Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 751090, 6 pages
doi:10.1155/2009/751090

Research Article

On T-Stability of Picard Iteration in Cone Metric Spaces

M. Asadi,[1] H. Soleimani,[1] S. M. Vaezpour,[2, 3] and B. E. Rhoades[4]

[1] *Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran 14778 93855, Iran*
[2] *Department of Mathematics, Amirkabir University of Technology, Tehran 15916 34311, Iran*
[3] *Department of Mathematics, Newcastle University, Newcastle, NSW 2308, Australia*
[4] *Department of Mathematics, Indiana University, Bloomington, IN 46205, USA*

Correspondence should be addressed to S. M. Vaezpour, vaez@aut.ac.ir

Received 28 March 2009; Revised 28 September 2009; Accepted 19 October 2009

Recommended by Brailey Sims

The aim of this work is to investigate the T-stability of Picard's iteration procedures in cone metric spaces and give an application.

Copyright © 2009 M. Asadi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction and Preliminary

Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies the following:

(i) P is closed, nonempty, and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; by defining, $x \leq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

In the following we always suppose that E is a real Banach space, P is a cone in E, and $\leq$ is the partial ordering with respect to P.

Definition 1.1 (see [1]). Let X be a nonempty set. Assume that the mapping $d : X \times X \to E$ satisfies the following:

(i) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y)=0$ if and only if $x=y$,

(ii) $d(x,y)=d(y,x)$ for all $x,y \in X$,

(iii) $d(x,y) \leq d(x,z)+d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Definition 1.2. Let $T: X \to X$ be a map for which there exist real numbers a,b,c satisfying $0<a<1$, $0<b<1/2$, $0<c<1/2$. Then T is called a *Zamfirescu operator* if, for each pair $x,y \in X$, T satisfies at least one of the following conditions:

(1) $d(Tx,Ty) \leq a d(x,y)$,

(2) $d(Tx,Ty) \leq b(d(x,Tx)+d(y,Ty))$,

(3) $d(Tx,Ty) \leq c(d(x,Ty)+d(y,Tx))$.

Every Zamfirescu operator T satisfies the inequality:

$$d(Tx,Ty) \leq \delta d(x,y) + 2\delta d(x,Tx) \tag{1.1}$$

for all $x,y \in X$, where $\delta = \max\{a, b/(1-b), c/(1-c)\}$, with $0<\delta<1$. For normed spaces see [2].

Lemma 1.3 (see [3]). *Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n, \tag{1.2}$$

where $\lambda_n \in (0,1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $b_n/\lambda_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Remark 1.4. Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences satisfying the following inequality:

$$a_{n+1} \leq \lambda a_{n-m} + b_n, \tag{1.3}$$

where $\lambda \in (0,1)$, for all $n \geq n_0$ and for some positive integer number m. If $b_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5. *Let P be a normal cone with constant K, and let $\{a_n\}$ and $\{b_n\}$ be sequences in E satisfying the following inequality:*

$$a_{n+1} \leq h a_n + b_n, \tag{1.4}$$

where $h \in (0,1)$ and $b_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. Let m be a positive integer such that $h^m K < 1$. By recursion we have

$$a_{n+1} \leq b_n + h b_{n-1} + \cdots + h^m b_{n-m} + h^{m+1} a_{n-m}, \tag{1.5}$$

so

$$\|a_{n+1}\| \leq K\|b_n + hb_{n-1} + \cdots + h^m b_{n-m}\| + h^{m+1}K\|a_{n-m}\|, \tag{1.6}$$

and then by Remark 1.4 $\|a_n\| \to 0$. Therefore $a_n \to 0$. □

2. T-Stability in Cone Metric Spaces

Let (X, d) be a cone metric space, and T a self-map of X. Let x_0 be a point of X, and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T, which yields a sequence $\{x_n\}$ of points from X.

Definition 2.1 (see [4]). The iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable with respect to T if $\{x_n\}$ converges to a fixed point q of T and whenever $\{y_n\}$ is a sequence in X with $\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$ we have $\lim_{n\to\infty} y_n = q$.

In practice, such a sequence $\{y_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T, x_n)$. Let $y_0 = x_0$. Now $x_1 = f(T, x_0)$. Because of rounding or discretization in the function T, a new value y_1 approximately equal to x_1 might be obtained instead of the true value of $f(T, x_0)$. Then to approximate y_2, the value $f(T, y_1)$ is computed to yield y_2, an approximation of $f(T, y_1)$. This computation is continued to obtain $\{y_n\}$ an approximate sequence of $\{x_n\}$.

