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Research Article

Construction of Fixed Points by Some Iterative Schemes

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We obtain strong convergence theorems of two modifications of Mann iteration processes with errors in the doubly sequence setting. Furthermore, we establish some weakly convergence theorems for doubly sequence Mann's iteration scheme with errors in a uniformly convex Banach space by a Frechét differentiable norm.

1. Introduction

Let X be a real Banach space and let C be a nonempty closed convex subset of X. A self-mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $\text{Fix}(T)$ the set of fixed points of T; that is, $\text{Fix}(T) = \{x \in C : Tx = x\}$. It is assumed throughout this paper that T is a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Construction of fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing (see [1–3]). One way to overcome this difficulty is to use Mann's iteration method that produces a sequence $\{x_n\}$ via the recursive sequence manner:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0. \tag{1.1}$$

Reich [4] proved that if X is a uniformly convex Banach space with a Frechét differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However, this scheme has only weak convergence even in a Hilbert space (see [5]). Some attempts to modify Mann's iteration method (1.1) so that strong convergence is guaranteed have recently been made.

The following modification of Mann's iteration method (1.1) in a Hilbert space H is given by Nakajo and Takahashi [6]:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n}(x_0), \end{aligned} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded from one, then $\{x_n\}$ defined by (1.2) converges strongly to $P_{\mathrm{Fix}(T)}(x_0)$. Their argument does not work outside the Hilbert space setting. Also, at each iteration step, an additional projection is needed to calculate.

Let C be a closed convex subset of a Banach space and $T : C \to C$ is a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Define $\{x_n\}$ in the following way:

$$\begin{aligned} x_0 &= x \in X, \\ y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\ x_{n+1} &= \beta_n u + (1-\beta_n) y_n, \end{aligned} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element in C, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. It is proved, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.3) converges to a fixed point of T (see [7]).

The second modification of Mann's iteration method (1.1) is an adaption to (1.3) for finding a zero of an m-accretive operator A, for which we assume that the zero set $A^{-1}(0) \neq \emptyset$.

The iteration process $\{x_n\}$ is given by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= J_{r_n} x_n, \\ x_{n+1} &= \beta_n u + (1-\beta_n) y_n, \end{aligned} \tag{1.4}$$

where for each $r > 0$, $J_r = (I + rA)^{-1}$ is the resolvent of A. In [7], it is proved, in a uniformly smooth Banach space and under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{r_n\}$, that $\{x_n\}$ defined by (1.4) converges strongly to a zero of A.

2. Preliminaries

Let X be a real Banach space. Recall that the (normalized) duality map J from X into X^*, the dual space of X, is given by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X. \tag{2.1}$$

Now, we define Opial's condition in the sense of doubly sequence.

Definition 2.1. A Banach space X is said to satisfy Opial's condition if for any sequence $\{x_{k,n}\}$ in X, $x_{k,n} \rightharpoonup x$ implies that

$$\lim_{k,n\to\infty} \sup \|x_{k,n} - x\| < \lim_{k,n\to\infty} \sup \|x_{k,n} - y\| \quad \forall y \in X \text{ with } y \neq x, \tag{2.2}$$

where $x_{k,n} \rightharpoonup x$ denotes that $\{x_{k,n}\}$ converges weakly to x.

We are going to work in uniformly smooth Banach spaces that can be characterized by duality mappings as follows (see [8] for more details).

Lemma 2.2 (see [8]). *A Banach space X is uniformly smooth if and only if the duality map J is single-valued and norm-to-norm uniformly continuous on bounded sets of X.*

Lemma 2.3 (see [8]). *In a Banach space X, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad x, y \in X, \tag{2.3}$$

where $j(x + y) \in J(x + y)$.

If C and D are nonempty subsets of a Banach space X such that C is a nonempty closed convex subset and $D \subset C$, then the map $Q : C \to D$ is called a retraction from C onto D provided $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is sunny [1, 4] provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. A sunny nonexpansive retraction plays an important role in our argument.

If X is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0 \quad \forall x \in C, \ y \in D. \tag{2.4}$$

Lemma 2.4 (see [9]). *Let X be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to \mathrm{Fix}(T)$ by $Qu = s\text{-}\lim_{t\to 0} x_t$. Then, Q is the unique sunny nonexpansive retract from C onto $\mathrm{Fix}(T)$; that is, Q satisfies the property*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0, \quad \forall u \in C, \ z \in \mathrm{Fix}(T). \tag{2.5}$$

Lemma 2.5 (see [10, 11]). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\sigma_n, \quad n \geq 0, \tag{2.6}$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ are such that

(i) $\lim_{n\to\infty} \gamma_n = 0$, *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either* $\lim_{n\to\infty} \sup \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then, $\{a_n\}_{n=0}^{\infty}$ converges to zero.

Lemma 2.6 (see [8]). *Assume that X has a weakly continuous duality map J_φ with gauge φ. Then, A is demiclosed in the sense that A is closed in the product space $X_w \times X$, where X is equipped with the norm topology and X_w with the weak topology. That is, if $(x_n, y_n) \in A$, $x_n \rightharpoonup x$, $y_n \to y$, then $(x, y) \in A$.*

Lemma 2.7 (see [12]). *Let X be a Banach space and $\gamma \geq 2$. Then,*

(i) *X is uniformly convex if and only if, for any positive number r, there is a strictly increasing continuous function $g_r : [0,\infty) \to [0,\infty)$, $g_r(0) = 0$, such that*

$$\|tx + (1-t)y\|^\gamma \leq t\|x\|^\gamma + (1-t)\|y\|^\gamma - W_\gamma(t)g_r(\|x-y\|), \tag{2.7}$$

where $t \in [0,1]$, $x, y \in B_r := \{u \in X : \|u\| \leq r\}$, the closed ball of X centered at the origin with radius r, and $W_\gamma(t) = t^\gamma(1-t) + t(1-t)^\gamma$.

