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Research Article

Fixed Points of Multivalued Maps in Modular Function Spaces

Marwan A. Kutbi and Abdul Latif

Department of Mathematics, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Abdul Latif, latifmath@yahoo.com

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The purpose of this paper is to study the existence of fixed points for contractive-type and nonexpansive-type multivalued maps in the setting of modular function spaces. We also discuss the concept of w-modular function and prove fixed point results for *weakly*-modular contractive maps in modular function spaces. These results extend several similar results proved in metric and Banach spaces settings.

1. Introduction and Preliminaries

The well-known Banach fixed point theorem on complete metric spaces (specifically, each contraction self-map of a complete metric space has a unique fixed point) has been extended and generalized in different directions. For example, see Edelstein [1, 2], Kasahara [3], Rhoades [4], Siddiq and Ansari [5], and others. One of its generalizations is for nonexpansive single-valued maps on certain subsets of a Banach space. Indeed, these fixed points are not necessarily unique. See, for example, Browder [6–8] and Kirk [9]. Fixed point theorems for contractive and nonexpansive multivalued maps have also been established by several authors. Let H denote the Hausdorff metric on the space of all bounded nonempty subsets of a metric space (X, d). A multivalued map $J : X \to 2^X$ (where 2^X denotes the collection of all nonempty subsets of X) with bounded subsets as values is called contractive [10] if

$$H(J(x), J(y)) \leq hd(x, y) \tag{1.1}$$

for all $x, y \in X$ and for a fixed number $h \in [0, 1)$. If the Lipschitz constant $h = 1$, then J is called a multivalued nonexpansive mapping [11]. Nadler [10], Markin [11], Lami-Dozo [12], and others proved fixed point theorems for these maps under certain conditions in the setting of metric and Banach spaces. Note that an element $x \in X$ is called a fixed point of a multivalued map $J : X \to 2^X$ if $x \in J(x)$. Among others, without using the concept of the Hausdorff metric, Husain and Tarafdar [13] introduced the notion of a nonexpansive-type multivalued map and proved a fixed point theorem on compact intervals of the real line. Using such type of notions Husain and Latif [14] extended their result to general Banach space setting.

The fixed point results in modular function spaces were given by Khamsi et al. [15]. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated in modular spaces. For instance, fixed point theorems are proved in [15, 16] for nonexpansive maps.

In this paper, we define nonexpansive-type and contractive-type multivalued maps in modular function spaces, investigate the existence of fixed points of such mappings, and prove similar results found in [17].

Now, we recall some basic notions and facts about modular spaces as formulated by Kozlowski [18]. For more details the reader may consult [15, 16].

Let Ω be a nonempty set and let Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$.

Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, that is, all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 1.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$,

(P_2) $\rho(f, E) \leq \rho(g, E)$ whenever $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$, $f, g \in \mathcal{E}$ and $E \in \Sigma$,

(P_3) $\rho(f, \cdot) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$,

(P_4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,

(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$, and

(P_6) for any $\alpha > 0$, $\rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is, $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \in \mathcal{P}$ and decreases to $\emptyset$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E);\ g \in \varepsilon,\ |g(\omega)| \leq |f(\omega)|, \text{ for every } \omega \in \Omega\}. \tag{1.2}$$

For the sake of simplicity we write $\rho(f)$ instead of $\rho(f, \Omega)$.

Definition 1.2. A set E is said to be ρ-null if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $p(w)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{w \in \Omega : p(w)\text{does not hold}\}$ is ρ-null.

Definition 1.3. A modular function ρ is called σ-finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(K_n) < \infty$ and $\Omega = \bigcup K_n$. It is easy to see that the functional

$\rho : \mathcal{M} \to [0, \infty]$ is a modular and satisfies the following properties:

(i) $\rho(f) = 0$ if and only if $f = 0$ ρ-a.e.,

(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$, and

(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$.

In addition, if the following property is satisfied,

(iii)′ $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ and, $f, g \in \mathcal{M}$,

we say that ρ is a convex modular.