One of the most popular iteration procedures for approximating a fixed point of T is Picard's iteration defined by $x_{n+1} = Tx_n$. If the conditions of Definition 2.1 hold for $x_{n+1} = Tx_n$, then we will say that Picard's iteration is T-stable.

Recently Qing and Rhoades [5] established a result for the T-stability of Picard's iteration in metric spaces. Here we are going to generalize their result to cone metric spaces and present an application.

Theorem 2.2. *Let (X, d) be cone metric space, P a normal cone, and $T : X \to X$ with $F(T) \neq \emptyset$. If there exist numbers $a \geq 0$ and $0 \leq b < 1$, such that*

$$d(Tx, q) \leq ad(x, Tx) + bd(x, q) \tag{2.1}$$

for each $x \in X$, $q \in F(T)$ and in addition, whenever $\{y_n\}$ is a sequence with $d(y_n, Ty_n) \to 0$ as $n \to \infty$, then Picard's iteration is T-stable.

Proof. Suppose $\{y_n\} \subseteq X$, $c_n = d(y_{n+1}, Ty_n)$ and $c_n \to 0$. We shall show that $y_n \to q$. Since

$$d(y_{n+1}, q) \leq d(y_{n+1}, Ty_n) + d(Ty_n, q) \leq c_n + ad(y_n, Ty_n) + bd(y_n, q), \tag{2.2}$$

if we put $a_n := d(Ty_n, q)$ and $b_n := c_n + ad(y_n, Ty_n)$ in Lemma 1.5, then we have $y_n \to q$.

Note that the fixed point q of T is unique. Because if p is another fixed point of T, then

$$d(p, q) = d(Tp, q) \leq ad(p, Tp) + bd(p, q) = bd(p, q), \tag{2.3}$$

which implies $p = q$. □

Corollary 2.3. *Let (X, d) be a cone metric space, P a normal cone, and $T : X \to X$ with $q \in F(T)$. If there exists a number $\lambda \in [0, 1)$, such that $d(Tx, Ty) \leq \lambda d(x, y)$, for each $x, y \in X$, then Picard's iteration is T-stable.*

Corollary 2.4. *Let (X, d) be a cone metric space, P a normal cone, and $T : X \to X$ is a Zamfirescu operator with $F(T) \neq \emptyset$ and whenever $\{y_n\}$ is a sequence with $d(y_n, Ty_n) \to 0$ as $n \to \infty$, then Picard's iteration is T-stable.*

Definition 2.5 (see [6]). Let (X, d) be a cone metric space. A map $T : X \to X$ is called a quasicontraction if for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists $u \in C(T; x, y) \equiv \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, such that $d(Tx, Ty) \leq \lambda u$.

Lemma 2.6. *If T is a quasicontraction with $0 < \lambda < 1/2$, then T is a Zamfirescu operator and so satisfies* (2.1).

Proof. Let $\lambda \in (0, 1/2)$ for every $x, y \in X$ we have $d(Tx, Ty) \leq \lambda u$ for some $u \in C(T; x, y)$. In the case that $u = d(x, Ty)$ we have

$$d(Tx, Ty) \leq \lambda d(x, Ty) \leq \lambda d(x, Tx) + \lambda d(Tx, Ty). \tag{2.4}$$

So

$$d(Tx, Ty) \leq \frac{\lambda}{1-\lambda} d(x, Tx) \leq 2\frac{\lambda}{1-\lambda} d(x, Tx) + \frac{\lambda}{1-\lambda} d(x, y). \tag{2.5}$$

Put $\delta := \lambda/(1-\lambda)$ so $0 < \delta < 1$. The other cases are similarly proved. Therefore T is a Zamfirescu operator. □

Theorem 2.7. *Let (X, d) be a nonempty complete cone metric space, P be a normal cone, and T a quasicontraction and self map of X with some $0 < \lambda < 1/2$. Then Picard's iteration is T-stable.*