(ii) *X is γ-uniformly convex if and only if there holds the inequality*

$$\|tx + (1-t)y\|^\gamma \leq t\|x\|^\gamma + (1-t)\|y\|^\gamma - c_\gamma W_\gamma(t)\|x-y\|^\gamma, \quad t \in [0,1],\ x, y \in X, \tag{2.8}$$

where $c_\gamma > 0$ is a constant.

Lemma 2.8 (see [4]). *Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, and let (T_n) be a sequence of nonexpansive self mapping of C with a nonempty common fixed point set F. If $x_1 \in C$ and $x_{n+1} = T_n x_n$ for $n \geq 1$, then $\lim_{n\to\infty}\langle x_n, J(f_1 - f_2)\rangle$ exists for all $f_1, f_2 \in F$. In particular, $\langle q_1 - q_2, J(f_1 - f_2)\rangle = 0$, where $f_1, f_2 \in F$ and q_1, q_2 are weak limit points of $\{x_n\}$.*

Lemma 2.9 (the demiclosedness principle of nonexpansive mappings [13]). *Let T be a nonexpansive selfmapping of a closed convex subset of E of a uniformly convex Banach space. Suppose that T has a fixed point. Then $I - T$ is demiclosed. This means that*

$$\{x_n\} \subset E, \quad x_n \rightharpoonup x, \quad (I-T)x_n \to y \Longrightarrow (I-T)x = y. \tag{2.9}$$

In 2005, Kim and Xu [7], proved the following theorem.

Theorem A. *Let C be a closed convex subset of a uniformly smooth Banach space X, and let $T : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Given a point $u \in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$, the following conditions are satisfied.*

(i) $\alpha_n \to 0,\ \beta_n \to 0,$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \sum_{n=0}^{\infty} \beta_n = \infty,$

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by

$$\begin{aligned} x_0 &\in C \quad \text{arbitrarily}, \\ y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \quad n \geq 0, \\ x_{n+1} &= \beta_n u + (1-\beta_n) y_n, \quad n \geq 0. \end{aligned} \tag{2.10}$$

Then $\{x_n\}_{n=0}^{\infty}$ is strongly converges to a fixed point of T.

Recently, the study of fixed points by doubly Mann iteration process began by Moore (see [14]). In [15, 16], we introduced the concept of Mann-type doubly sequence iteration with errors, then we obtained some fixed point theorems for some different classes of mappings. In this paper, we will continue our study in the doubly sequence setting. We propose two modifications of the doubly Mann iteration process with errors in uniformly smooth Banach spaces: one for nonexpansive mappings and the other for the resolvent of accretive operators. The two modified doubly Mann iterations are proved to have strong convergence. Also, we append this paper by obtaining weak convergence theorems for Mann's doubly sequence iteration with errors in a uniformly convex Banach space by a Fréchet differentiable norm. Our results in this paper extend, generalize, and improve a lot of known results (see, e.g., [4, 7, 8, 17]). Our generalizations and improvements are in the use of doubly sequence settings as well as by adding the error part in the iteration processes.

3. A Fixed Point of Nonexpansive Mappings

In this section, we propose a modification of doubly Mann's iteration method with errors to have strong convergence. Modified doubly Mann's iteration process is a convex combination of a fixed point in C, and doubly Mann's iteration process with errors can be defined as

$$\begin{aligned} x_{0,0} &= x \in C \quad \text{arbitrarily}, \\ y_{k,n} &= \alpha_n x_{k,n} + (1-\alpha_n) T x_{k,n} + \alpha_n w_{k,n}, \quad k, n \geq 0, \\ x_{k,n+1} &= \beta_n u + (1-\beta_n) y_{k,n} + \beta_n v_{k,n}, \quad k, n \geq 0. \end{aligned} \tag{3.1}$$

The advantage of this modification is that not only strong convergence is guaranteed, but also computations of iteration processes are not substantially increased.

Now, we will generalize and extend Theorem A by using scheme (3.1).

Theorem 3.1. *Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Given a point $u \in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$, the following conditions are satisfied.*

(i) $\alpha_n \to 0$, $\beta_n \to 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n = \infty$.

Define a sequence $\{x_{k,n}\}_{k,n=0}^{\infty}$ *in* C *by* (3.1). *Then,* $\{x_{k,n}\}_{k,n=0}^{\infty}$ *converges strongly to a fixed point of* T.