The modular ρ defines a corresponding modular space, that is, the vector space L_ρ given by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \tag{1.3}$$

When ρ is convex, the formula

$$\|f\|_p = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1\right\} \tag{1.4}$$

defines a norm in the modular space L_ρ which is frequently called the Luxemburg norm. We can also consider the space

$$E_\rho = \{f \in \mathcal{M}; \rho(\alpha f, A_n) \to 0 \text{ as } n \to \infty \text{ for every } A_n \in \Sigma \text{ that decreases to } \emptyset \text{ and } \alpha > 0\}. \tag{1.5}$$

Definition 1.4. A function modular is said to satisfy the Δ_2-condition if $\sup_{n\geq 1}\rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{f_n\}_{n\geq 1} \subset \mathcal{M}$, $D_k \in \Sigma$ decreases to $\emptyset$ and $\sup_{n\geq 1}\rho(f_n, D_k) \to 0$ as $k \to \infty$.

We know from [18] that $E_\rho = L_\rho$ when ρ satisfies the Δ_2-condition.

Definition 1.5. A function modular is said to satisfy the Δ_2-type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \leq K\rho(f)$.

In general, Δ_2-type condition and Δ_2-condition are not equivalent, even though it is obvious that Δ_2-type condition implies Δ_2-condition on the modular space L_ρ.

Definition 1.6. Let Ł$_\rho$ be a modular space.

(1) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(2) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega; f_n(\omega) \nrightarrow f(\omega)\}$ is ρ-null.

(3) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞.

(4) A subset C of L_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

(5) A subset C of L_ρ is called ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of C always belongs to C.

(6) A subset C of L_ρ is called ρ-a.e. compact if every sequence in C has a ρ-a.e. convergent subsequence in C.

(7) A subset C of L_ρ is called ρ-bounded if

$$\delta_\rho(C) = \sup\{\rho(f-g); f, g \in C\} < \infty. \tag{1.6}$$

We recall two basic results (see [15]) in the theory of modular spaces.

(i) If there exists a number $\alpha > 0$ such that $\rho(\alpha(f_n - f)) \to 0$, then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to f\, \rho$-a.e.

(ii) (Lebesgue's Theorem) If $f_n, f \in \mathcal{M}$, $f_n \to f\, \rho$-a.e. and there exists a function $g \in E_\rho$ such that $|f_n| \leq |g| \rho$-a.e. for all n, then $\|f_n - f\|_p \to 0$.

We know, by [15, 16] that under Δ_2-condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2-type condition. In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2-type condition.

Definition 1.7. Let ρ be as aforementioned. We define a growth function ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)},\ f \in L_\rho \setminus \{0\}\right\} \quad \forall 0 \leq t < \infty. \tag{1.7}$$

We have the following:

Lemma 1.8 (see [19]). *Let ρ be as aforementioned. Then the growth function ω has the following properties:*

(1) $\omega(t) < \infty$,$\forall t \in [0, \infty)$,

(2) $\omega : [0, \infty) \to [0, \infty)$ *is a convex, strictly increasing function. So, it is continuous,*

(3) $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$,

(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0, \infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 1.9 (see [19]). *Let ρ be a convex function modular satisfying the Δ_2-type condition. Then*

$$\|f\|_p \leq \frac{1}{\omega^{-1}(1/\rho(f))} \quad \textit{whenever} \quad f \in L_\rho. \tag{1.8}$$

The next lemma will be of major interest throughout this work.

Lemma 1.10 (see [16]). *Let ρ be a function modular satisfying the Δ_2-condition and let $\{f_n\}$ be a sequence in L_ρ such that $f_n \overset{\rho-a.e}{\rightarrow} f \in L_\rho$, and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then,*

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f - g) \quad \forall g \in L_\rho. \tag{1.9}$$

Moreover, one has

$$\rho(f) \leq \liminf_{n\to\infty} \rho(f_n). \tag{1.10}$$

2. Fixed Points of Contractive-Type and Nonexpansive-Type Maps

In the sequel we assume that ρ is a convex, σ-finite modular function satisfying the Δ_2-type condition, and C is a nonempty ρ-bounded subset of the modular function space L_ρ. We denote that $\mathcal{C}(C)$ is a collection of all nonempty ρ-closed subsets of C, and $\mathcal{K}(C)$ is a collection of all nonempty ρ-compact subsets of C.

We say that a multivalued map $T : C \to 2^C$ is ρ-contractive-type if there exists $k \in (0,1)$ such that for any $f, g \in C$ and for any $F \in T(f)$, there exists $G \in T(g)$ such that

$$\rho(F - G) \leq k\rho(f - g), \tag{2.1}$$

and ρ-nonexpansive-type if for any $f, g \in C$ and for any $F \in T(f)$, there exists $G \in T(g)$ such that

$$\rho(F - G) \leq \rho(f - g). \tag{2.2}$$

We have the following fixed point theorem (for which a similar result may be found in [17]).