Proof. By [6, Theorem 2.1], T has a unique fixed point $q \in X$. Also T satisfies (2.1). So by Theorem 2.2 it is enough to show that $d(y_n, Ty_n) \to 0$. We have

$$d(y_n, Ty_n) \leq d(y_n, Ty_{n-1}) + d(Ty_{n-1}, Ty_n). \tag{2.6}$$

Put $b_n := d(y_n, Ty_n), c_n := d(y_{n+1}, Ty_n)$ and $d_n := d(Ty_{n-1}, Ty_n)$. Therefore $c_n \to 0$ as $n \to \infty$ and

$$b_n \leq c_{n-1} + d_n \leq c_{n-1} + \lambda u_n, \tag{2.7}$$

where

$$u_n \in C(T, y_{n-1}, y_n) = \{d(y_{n-1}, y_n), d(y_{n-1}, Ty_{n-1}), d(y_n, Ty_n), d(y_{n-1}, Ty_n), d(y_n, Ty_{n-1})\}. \tag{2.8}$$

Hence we have $u_n = b_n$ or $u_n \leq sb_{n-1} + lc_{n-1}$ where $s = 0, 1$ or $1/(1-\lambda)$ and $l = 1$ or $1+\lambda$. Therefore by (2.7), $b_n \leq (\lambda l + 1)c_{n-1} + \lambda s b_{n-1}$ by $0 \leq \lambda s < 1$. Now by Lemma 1.5 we have $b_n \to 0$. □

3. An Application

Theorem 3.1. *Let* $X := (C[0,1], \mathbb{R})$ *with* $\|f\|_\infty := \sup_{0\leq x\leq 1}|f(x)|$ *for* $f \in X$ *and let* T *be a self map of* X *defined by* $Tf(x) = \int_0^1 F(x, f(t))dt$ *where*

(a) $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is a continuous function,*

(b) *the partial derivative* F_y *of* F *with respect to* y *exists and* $|F_y(x,y)| \leq L$ *for some* $L \in [0,1)$,

(c) *for every real number* $0 \leq a < 1$ *one has* $ax \leq F(x, ay)$ *for every* $x, y \in [0,1]$.

Let $P := \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0\}$ *be a normal cone and* (X, d) *the complete cone metric space defined by* $d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$ *where* $\alpha \geq 0$. *Then,*

(i) *Picard's iteration is* T*-stable if* $0 \leq L < 1/2$,

(ii) *Picard's iteration fails to be* T*-stable if* $1/2 \leq L < 1$ *and* $\int_0^1 F(x,t)dt \neq x$.

Proof. (i) We have T being a continuous quasicontraction map with $0 \leq \lambda := L < 1/2$; so by Theorem 2.7, Picard's iteration is T-stable.

(ii) Put $y_n(x) := nx/(n+1)$ so $y_n \in X$ and $d(y_n, h) \to 0$, where $h(x) = x$. Also $d(y_{n+1}, Ty_n) \to 0$, since

$$\begin{aligned}\|y_{n+1} - Ty_n\|_\infty &= \sup_{0\leq x\leq 1}\left|\frac{n+1}{n+2}x - \int_0^1 F\left(x, \frac{nt}{n+1}\right)dt\right| \\ &\leq \sup_{0\leq x\leq 1}\left|\frac{n+1}{n+2}x - \frac{nx}{n+1}\right| \longrightarrow 0,\end{aligned} \tag{3.1}$$

as $n \to \infty$. But $y_n \to h$ and h is not a fixed point for T. Therefore Picard's iteration is not T-stable. □

Example 3.2. Let $F_1(x,y) := x + y/4$ and $F_2(x,y) := x + y/2$. Therefore F_1 and F_2 satisfy the hypothesis of Theorem 3.1 where F_1 has property (i) and F_2 has property (ii). So the self maps T_1, T_2 of X defined by $T_1 f(x) = x + (1/4)\int_0^1 f(t)dt$ and $T_2 f(x) = x + (1/2)\int_0^1 f(t)dt$ have unique fixed points but Picard's iteration is T-stable for T_1 but not T-stable for T_2.

References

[1] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[2] X. Zhiqun, "Remarks of equivalence among Picard, Mann, and Ishikawa iterations in normed spaces," *Fixed Point Theory and Applications*, vol. 2007, Article ID 61434, 5 pages, 2007.

[3] F. P. Vasilev, *Numerical Methodes for Solving Extremal Problems*, Nauka, Moscow, Russian, 2nd edition, 1988.

[4] A. M. Harder and T. L. Hicks, "Stability results for fixed point iteration procedures," *Mathematica Japonica*, vol. 33, no. 5, pp. 693–706, 1988.

[5] Y. Qing and B. E. Rhoades, “T-stability of Picard iteration in metric spaces,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 418971, 4 pages, 2008.

[6] D. Ilić and V. Rakoćević, “Quasi-contraction on a cone metric space,” *Applied Mathematics Letters*, vol. 22, no. 5, pp. 728–731, 2009.