Proof. First, we observe that $\{x_{k,n}\}_{k,n=0}^{\infty}$ is bounded. Indeed, if we take a fixed point p of T noting that

$$\begin{aligned}\|y_{k,n} - p\| &= \|\alpha_n x_{k,n} + (1-\alpha_n)Tx_{k,n} + \alpha_n w_{k,n} - p\| \\ &\le \alpha_n\|x_{k,n} - p\| + (1-\alpha_n)\|Tx_{k,n} - p\| + \alpha_n\|w_{k,n}\| \\ &= \|x_{k,n} - p\| + \alpha_n\|w_{k,n}\|,\end{aligned} \tag{3.2}$$

we obtain

$$\begin{aligned}\|x_{k,n+1} - p\| &= \|\beta_n u + (1-\beta_n)y_{k,n} + \beta_n v_{k,n} - p\| \\ &\le \beta_n\|u-p\| + (1-\beta_n)\|y_{k,n} - p\| + \beta_n\|v_{k,n}\| \\ &\le \beta_n\|u-p\| + (1-\beta_n)(\|x_{k,n} - p\| + \alpha_n\|w_{k,n}\|) + \beta_n\|v_{k,n}\| \\ &\le \max\{\|x_{k,n} - p\|, \|u-p\|\} + \beta_n\|v_{k,n}\| + (1-\beta_n)\alpha_n\|w_{k,n}\|.\end{aligned} \tag{3.3}$$

Now, an induction yields

$$\|x_{k,n} - p\| \le \max\{\|x_{0,0} - p\|, \|u-p\|, \|v_{0,0}\|\} \quad k,n \ge 0. \tag{3.4}$$

Hence, $\{x_{k,n}\}$ is bounded, so is $\{y_{k,n}\}$. As a result, we obtain by condition (i)

$$\begin{aligned}\|x_{k,n+1} - y_{k,n}\| &= \|\beta_n u - \beta_n y_{k,n} + \beta_n v_{k,n}\| \\ &\le \beta_n\|u - y_{k,n}\| + \beta_n\|v_{k,n}\| \longrightarrow 0.\end{aligned} \tag{3.5}$$

We next show that

$$\|x_{k,n} - Tx_{k,n}\| \longrightarrow 0. \tag{3.6}$$

It suffices to show that

$$\|x_{k,n+1} - x_{k,n}\| \longrightarrow 0. \tag{3.7}$$

Indeed, if (3.7) holds, in view of (3.5), we obtain

$$\begin{aligned}\|x_{k,n} - Tx_{k,n}\| &\le \|x_{k,n} - x_{k,n+1}\| + \|x_{k,n+1} - y_{k,n}\| + \|y_{k,n} - Tx_{k,n}\| \\ &\le \|x_{k,n} - x_{k,n+1}\| + \|x_{k,n+1} - y_{k,n}\| + \alpha_n\|x_{k,n} - Tx_{k,n}\| + \alpha_n\|w_{k,n}\| \longrightarrow 0.\end{aligned} \tag{3.8}$$

Hence, (3.6) holds. In order to prove (3.7), we calculate

$$\begin{aligned} x_{k,n+1}-x_{k,n} &= (\beta_n-\beta_{n-1})(u-Tx_{n-1})+(1-\beta_n)\alpha_n(x_{k,n}-x_{k,n-1}) \\ &\quad+\left[(\alpha_n-\alpha_{n-1})(1-\beta_n)-(\beta_n-\beta_{n-1})\alpha_{n-1}\right](x_{k,n-1}-Tx_{k,n-1}) \\ &\quad+(1-\alpha_n)(1-\beta_n)(Tx_{k,n}-Tx_{k,n-1})+(1-\beta_n)\alpha_n w_{k,n} \\ &\quad+\beta_n v_{k,n}-(1-\beta_n)\alpha_{n-1}w_{k,n-1}-\beta_{n-1}v_{k,n-1}. \end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned} \|x_{k,n+1}-x_{k,n}\| &\le (1-\alpha_n)(1-\beta_n)\|Tx_{k,n}-Tx_{k,n-1}\|+(1-\beta_n)\alpha_n\|x_{k,n}-x_{k,n-1}\| \\ &\quad+\left|(\alpha_n-\alpha_{n-1})(1-\beta_n)-(\beta_n-\beta_{n-1})\alpha_{n-1}\right|\|x_{k,n-1}-Tx_{k,n-1}\| \\ &\quad+|\beta_n-\beta_{n-1}|\|u-Tx_{k,n-1}\|+(1-\beta_n)\alpha_n\|w_{k,n}\|+\beta_n\|v_{k,n}\| \\ &\quad-(1-\beta_n)\alpha_{n-1}\|w_{k,n-1}\|-\beta_{n-1}\|v_{k,n-1}\|. \end{aligned} \tag{3.10}$$

Hence, by assumptions (i)-(ii), we obtain $\|x_{k,n+1}-x_{k,n}\| \longrightarrow 0$.

Next, we claim that

$$\lim_{k,n\to\infty}\sup\langle u-q, J(x_{k,n}-q)\rangle \le 0, \tag{3.11}$$

where $q=Q(u)=s-\lim_{t\to 0}z_t$ with z_t being the fixed point of the contraction $z \longrightarrow tu+(1-t)Tz$. In order to prove (3.11), we need some more information on q, which is obtained from that of z_t (cf. [18]). Indeed, z_t solves the fixed point equation

$$z_t = tu+(1-t)Tz_t+tv. \tag{3.12}$$

Thus we have

$$z_t-x_{k,n}=(1-t)(Tz_t-x_{k,n})+t(u-x_{k,n})+tv. \tag{3.13}$$

We apply Lemma 2.3 to get

$$\begin{aligned} \|z_t-x_{k,n}\|^2 &\le (1-t)^2\|Tz_t-x_{k,n}\|^2+2t\langle u+v-x_{k,n}, J(z_t-x_{k,n})\rangle \\ &\le (1-2t+t^2)\|z_t-x_{k,n}\|+a_n(t)+2t\langle u+v-z_t, J(z_t-x_{k,n})\rangle+2t\|z_t-x_{k,n}\|^2, \end{aligned} \tag{3.14}$$

$$a_n(t)=(2\|z_t-x_{k,n}\|+\|x_{k,n}-Tx_{k,n}\|)\|x_{k,n}-Tx_{k,n}\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.15}$$