Theorem 2.1. *Let C be a nonempty ρ-closed subset of the modular function space L_ρ. Then any $T : C \to \mathcal{C}(C)$ ρ-contractive-type map has a fixed point, that is, there exists $f \in C$ such that $f \in T(f)$.*

Proof. Let $f_0 \in C$. Without loss of generality, assume that f_0 is not a fixed point of T. Then there exists $f_1 \in T(f_0)$ such that $f_1 \neq f_0$. Hence $\rho(f_0, f_1) > 0$. Since T is ρ-contractive-type, then there exists $f_2 \in T(f_1)$ such that

$$\rho(f_1 - f_2) \leq k\rho(f_0 - f_1). \tag{2.3}$$

By induction, one can easily construct a sequence $\{f_n\} \in C$ such that $f_{n+1} \in T(f_n)$ and

$$\rho(f_{n+1} - f_n) \leq k\rho(f_n - f_{n-1}), \tag{2.4}$$

for any $n \geq 1$. In particular we have

$$\rho(f_{n+1} - f_n) \leq k^n \rho(f_1 - f_0). \tag{2.5}$$

Without loss of generality, we may assume $\rho(f_{n+1}, f_n) \neq 0$, otherwise f_n is a fixed point of T. Hence

$$\frac{1}{k^n \rho(f_1 - f_0)} \leq \frac{1}{\rho(f_{n+1} - f_n)} \tag{2.6}$$

Using Lemma 1.9, we get

$$\|f_{n+1} - f_n\|_\rho \leq \frac{1}{\omega^{-1}(1/\rho(f_{n+1} - f_n))}. \tag{2.7}$$

Using the properties of $\omega(t)$, we get

$$\omega^{-1}\left(\frac{1}{k^n \rho(f_1 - f_0)}\right) \leq \omega^{-1}\left(\frac{1}{\rho(f_{n+1} - f_n)}\right). \tag{2.8}$$

So

$$\omega^{-1}\left(\frac{1}{k}\right)^n \omega^{-1}\left(\frac{1}{\rho(f_1 - f_0)}\right) \leq \omega^{-1}\left(\frac{1}{\rho(f_{n+1} - f_n)}\right), \tag{2.9}$$

which implies

$$\|f_{n+1} - f_n\|_\rho \leq \frac{1}{\omega^{-1}(1/k)^n \omega^{-1}(1/\rho(f_1 - f_0))}. \tag{2.10}$$

Since $\omega(1) = 1$ and $k < 1$, then $1 < \omega^{-1}(1/k)$. This forces $\{f_n\}$ to be $\|\cdot\|_\rho$-Cauchy. Hence the sequence $\{f_n\}\|\cdot\|_\rho$-converges to some $f \in L_\rho$. Since ρ satisfies the Δ_2-condition, then $\{f_n\}\rho$-converges to f. Since C is ρ-closed, then $f \in C$. Let us prove that f is indeed a fixed point of T. Since T is a ρ-contractive-type mapping, then for any $n \geq 1$, there exists $F_n \in T(f)$ such that

$$\rho(f_{n+1} - F_n) \leq k\rho(f_n - f). \tag{2.11}$$

Hence $\{\rho(f_{n+1} - F_n)\}$ converges to 0. Since ρ satisfies the Δ_2-condition, we have $\{\|f_{n+1} - F_n\|_\rho\}$ converges to 0. Since $\{f_n\}\|\cdot\|_\rho$-converges to f, then $\{F_n\}\|\cdot\|_\rho$-converges to f. Hence $\{F_n\}\rho$-converges to f. Since $T(f)$ is ρ-closed and $\{F_n\} \in T(f)$, we get $f \in T(f)$. □

Remark 2.2. Consider the multivalued map $T_A(f) = A$, where A is a nonempty ρ-closed subset of C. Then it is easy to show that T_A is a ρ-contractive-type map. The set of all fixed point of T_A is exactly the set A. In particular, ρ-contractive-type maps may not have a unique fixed point.

As an application of the above theorem, we have the following result.

