambda x$ for $x \in U_\delta$ with $f(x) \notin \overline{U}$ and $\lambda > 1$. Hence, $x \neq H(x,t)$ for every $x \in U_\delta$, with $f(x) \notin \overline{U}$, and $t \in [0,1]$.

(P3) Since f is weakly contractive, there exists a compactly positive function $\psi : X \times X \to (0,\infty)$ such that $d(f(x), f(y)) \leq d(x,y) - \psi(x,y)$ for every $x, y \in \overline{U}$. Then, if $x, y \in \overline{U}$ and $t \in [0,1]$,

$$\begin{aligned} d(H(x,t), H(y,t)) &= t\|f(x) - f(y)\| \\ &\leq d(f(x), f(y)) \\ &\leq d(x,y) - \psi(x,y). \end{aligned} \tag{2.7}$$

(P4) Since $f(\overline{U})$ is bounded, there exists $M \geq 0$ such that $\|f(x)\| \leq M$ for all $x \in \overline{U}$. Hence,

$$\begin{aligned} d(H(x,t), H(x,s)) &= \|f(x)\||t-s| \\ &\leq M|t-s| \\ &= |\phi(t) - \phi(s)|, \end{aligned} \tag{2.8}$$

where $\phi : [0,1] \to \mathbb{R}$ is the continuous function defined as $\phi(t) = Mt$.

(P5) Suppose that for some $x \in \partial U$ and $\lambda < 1$ we have that $H(x,\lambda) \in \partial U$. Then, $f(x) \neq 0$ since $H(x,\lambda) = \lambda f(x)$, $0 \in U$ and U is open. Let us see that $H(x,1) \notin \overline{U}$: suppose, on the contrary, that $H(x,1) \in \overline{U}$, that is, $f(x) \in \overline{U}$ and define

$$\widehat{\lambda} := \sup\left\{t \geq 1 : t f(x) \in \overline{U}\right\}. \tag{2.9}$$

Then, it is easy to see that $\widehat{\lambda} f(x) \in \partial U$, which contradicts that U is strictly star shaped, since we also have that $\lambda f(x) \in \partial U$. □

3. A Class of Generalized Contractions

A multitude of generalizations and variants of Banach's contractive condition have been given after Banach's theorem (see, e.g., Rhoades [13]) and, recently, Agarwal and O'Regan [11] have given a homotopy result (thus generalizing a fixed point theorem of Hardy and Rogers [14]) under the following generalized contractive condition: there exists $a \in (0,1)$ such that for all $x, y \in X$

$$d(f(x), f(y)) \leq a \max\left\{d(x,y), d(x,f(x)), d(y,f(y)), \frac{1}{2}[d(x,f(y)) + d(y,f(x))]\right\}. \tag{3.1}$$

In this section we give a homotopy result for this class of mappings under the condition (I-C). In the proof of our theorem we shall use the following result [11].

Lemma 3.1. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$, and $h : \overline{B(x_0,r)} \to X$. Suppose that there exists $a \in (0,1)$ such that for $x,y \in \overline{B(x_0,r)}$ one has*

$$d(h(x),h(y)) \leq a \max\left\{d(x,y), d(x,h(x)), d(y,h(y)), \frac{1}{2}[d(x,h(y)) + d(y,h(x))]\right\}, \quad d(x_0,h(x_0)) < (1-a)r. \tag{3.2}$$

Then there exists $x \in \overline{B(x_0,r)}$ with $x = h(x)$.

The proof of the following theorem is very similar to the proof of Theorem 2.4, and we give a sketch of it.