It follows that

$$\langle z_t-u, J(z_t-x_{k,n})\rangle \le \frac{t}{2}\|z_t-x_{k,n}\|^2+\frac{1}{2t}a_n(t). \tag{3.16}$$

Letting $n \to \infty$ in (3.16) and noting (3.15) yield

$$\lim_{n\to\infty} \sup \langle z_t - u, J(z_t - x_{k,n})\rangle \le \frac{t}{2}M, \tag{3.17}$$

where $M > 0$ is a constant such that $M \ge \|z_t - x_{k,n}\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Since the set $\{z_t - x_{k,n}\}$ is bounded, the duality map J is norm-to-norm uniformly continuous on bounded sets of X (Lemma 2.2), and z_t strongly converges to q. By letting $t \to 0$ in (3.17), thus (3.11) is therefore proved. Finally, we show that $x_{k,n} \to q$ strongly and this concludes the proof. Indeed, using Lemma 2.3 again, we obtain

$$\begin{aligned}
\|x_{k,n+1} - q\|^2 &= \|(1-\beta_n)y_{k,n} + \beta_n u + \beta_n v_{k,n} - q\|^2 \\
&= \|(1-\beta_n)(y_{k,n} - q) + \beta_n(u-q) + \beta_n v_{k,n}\|^2 \\
&\le (1-\beta_n)^2\|y_{k,n} - q\|^2 + 2\beta_n\langle u + v_{k,n} - q, J(x_{k,n+1} - q)\rangle \\
&\le (1-\beta_n)^2\{(\|x_{k,n} - q\| + \alpha_n\|w_{k,n}\|)^2\} + 2\beta_n\langle u + v_n - q, J(x_{k,n+1} - q)\rangle.
\end{aligned} \tag{3.18}$$

Now we apply Lemma 2.5, and using (3.11) we obtain that $\|x_{k,n} - q\| \to 0$. □

We support our results by giving the following examples.

Example 3.2. Let $T : [0,1] \times [0,1] \to [0,1] \times [0,1]$ be given by $Tx = x$. Then, the modified doubly Mann's iteration process with errors converges to the fixed point $x^* = (0,0)$, and both Picard and Mann iteration processes converge to the same point too.

Proof. (I) Doubly Picards iteration converges.

For every point in $(0,1] \times (0,1]$ is a fixed point of T. Let $b_{0,0}$ be a point in $(0,1] \times (0,1]$, then

$$b_{k+1,k+1} = Tb_{k,k} = T^n b_{0,0} = b_{0,0}. \tag{3.19}$$

Hence,

$$\lim_{k\to\infty} b_{k,k} = b_{0,0}. \tag{3.20}$$

Let $(x,y) - (a,b) = (|x-a|, |y-b|)$, for all $(x,y),(a,b) \in (0,1] \times (0,1]$. Take $p_{0,0} = (0,0)$ and $p_{k,k} = (1/k, 1/k)$. Thus

$$\delta_{k,k} = p_{k+1,k+1} - Tp_{k,k} = \left(\frac{1}{k(k+1)}, \frac{1}{k(k+1)}\right) \longrightarrow (0,0). \tag{3.21}$$

(II) Doubly Mann's iteration converges.

Let $e_{0,0}$ be a point in $(0,1] \times (0,1]$, then

$$e_{k+1,k+1} = (1-\alpha_k)e_{k,k} + \alpha_k e_{k,k} = e_{k,k} = \cdots = e_{0,0}. \tag{3.22}$$

Since doubly Mann's iteration is defined by

$$e_{k+1,k+1} = (1-\alpha_k)e_{k,k} + \alpha_k T e_{k,k}. \tag{3.23}$$

Take $u_{0,0} = e_{0,0},\ u_{k,k} = (1/(k+1), 1/(k+1))$ to obtain

$$\begin{aligned} \varepsilon_{k,k} &= u_{k+1,k+1} - (1-\alpha_k)u_{k,k} + \alpha_k T u_{k,k} \\ &= \left(\frac{1}{(k+1)(k+2)}, \frac{1}{(k+1)(k+2)}\right) \longrightarrow (0,0). \end{aligned} \tag{3.24}$$

(III) Modified doubly Mann's iteration process with errors converges because the sequence $e_{k,k+1} \to (0,0)$ as we can see and by using (3.1), we obtain

$$\begin{aligned} y_{k,k} &= \alpha_k e_{k,k} + (1-\alpha_k)e_{k,k} + \alpha_k w_{k,k} \\ &= e_{k,k} + \alpha_k w_{k,k}. \end{aligned} \tag{3.25}$$

In (3.1), we suppose that $u = e_{k,k}$,

$$\begin{aligned} e_{k,k+1} &= \beta_k u + (1-\beta_k)(e_{k,k} + \alpha_k w_{k,k}) + \beta_k v_{k,k} \\ &= e_{k,k} + (1-\beta_k)\alpha_k w_{k,k} + \beta_k v_{k,k}, \\ e_{k,k+1} - e_{k,k} &= (1-\beta_k)\alpha_k w_{k,k} + \beta_k v_{k,k}. \end{aligned} \tag{3.26}$$

Let $k \to \infty$ and using Theorem 3.1 (T is nonexpansive), we obtain $e_{k,k+1} - e_{k,k} = (0,0)$. □

Example 3.3. Let $T : [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)$ be given by $Tx = x/4$. Then the doubly Mann's iteration converges to the fixed point of $x^* = (0,0)$ but modified doubly Mann's iteration process with errors does not converge.