Proposition 2.3. *Let C be a ρ-closed convex subset of the modular function space L_ρ. Let $T : C \to \mathcal{C}(C)$ be ρ-nonexpansive-type map. Then there exists an approximate fixed points sequence $\{f_n\}$ in C, that is, for any $n \geq 1$ there exists $F_n \in T(f_n)$ such that*

$$\lim_{n\to\infty} \rho\left(f_n - F_n\right) = 0. \tag{2.12}$$

In particular one has $\lim_{n\to\infty}\text{dist}_\rho(f_n, T(f_n)) = 0$, *where*

$$\text{dist}_\rho\left(f_n, T\left(f_n\right)\right) = \inf\left\{\rho\left(f_n - g\right);\ g \in T\left(f_n\right)\right\}. \tag{2.13}$$

Proof. Let $\lambda \in (0,1)$ and let f_0 be a fixed point in C. For each $f \in C$, define a map

$$T_\lambda(f) = \lambda f_0 + (1-\lambda)T(f) = \left\{\lambda f_0 + (1-\lambda)g;\ g \in T(f)\right\}. \tag{2.14}$$

Note that $T_\lambda(f)$ is nonempty and ρ-closed subset of C because $T(f)$ is ρ-closed and C is convex. Since T is a ρ-nonexpansive-type map, for each $f, g \in C$ and for any $F \in T(f)$, there exists $G \in T(g)$ such that

$$\rho(F - G) \leq \rho(f - g). \tag{2.15}$$

Since ρ is convex we get

$$\rho\left(\left(\lambda f_0 + (1-\lambda)F\right) - \left(\lambda f_0 + (1-\lambda)G\right)\right) = \rho((1-\lambda)(F-G)) \leq (1-\lambda)\rho(F-G), \tag{2.16}$$

which implies

$$\rho\left(\left(\lambda f_0 + (1-\lambda)F\right) - \left(\lambda f_0 + (1-\lambda)G\right)\right) \leq (1-\lambda)\rho(f-g). \tag{2.17}$$

In other words, the map T_λ is a ρ-contractive-type. Theorem 2.1 implies the existence of a fixed point f_λ of T_λ, thus there exists $F_\lambda \in T(f_\lambda)$ such that

$$f_\lambda = \lambda f_0 + (1-\lambda)F_\lambda. \tag{2.18}$$

In particular, we have

$$\rho\left(f_\lambda - F_\lambda\right) = \rho\lambda\left(f_0 - F_\lambda\right) \leq \lambda\rho\left(f_0 - F_\lambda\right) \leq \lambda\delta_\rho(C), \tag{2.19}$$

where $\delta_\rho(C) = \sup_{f,g\in C}\rho(f-g)$ is the ρ-diameter of C. Note that since C is ρ-bounded, then $\delta_\rho(C) < \infty$. If we choose $\lambda = 1/n$, for $n \geq 1$ and write $f_n = f_{\lambda_n}$ and $F_n = F_{\lambda_n}$, we get

$$\rho(f_n - F_n) \leq \frac{\delta_\rho(C)}{n}, \tag{2.20}$$

for any $n \geq 1$, which implies $\lim_{n\to\infty}\rho(f_n - F_n) = 0$. □

Using the above result, we are now ready to prove the main fixed point result for ρ-nonexpansive-type multivalued maps.

Theorem 2.4. *Let C be a nonempty ρ-closed convex subset of the modular function space L_ρ. Assume that C is ρ-a.e. compact. Then each ρ-nonexpansive-type map $T : C \to \mathcal{K}(C)$ has a fixed point.*

Proof. Proposition 2.3 ensures the existence of a sequence $\{f_n\}$ in C and a sequence $\{F_n\}$ such that $F_n \in T(f_n)$ and $\lim_{n\to\infty}\rho(f_n - F_n) = 0$. Without loss of generality we may assume that $\{f_n\}\rho$-a.e. converges to $f \in C$ and $\{F_n\}\rho$-a.e. converges to $F \in C$. Lemma 1.10 implies

$$\rho(f - F) \leq \liminf_{n\to\infty}\rho(f_n - F_n) = 0. \tag{2.21}$$

Hence $f = F$. Since T is a ρ-nonexpansive-type map, then there exists a sequence $\{G_n\} \in T(f)$ such that

$$\rho(F_n - G_n) \leq \rho(f_n - f), \tag{2.22}$$

for all $n \geq 1$. Since $T(f)$ is ρ-compact, we may assume that $\{G_n\}$ is ρ-convergent to some $h \in T(f)$. Lemma 1.10 implies

$$\liminf_{n\to\infty}\rho(f_n - f) + \rho(f - h) = \liminf_{n\to\infty}\rho(f_n - h). \tag{2.23}$$