Theorem 3.2. *Let (X,d) be a complete metric space, and U an open subset of X. Let $f,g : \overline{U} \to X$ be two maps such that there exists $H : \overline{U} \times [0,1] \to X$ with the following properties:*

(P1) *$H(x,1) = f(x)$ and $H(x,0) = g(x)$ for every $x \in \overline{U}$;*

(P2) *there exists $\delta > 0$ such that $x \neq H(x,t)$ for every $x \in U_\delta$, with $f(x) \notin \overline{U}$, and $t \in [0,1]$, where $U_\delta = \{x \in U : \operatorname{dist}(x,\partial U) < \delta\}$;*

(P3) *there exists $a \in (0,1)$ such that for all $x,y \in \overline{U}$ and $\lambda \in [0,1]$ one has*

$$\begin{aligned} &d(H(x,\lambda),H(y,\lambda)) \\ &\quad\leq a \max\left\{d(x,y), d(x,H(x,\lambda)), d(y,H(y,\lambda)), \frac{1}{2}[d(x,H(y,\lambda)) + d(y,H(x,\lambda))]\right\}; \end{aligned} \tag{3.3}$$

(P4) *there exists a continuos function $\phi : [0,1] \to \mathbb{R}$ such that, for every $x \in \overline{U}$ and $t,s \in [0,1]$, $d(H(x,t),H(x,s)) \leq |\phi(t) - \phi(s)|$;*

(P5) *if $x \in \partial U$ and $0 \leq \lambda < 1$, with $H(x,\lambda) \in \partial U$, then $H(x,1) \notin \overline{U}$.*

If g has a fixed point in U, then f has a fixed point in $\overline{U}$.

Proof. Suppose that f does not have any fixed point in $\overline{U}$ and consider the nonempty set

$$A = \{\lambda \in [0,1] : H(x,\lambda) = x \text{ for some } x \in U\}. \tag{3.4}$$

We will arrive to a contradiction by showing that $A = [0,1]$, and for this we only need prove that A is closed and open in $[0,1]$.

To show that A is closed in $[0,1]$, consider a sequence $\{\lambda_n\}$ in A, with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$, and show that $\lambda \in A$; that is, that there exists $x_0 \in U$ with $H(x_0,\lambda) = x_0$. To prove that x_0 exists, take any sequence $\{x_n\}$ in U with $x_n = H(x_n,\lambda_n)$, prove that $\{x_n\}$ is Cauchy, and define x_0 as the limit of $\{x_n\}$, as $n \to \infty$.

That $\{x_n\}$ is a Cauchy sequence, as well as $x_0 = H(x_0,\lambda)$, follows from standard arguments which can be seen in [11, Theorem 3.1]. It remains to show that $x_0 \in U$.

To prove this, suppose that it is not true and arrive to a contradiction as follows: we have that $H(x_0, \lambda) = x_0 \in \overline{U} \setminus U = \partial U$, and also that $0 \leq \lambda < 1$, because f does not have any fixed point in $\overline{U}$. Then, by (P5) $f(x_0) \notin \partial U$. On the other hand, $f(x_0) = \lim f(x_n) \in \overline{U}$ because $f(x_n) \in \overline{U}$ for n large enough. To be convinced of it, just apply (P2): since $x_0 \in \partial U$, $\{x_n\} \to x_0$ and $x_n \in U$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U_\delta$ for all $n \geq n_0$. Then, $f(x_n) \in \overline{U}$ for all $n \geq n_0$ since $x_n = H(x_n, \lambda_n)$.

To prove that A is open argue as in Theorem 2.4, use Lemma 3.1 instead of Lemma 2.3. □

As an immediate consequence, we obtain the following result, whose proof is omitted because it is analogous to the proof of Theorem 2.5.

Theorem 3.3. *Suppose that U is an open and strictly star shaped subset of a Banach space $(X, \|\cdot\|)$, with $0 \in U$, and that $f : \overline{U} \to X$ is map with $f(\overline{U})$ being bounded. Assume also that there exists $a \in (0,1)$ such that for all $x, y \in \overline{U}$ and $\lambda \in [0,1]$ one has*

$$\begin{aligned} &d(\lambda f(x), \lambda f(y)) \\ &\quad \leq a \max\left\{ d(x,y), d(x, \lambda f(x)), d(y, \lambda f(y)), \frac{1}{2}\left[d(x, \lambda f(y)) + d(y, \lambda f(x))\right] \right\}. \end{aligned} \tag{3.5}$$

If f satisfies the condition (I-C)*, then f has a fixed point in $\overline{U}$.*

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