Proof. (I) Doubly Mann's iteration converges because the sequence $e_{k,k} \to (0,0)$ as we can see,

$$\begin{aligned} e_{k+1,k+1} &= (1-\alpha_k)e_{k,k} + \alpha_k \frac{e_{k,k}}{4} \\ &= \left(1 - \frac{3\alpha_k}{4}\right)e_{k,k} \\ &= \prod_{m=1}^{n}\left(1 - \frac{3\alpha_m}{4}\right)e_{0,0} \\ &\le \exp\left(-\frac{3}{4}\sum_{k=1}^{n}\alpha_k\right) \longrightarrow (0,0). \end{aligned} \tag{3.27}$$

The last inequality is true because $1 - x \le \exp(-x)$, for all $x \ge 0$ and $\sum_{k=1}^{n} \alpha_k = \infty$.

(II) The origin is the unique fixed point of T.

(III) Note that, modified doubly Mann's iteration process with errors does not converge to the fixed point of T, because the sequence $e_{k,k+1} \nrightarrow (0,0)$ as we can see and by using (3.1), we obtain

$$y_{k,k} = \alpha_k e_{k,k} + (1-\alpha_k)\frac{e_{k,k}}{4} + \alpha_k w_{k,k} = \left(\frac{1+3\alpha_k}{4}\right)e_{k,k} + \alpha_k w_{k,k}. \tag{3.28}$$

Putting $u = e_{k,k}$,

$$e_{k,k+1} = \beta_k e_{k,k} + (1-\beta_k)\left(\left(\frac{1+3\alpha_k}{4}\right)e_{k,k} + \alpha_k w_{k,k}\right) + \beta_k v_{k,k}. \tag{3.29}$$

Letting $k \to \infty$, we deduce that $e_{k,k+1} \nrightarrow (0,0)$. □

4. Convergence to a Zero of Accretive Operator

In this section, we prove a convergence theorem for m-accretive operator in Banach spaces. Let X be a real Banach space. Recall that, the (possibly multivalued) operator A with domain $D(A)$ and range $R(A)$ in X is accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,2)$, there exists a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j\rangle \geq 0. \tag{4.1}$$

An accretive operator A is m-accretive if $R(I+rA) = X$ for each $r > 0$. Throughout this section, we always assume that A is m-accretive and has a zero. The set of zeros of A is denoted by F. Hence,

$$F = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0). \tag{4.2}$$

For each $r > 0$, we denote by J_r the resolvent of A, that is, $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : X \to X$ is nonexpansive and $\mathrm{Fix}(J_r) = F$ for all $r > 0$. We need the resolvent identity (see [19, 20] for more information).

Lemma 4.1 ([7] (the resolvent identity)). *For $\lambda > 0$, $\mu > 0$ and $x \in X$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right). \tag{4.3}$$

Theorem 4.2. *Assume that X is a uniformly smooth Banach space, and A is an m-accretive operator in X such that $A^{-1}(0) \neq \varnothing$. Let $\{x_{k,n}\}$ be defined by*

$$\begin{aligned} x_{0,0} &= x \in X, \\ y_{k,n} &= J_{r_n} x_{k,n}, \\ x_{k,n+1} &= \alpha_n u + (1-\alpha_n) y_{k,n} + \alpha_n w_{k,n}. \end{aligned} \tag{4.4}$$

Suppose $\{\alpha_n\}$ *and* $\{r_n\}$ *satisfy the conditions,*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$,

(iii) $r_n \geq \varepsilon$ *for some* $\varepsilon > 0$ *and for all* $n \geq 1$. *Also assume that*

$$\sum_{n=1}^{\infty}\left|1-\frac{r_{n-1}}{r_n}\right| < \infty. \tag{4.5}$$

Then, $\{x_{k,n}\}$ *converges strongly to a zero of* A.

Proof. First of all we show that $\{x_{k,n}\}$ is bounded. Take $p \in F = A^{-1}(0)$. It follows that

$$\begin{aligned}\left\|x_{k,n+1}-p\right\| &= \left\|\alpha_n u + (1-\alpha_n)J_{r_n}x_{k,n} + \alpha_n w_{k,n} - p\right\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\left\|x_{k,n}-p\right\| + \alpha_n\left\|w_{k,n}\right\|.\end{aligned} \tag{4.6}$$

By induction, we get that

$$\left\|x_{k,n}-p\right\| \leq \max\left\{\left\|x_{0,0}-p\right\|, \|u-p\|, \left\|w_{0,0}\right\|\right\} \quad k,n \geq 0. \tag{4.7}$$