Since ρ satisfies the Δ_2-condition, then

$$\begin{aligned}\liminf_{n\to\infty}\rho(f_n - h) &= \liminf_{n\to\infty}\rho(f_n - F_n + F_n - G_n + G_n - h)\\ &= \liminf_{n\to\infty}\rho(F_n - G_n)\end{aligned} \tag{2.24}$$

(see, [20]). Since $\rho(F_n - G_n) \leq \rho(f_n - f)$, we get

$$\liminf_{n\to\infty}\rho(f_n - h) \leq \liminf_{n\to\infty}\rho(f_n - f), \tag{2.25}$$

which implies

$$\liminf_{n\to\infty}\rho(f_n - f) + \rho(f - h) \leq \liminf_{n\to\infty}\rho(f_n - f). \tag{2.26}$$

Hence $\rho(f-h)=0$ or $f=h$. Hence $f\in T(f)$; that is, f is a fixed point of T. $\square$

Proposition 2.3 and Theorem 2.4 are also hold if we assume that C is starshaped instead of Convex. (A set C is called starshaped if there exists $f_0\in C$ such that $\lambda f_0-(1-\lambda)f\in C$ provided $f\in C$ and $\lambda\in[0,1]$.)

3. Fixed Points of w-Contractive-Type Maps

In [21] the authors introduced the concept of w-distance in metric spaces which they connected to the existence of fixed point of single and multivalued maps (see also [22]). Similarly we extend their definition and results to modular spaces. Indeed let ρ be a convex, σ-finite modular function. A function $p : L_\rho\times L_\rho \to [0,\infty)$ is called w-modular on the modular function space L_ρ if the following are satisfied:

(1) $p(f,g)\le p(f,h)+p(h,g)$ for any $f,g,h\in L_\rho$;

(2) for any $f\in L_\rho$, $p(f,\cdot) : L_\rho \to [0,\infty)$ is lower semicontinuous; that is, if $\{g_n\}\rho$-converges to g, then

$$p(f,g)\le \liminf_{n\to\infty} p(f,g_n), \tag{3.1}$$

(3) for any $\varepsilon>0$, there exists $\delta>0$ such that $p(f,g)\le\delta$ and $p(f,h)\le\delta$ imply $\rho(g,h)\le\varepsilon$.

As it was done in [21], we need the following technical lemma.

Lemma 3.1. *Let $p(\cdot,\cdot)$ be w-modular on the modular function space L_ρ. Let $\{f_n\}$ and $\{g_n\}$ be sequences in L_ρ, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and $f,g,h\in L_\rho$. Then the following hold:*

(1) *if $p(f_n,g)\le\alpha_n$ and $p(f_n,h)\le\beta_n$, for all $n\ge1$, then $g=h$; in particular if $p(f,g)=0$ and $p(f,h)=0$, then $g=h$;*

(2) *if $p(f_n,g_n)\le\alpha_n$ and $p(f_n,h)\le\beta_n$, for any $n\ge1$, then $\{g_n\}\rho$-converges to h;*

(3) *if $p(f_n,f_m)\le\alpha_n$ for any $n,m\ge1$ with $m>n$, then $\{f_n\}$ is a ρ-Cauchy sequence;*

(4) *if $p(g,f_n)\le\alpha_n$ for any $n\ge1$, then $\{f_n\}$ is a ρ-Cauchy sequence.*

The proof is easy and similar to the one given in [21]. Now we are ready to give the first fixed point result in this setting. Let C be a nonempty ρ-closed subset of the modular function space L_ρ. We say that a multivalued map $T : C \to \mathcal{C}(C)$ is weakly ρ-contractive-type map if there exists w-modular $p(\cdot,\cdot)$ on L_ρ and $k\in[0,1)$ such that for any $f,g\in C$ and any $F\in T(f)$, there exists $G\in T(g)$ such that $p(F,G)\le kp(f,g)$.