This implies that $\{x_{k,n}\}$ is bounded. Then, it follows that

$$\left\|x_{k,n+1}-J_{r_n}x_{k,n}\right\| \longrightarrow 0. \tag{4.8}$$

A simple calculation shows that

$$x_{k,n+1}-x_{k,n} = (\alpha_n-\alpha_{n-1})(u-y_{k,n-1}) + (1-\alpha_n)(y_{k,n}-y_{k,n-1}) + \alpha_n w_{k,n} - \alpha_{n-1}w_{k,n-1}. \tag{4.9}$$

The resolvent identity (4.3) implies that

$$y_{k,n} = J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_{k,n} + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_{k,n}\right), \tag{4.10}$$

which in turn implies that

$$
\begin{aligned}
\left\|y_{k,n}-y_{k,n-1}\right\| &= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_{k,n}+\left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_{k,n}\right)-J_{r_{n-1}}x_{k,n-1}\right\|,\\
&= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_{k,n}-x_{k,n-1}\right)+\left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_{k,n}\right\|\\
&= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_{k,n}-x_{k,n}+x_{k,n}-x_{k,n-1}\right)+\left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_{k,n}\right\|\\
&= \left\|J_{r_{n-1}}\left[\left(\frac{r_{n-1}}{r_n}-1\right)x_{k,n}+\left(x_{k,n}-x_{k,n-1}\right)\right]+\left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_{k,n}\right\|\\
&= \left\|\left(1-\frac{r_{n-1}}{r_n}\right)\left(J_{r_{n-1}}-x_{k,n}\right)+J_{r_{n-1}}\left(x_{k,n}-x_{k,n-1}\right)\right\|\\
&\le \left|1-\frac{r_{n-1}}{r_n}\right|\left\|J_{r_n}x_{k,n}-J_{r_{n-1}}x_{k,n}\right\|+\left\|J_{r_{n-1}}x_{k,n}-J_{r_{n-1}}x_{k,n-1}\right\|\\
&\le \left|1-\frac{r_{n-1}}{r_n}\right|\left\|J_{r_n}x_{k,n}-J_{r_{n-1}}x_{k,n}\right\|+\left\|x_{k,n}-x_{k,n-1}\right\|.
\end{aligned}
\tag{4.11}
$$

Combining (4.9) and (4.11), we obtain

$$
\begin{aligned}
\left\|x_{k,n+1}-x_{k,n}\right\| \le{}& \left(1-\alpha_n\right)\left\|x_{k,n}-x_{k,n-1}\right\|+M\left(\left|\alpha_n-\alpha_{n-1}\right|+\left|1-\frac{r_{n-1}}{r_n}\right|\right)\\
&+\alpha_n\left\|w_{k,n}\right\|+\alpha_{n-1}\left\|w_{k,n-1}\right\|,
\end{aligned}
\tag{4.12}
$$

where M is a constant such that $M \ge \max\{\|u-y_{k,n}\|, \|J_r x_{k,n}-x_{k,n}\|\}$ for all $n \ge 0$ and $r > 0$. By assumptions (i)–(iii) in the theorem, we have that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, and $(|\alpha_n-\alpha_{n-1}|+|1-r_{n-1}/r_n|) < \infty$. Hence, Lemma 2.5 is applicable to (4.12), and we conclude that $\|x_{k,n+1}-x_{k,n}\| \to 0$.

Take a fixed number r such that $\varepsilon > r > 0$. Again from the resolvent identity (4.3), we find

$$
\begin{aligned}
\left\|J_{r_n}x_{k,n}-J_r x_{k,n}\right\| &= \left\|J_r\left(\frac{r}{r_n}x_{k,n}+\left(1-\frac{r}{r_n}\right)J_{r_n}x_{k,n}\right)-J_r x_{k,n}\right\|\\
&\le \left(1-\frac{r}{r_n}\right)\left\|x_{k,n}-J_{r_n}x_{k,n}\right\|\\
&\le \left\|x_{k,n}-x_{k,n+1}\right\|+\left\|x_{k,n+1}-J_{r_n}x_{k,n}\right\| \longrightarrow 0.
\end{aligned}
\tag{4.13}
$$

It follows that

$$
\begin{aligned}
\left\|x_{k,n+1}-J_r x_{k,n+1}\right\| &\le \left\|x_{k,n+1}-J_{r_n}x_{k,n}\right\|+\left\|J_{r_n}x_{k,n}-J_r x_{k,n}\right\|+\left\|J_r x_{k,n}-J_r x_{k,n+1}\right\|\\
&\le \left\|x_{k,n+1}-J_{r_n}x_{k,n}\right\|+\left\|J_{r_n}x_{k,n}-J_r x_{k,n}\right\|+\left\|x_{k,n}-x_{k,n+1}\right\|.
\end{aligned}
\tag{4.14}
$$

Hence,

$$\|x_{k,n} - J_r x_{k,n}\| \longrightarrow 0. \tag{4.15}$$

Since in a uniformly smooth Banach space the sunny nonexpansive retract Q from X onto the fixed point set $\mathrm{Fix}(J_r)(= F = A^{-1}(0))$ of J_r is unique, it must be obtained from Reich's theorem (Lemma 2.4). Namely, $Q(u) = s - \lim_{t\to 0} z_t,\ u \in X$, where $t \in (0,1)$ and $z_t \in X$ solve the fixed point equation

$$z_t - x_{k,n} = t(u - x_{k,n}) + (1-t)(J_r x_t - x_{k,n}). \tag{4.16}$$

Applying Lemma 2.3, we get

$$\begin{aligned}\|z_t - x_{k,n+1}\|^2 &= (1-t)^2\|J_r z_t - x_{k,n}\|^2 + 2t\langle u - x_{k,n}, J(z_t - x_{k,n})\rangle \\ &\le (1-t)^2\|z_t - x_{k,n}\|^2 + a_n(t) + 2t\langle u - z_t, J(z_t - x_{k,n})\rangle + 2t\|z_t - x_{k,n}\|,\end{aligned} \tag{4.17}$$

where $a_n(t) = 2\|z_t - x_{k,n}\| \cdot \|J_r x_{k,n} - x_{k,n}\| + \|J_r x_{k,n} - x_{k,n}\|^2 \to 0$ by (4.15). It follows that

$$\langle z_t - u, J(z_t - x_{k,n})\rangle \le \frac{t}{2}\|z_t - x_{k,n}\|^2 + \frac{1}{2t}a_n(t). \tag{4.18}$$

Therefore, letting $k, n \to \infty$ in (4.18), we get

$$\lim_{k,n\to\infty} \sup \langle z_t - u, J(z_t - x_{k,n})\rangle \le \frac{t}{2}M, \tag{4.19}$$

where M is a constant such that $M \ge \|z_t - x_{k,n}\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Since $z_t \to Q(u)$ strongly and the duality map J is norm-to-norm uniformly continuous on bounded sets of X, it follows that (by letting $t \to 0$ in (4.19))

$$\lim_{k,n\to\infty} \sup \langle u - Q(u), J(x_{k,n} - Q(u))\rangle \le 0, \tag{4.20}$$

$$\begin{aligned}\|x_{k,n+1} - Q(u)\|^2 &= \|\alpha_n(u - Q(u)) + (1-\alpha_n)(J_{r_n}x_{k,n} - Q(u))\|^2 \\ &\le (1-\alpha_n)^2\|J_{r_n}x_{k,n} - Q(u)\|^2 + 2\alpha_n\langle u - Q(u), J(x_{k,n+1} - Q(u))\rangle \\ &\le (1-\alpha_n)\|x_{k,n} - Q(u)\|^2 + 2\alpha_n\langle u - Q(u), J(x_{k,n+1} - Q(u))\rangle.\end{aligned} \tag{4.21}$$

Now we apply Lemma 2.5 and using (4.20), we obtain that $\|x_{k,n} - Q(u)\| \to 0$. □

5. Weakly Convergence Theorems

We next introduce the following iterative scheme. Given an initial $x_{0,0} \in C$, we define $(x_{k,n})$ by

$$x_{k,n+1} = \alpha_n x_{k,n} + (1-\alpha_n) J_{r_n} x_{k,n} + \alpha_n u_{k,n}, \quad k, n \geq 0. \tag{5.1}$$

Theorem 5.1. *Let X be a uniformly convex Banach space with a Frechét differentiable norm. Assume that X has a weakly continuous duality map J_φ with gauge φ. Assume also that*

(i) $\alpha_n \longrightarrow 0$,

(ii) $r_n \longrightarrow \infty$.

Then, the scheme (5.1) *converges weakly to a point q in F.*

Proof. First, we observe that for any $p \in F$, the sequence $\{\|x_{k,n} - p\|\}$ is nonincreasing.

Indeed, we have by nonexpansivity of J_{r_n},

$$\begin{aligned} \|x_{k,n} - p\| &= \|\alpha_n x_{k,n} + (1-\alpha_n) J_{r_n} x_{k,n} + \alpha_n u_{k,n} - p\| \\ &\leq \alpha_n \|x_{k,n} - p\| + (1-\alpha_n)\|J_{r_n} x_{k,n} - p\| + \alpha_n \|u_{k,n}\| \\ &= \|x_{k,n} - p\| + \alpha_n \|u_{k,n}\|. \end{aligned} \tag{5.2}$$

In particular, $\{x_{k,n}\}$ is bounded, so is $\{J_{r_n} x_{k,n}\}$. Let $W_w(x_{k,n})$ be the set of weak limit point of the sequence $\{x_{k,n}\}$.

Note that we can rewrite the scheme (5.1) in the form

$$x_{k,n+1} = T_n x_{k,n}, \quad k, n \geq 0, \tag{5.3}$$

where T_n is the nonexpansive mapping given by

$$T_n x = \alpha_n x + (1-\alpha_n) J_{r_n} x + \alpha_n u, \quad x \in C. \tag{5.4}$$

Then, we have $F(T_n) = F(J_{r_n}) = F$ for $n \geq 1$. Hence, by Lemma 2.7, we get

$$\langle q_1 - q_2, J(f_1 - f_2) \rangle = 0, \quad q_1, q_2 \in W_w(x_{k,n}), \ f_1, f_2 \in F. \tag{5.5}$$

Therefore, $\{x_{k,n}\}$ will converge weakly to a point in F if we can show that $W_w(x_{k,n}) \subset F$. To show this, we take a point v in $W_w(x_{k,n})$. Then we have a subsequence $\{x_{k,n_i}\}$ of $\{x_{k,n}\}$ such that $x_{k,n_i} \rightharpoonup v$. Noting that

$$\begin{aligned} \|x_{k,n+1} - J_{r_n} x_{k,n}\| &= \|\alpha_n x_{k,n} - \alpha_n J_{r_n} x_{k,n} + \alpha_n u_{k,n}\| \\ &\leq \alpha_n \|x_{k,n} - J_{r_n} x_{k,n}\| + \alpha_n \|u_{k,n}\| \longrightarrow 0, \end{aligned} \tag{5.6}$$

we obtain

$$A_{r_{n_i}-1}x_{k,n_i-1} \subset AJ_{r_{n_i}-1}x_{k,n_i-1},$$
$$A_{r_{n_i}-1}x_{k,n_i-1} \longrightarrow 0, \qquad J_{r_{n_i}-1}x_{k,n_i-1} \rightharpoonup v. \tag{5.7}$$