Theorem 3.2. *Let C be a nonempty ρ-closed subset of the modular function space L_ρ. Then each weakly ρ-contractive-type map $T : C \to \mathcal{C}(C)$ has a fixed point $f\in C$, and $p(f,f)=0$.*

Proof. Let $p(\cdot,\cdot)$ be a w-modular and $k \in [0,1)$ associated to T, that is, for any $f, g \in C$ and any $F \in T(f)$, there exists $G \in T(g)$ such that $p(F,G) \le kp(f,g)$. Fix $f_0 \in C$ and $f_1 \in T(f_0)$. By induction one can construct a sequence $\{f_n\}$ such that $f_{n+1} \in T(f_n)$ and

$$p(f_n, f_{n+1}) \le kp(f_{n-1}, f_n), \tag{3.2}$$

for every $n \ge 1$. In particular we have $p(f_n, f_{n+1}) \le k^n p(f_0, f_1)$, for every $n \ge 1$. Using the properties of $p(\cdot,\cdot)$, we get

$$p(f_n, f_{n+h}) \le \frac{k^n}{1-k} p(f_0, f_1), \tag{3.3}$$

for any $n, h \ge 1$. Lemma 3.1 implies that the sequence $\{f_n\}$ is ρ-Cauchy. Hence $\{f_n\}\rho$-converges to some $f \in C$. Using the lower semicontinuity of p, we get

$$p(f_n, f) \le \liminf_{n\to\infty} p(f_n, f_{n+h}) \le \frac{k^n}{1-k} p(f_0, f_1), \tag{3.4}$$

for any $n \ge 1$. Since $f_n \in T(f_{n-1})$ and T is weakly ρ-contractive-type map, there exists $g_n \in T(f)$ such that

$$p(f_n, g_n) \le kp(f_{n-1}, f) \le \frac{k^n}{1-k} p(f_0, f_1), \tag{3.5}$$

for any $n \ge 2$. Lemma 3.1 implies that $\{g_n\}\rho$- converges to f as well. Since $T(f)$ is ρ-closed, then $f \in T(f)$, that is, f is a fixed point of T. Let us complete the proof by showing that $p(f,f) = 0$. Since $f \in T(f)$, there exists $h_1 \in T(f)$ such that $p(f,h_1) \le kp(f,f)$. By induction we can construct a sequence $\{h_n\}$ in C such that $h_{n+1} \in T(h_n)$ and $p(f, h_{n+1}) \le kp(f, h_n)$, for any $n \ge 1$. So we have $p(f, h_n) \le k^n p(f, f)$, for any $n \ge 1$. Lemma 3.1 implies that $\{h_n\}$ is ρ-Cauchy. Hence $\{h_n\}\rho$- converges to some $h \in C$. Using the lower semicontinuity of $p(\cdot,\cdot)$ we get

$$p(f, h) \le \liminf_{n\to\infty} p(f, h_n) \le 0. \tag{3.6}$$

Hence $p(f,h) = 0$. Then for any $n \ge 1$, we have

$$p(f_n, h) \le p(f_n, f) + p(f, h) \le \frac{k^n}{1-k} p(f_0, f_1). \tag{3.7}$$

Lemma 3.1 implies $f = h$, or $p(f,f) = 0$. □

Note that in the proof above we did not use the Δ_2-condition. The reason behind is that $p(\cdot,\cdot)$ satisfies the triangle inequality. If T is single valued, then we have little more information about the fixed point. Indeed, let C be a nonempty ρ-closed subset of the modular function space L_ρ. The map $T : C \to C$ is called a weakly ρ-contractive type map if there exists w-modular $p(\cdot,\cdot)$ on L_ρ and $k \in [0,1)$ such that for any $f, g \in C$; $p(T(f), T(g)) \le kp(f,g)$.

Theorem 3.3. *Let C be a nonempty ρ-closed subset of the modular function space L_ρ. Then each weakly ρ-contractive type map $T : C \to C$ has a unique fixed point $f \in C$, and $p(f, f) = 0$.*

Proof. Theorem 3.2 ensures the existence of a fixed point $f \in C$, that is, $T(f) = f$ and $p(f, f) = 0$. Let us show that f is the only fixed point of T. Assume that $h \in C$ is another fixed point of T. Then we must have $p(f, h) = 0$. Combining this with $p(f, f) = 0$, Lemma 3.1 implies $f = h$. □

Similar extensions of the results as found in [21–23] may be proved in our setting.

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