By Lemma 2.6, we conclude that $0 \in Av$, that is, $v \in F$. □

Theorem 5.2. *Let X be a uniformly convex Banach space which either has a Frechét differentiable norm or satisfies Opial's property. Assume for some $\epsilon > 0$,*

(i) *$\epsilon \le \alpha_n \le 1-\epsilon$ for $n \ge 1$,*

(ii) *$r_n \ge \epsilon$ for $n \ge 1$.*

Then, the scheme (5.1) *converges weakly to a point q in F.*

Proof. We have shown that $\lim_{k,n\to\infty}\|x_{k,n}-p\|$ exists for all $p \in F$. Applying Lemma 2.7(i), we have a strictly increasing continuous function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, such that

$$\begin{aligned}\|x_{k,n+1}-p\|^2 &= \|\alpha_n x_{k,n} + (1-\alpha_n)J_{r_n}x_{k,n} + \alpha_n u_{k,n} - p\|^2 \\ &= \|\alpha_n((x_{k,n}-p)+u_n) + (1-\alpha_n)(J_{r_n}x_{k,n}-p)\|^2 \\ &= \alpha_n\|x_{k,n}-p\|^2 + \alpha_n\|u_{k,n}\|^2 + (1-\alpha_n)\|J_{r_n}x_{k,n}-p\|^2 \\ &\quad - \alpha_n(1-\alpha_n)g(\|x_{k,n}-J_{r_n}x_{k,n}\|).\end{aligned} \tag{5.8}$$

This implies that

$$\alpha_n(1-\alpha_n)g(\|x_{k,n}-J_{r_n}x_{k,n}\|) \le \|x_{k,n}-p\|^2 - \|x_{k,n}-p\|^2. \tag{5.9}$$

Since $\alpha_n(1-\alpha_n) \ge \epsilon^2$, we obtain by (5.9) that

$$\sum_{k,n} g(\|x_{k,n}-J_{r_n}x_{k,n}\|) < \infty \Longrightarrow \lim_{k,n\to\infty}\|x_{k,n}-J_{r_n}x_{k,n}\| = 0. \tag{5.10}$$

For any fixed $\lambda \in (0,1)$, by Lemma 4.1, we have

$$J_{r_n}x_{k,n} = J_\lambda\left(\frac{\lambda}{r_n}x_{k,n} + \left(1-\frac{\lambda}{r_n}\right)J_{r_n}x_{k,n}\right). \tag{5.11}$$

We deduce that

$$\begin{aligned}\|J_{r_n}x_{k,n} - J_\lambda x_{k,n}\| &\le \left\|\left(\frac{\lambda}{r_n}x_{k,n} + \left(1-\frac{\lambda}{r_n}\right)J_{r_n}x_{k,n}\right) - x_{k,n}\right\| \\ &= \left(1-\frac{\lambda}{r_n}\right)\|x_{k,n}-J_{r_n}x_{k,n}\| \\ &\le \|x_{k,n}-J_{r_n}x_{k,n}\| \longrightarrow 0 \quad (n\longrightarrow\infty).\end{aligned} \tag{5.12}$$

Therefore we obtain by (5.9) that

$$\|x_{k,n} - J_\lambda x_{k,n}\| \longrightarrow 0 \quad (n \longrightarrow \infty), \ \lambda \in (0,1). \tag{5.13}$$

Apply Lemma 2.9 to find out that $W_w(x_{k,n}) \subset F(J_\lambda) = F$. It remains to show that $W_w(x_{k,n})$ is a singleton set. Towards this end, we take $p, q \in W_w(x_{k,n})$ and distinguish the two cases.

In case X has a Frechét differentiable norm, we apply Lemma 2.8 to get

$$\langle p - q, J(p-q)\rangle = 0, \tag{5.14}$$

hence, $p = q$. In case X satisfies Opial's condition, we can find two subsequences $\{x_{k,n_i}\}$, $\{x_{k,m_j}\}$ such that $x_{k,n_i} \rightharpoonup p$, $x_{k,m_j} \rightharpoonup q$. If $p \neq q$, Opial's property creates the contradiction,

$$\begin{aligned} \lim_{k,n\to\infty}\|x_{k,n} - p\| &= \lim_{k,n\to\infty}\|x_{k,n_i} - p\| < \lim_{k,n\to\infty}\|x_{k,n_i} - q\| \\ &= \lim_{k,n\to\infty}\|x_{k,m_j} - q\| < \lim_{k,n\to\infty}\|x_{k,n_j} - p\| \\ &= \lim_{k,n\to\infty}\|x_{k,n} - p\|. \end{aligned} \tag{5.15}$$

In either case, we have shown that $W_w(x_{k,n})$ consists of exact one point, which is clearly the weak limit of $\{x_{k,n}\}$. □

Remark 5.3. The schemes (3.1), (4.4), and (5.1) generalize and extend several iteration processes from literature (see [7, 8, 17, 21–25] and others).

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