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Research Article

Common Fixed Point Theorems in Menger Probabilistic Quasimetric Spaces

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We consider complete Menger probabilistic quasimetric space and prove common fixed point theorems for weakly compatible maps in this space.

1. Introduction and Preliminaries

K. Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [1]. The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physiological thresholds and physical quantities particularly in connections with both string and E-infinity theory; see [2–5]. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [6–10].

In the sequel, we will adopt usual terminology, notation, and conventions of the theory of Menger probabilistic metric spaces, as in [7, 8, 10]. Throughout this paper, the space of all probability distribution functions (in short, dfs) is denoted by $\Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0,1] : F$ is left-continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$, and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the df given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{1.1}$$

Definition 1.1 (see [1]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is t-norm if T is satisfying the following conditions:

(a) T is commutative and associative;

(b) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \leq T(c,d)$, whenever $a \leq c$ and $b \leq d$, and $a,b,c,d \in [0,1]$.

The following are the four basic t-norms:

$$\begin{aligned} T_M(x,y) &= \min(x,y), \\ T_P(x,y) &= x \cdot y, \\ T_L(x,y) &= \max(x+y-1,0), \\ T_D(x,y) &= \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{1.2}$$

Each t-norm T can be extended [11] (by associativity) in a unique way to an n-ary operation taking for $(x_1,\ldots,x_n) \in [0,1]^n$ the values $T^1(x_1,x_2) = T(x_1,x_2)$ and

$$T^n(x_1,\ldots,x_{n+1}) = T\left(T^{n-1}(x_1,\ldots,x_n), x_{n+1}\right) \tag{1.3}$$

for $n \geq 2$ and $x_i \in [0,1]$, for all $i \in \{1,2,\ldots,n+1\}$.

We also mention the following families of t-norms.

Definition 1.2. It is said that the t-norm T is of Hadžić-type (H-type for short) and $T \in \mathcal{H}$ if the family $\{T^n\}_{n\in\mathbb{N}}$ of its iterates defined, for each x in $[0,1]$, by

$$T^0(x) = 1, \qquad T^{n+1}(x) = T(T^n(x),x), \quad \forall n \geq 0, \tag{1.4}$$

is equicontinuous at $x = 1$, that is,

$$\forall \epsilon \in (0,1) \exists \delta \in (0,1) \text{ such that } x > 1-\delta \Longrightarrow T^n(x) > 1-\epsilon, \quad \forall n \geq 1. \tag{1.5}$$

There is a nice characterization of continuous t-norm T of the class $\mathcal{H}$ [12].

(i) If there exists a strictly increasing sequence $(b_n)_{n\in\mathbf{N}}$ in $[0,1]$ such that $\lim_{n\to\infty} b_n = 1$ and $T(b_n,b_n) = b_n$ $\forall n \in \mathbb{N}$, then T is of Hadžić-type.

(ii) If T is continuous and $T \in \mathcal{H}$, then there exists a sequence $(b_n)_{n\in\mathbb{N}}$ as in (i).The t-norm T_M is an trivial example of a t-norm of H-type, but there are t-norms T of Hadžić-type with $T \neq T_M$ (see, e.g., [13]).

Definition 1.3 (see [13]). If T is a t-norm and $(x_1, x_2, \ldots, x_n) \in [0,1]^n (n \in \mathbb{N})$, then $T_{i=1}^n x_i$ is defined recurrently by 1, if $n = 0$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 1$. If $(x_i)_{i\in\mathbb{N}}$ is a sequence of numbers from $[0,1]$, then $T_{i=1}^\infty x_i$ is defined as $\lim_{n\to\infty} T_{i=1}^n x_i$ (this limit always exists) and $T_{i=n}^\infty x_i$ as $T_{i=1}^\infty x_{n+i}$. In fixed point theory in probablistic metric spaces there are of particular interest the t-norms T and sequences $(x_n) \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$. Some examples of t-norms with the above property are given in the following proposition.

Proposition 1.4 (see [13]). (i) *For $T \geq T_L$ the following implication holds:*

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty. \tag{1.6}$$

(ii) *If $T \in \mathcal{H}$, then for every sequence $(x_n)_{n\in\mathbb{N}}$ in I such that* $\lim_{n\to\infty} x_n = 1$, *one has* $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$.

Note [14, Remark 13] that if T is a t-norm for which there exists $(x_n) \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$, then $\sup_{t<1} T(t,t) = 1$. Important class of t-norms is given in the following example.

Example 1.5. (i) The Dombi family of t-norms $(T_\lambda^D)_{\lambda\in[0,\infty]}$ is defined by

$$T_\lambda^D(x,y) = \begin{cases} T_D(x,y), & \lambda = 0, \\ T_M(x,y), & \lambda = \infty, \\ \dfrac{1}{1 + \left(((1-x)/x)^\lambda + ((1-y)/y)^\lambda\right)^{1/\lambda}}, & \lambda \in (0, \infty). \end{cases} \tag{1.7}$$

(ii) The Aczél-Alsina family of t-norms $(T_\lambda^{AA})_{\lambda\in[0,\infty]}$ is defined by

$$T_\lambda^{AA}(x,y) = \begin{cases} T_D(x,y), & \lambda = 0, \\ T_M(x,y), & \lambda = \infty, \\ e^{-\left((-\log\, x)^\lambda + (-\log\, y)^\lambda\right)^{1/\lambda}}, & \lambda \in (0, \infty). \end{cases} \tag{1.8}$$

(iii) Sugeno-Weber family of t-norms $(T_\lambda^{SW})_{\lambda\in[-1,\infty]}$ is defined by

$$T_\lambda^{SW}(x,y) = \begin{cases} T_D(x,y), & \lambda = -1, \\ T_P(x,y), & \lambda = \infty, \\ \max\left(0, \dfrac{x + y - 1 + \lambda xy}{1 + \lambda}\right), & \lambda \in (-1, \infty). \end{cases} \tag{1.9}$$

In [13] the following results are obtained.

(a) If $(T_\lambda^D)_{\lambda\in(0,\,\infty)}$ is the Dombi family of t-norms and $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements from $(0,\ 1]$ such that $\lim_{n\to\infty} x_n = 1$ then we have the following equivalence:

$$\sum_{i=1}^{\infty}(1-x_i)^\lambda < \infty \Longleftrightarrow \lim_{n\to\infty}(T_\lambda^D)_{i=n}^{\infty} x_i = 1. \tag{1.10}$$

(b) Equivalence (1.10) holds also for the family $(T_\lambda^{AA})_{\lambda\in(0,\,\infty)}$, that is,

$$\sum_{i=1}^{\infty}(1-x_i)^\lambda < \infty \Longleftrightarrow \lim_{n\to\infty}(T_\lambda^{AA})_{i=n}^{\infty} x_i = 1. \tag{1.11}$$

(c) If $(T_\lambda^{SW})_{\lambda\in(-1,\,\infty]}$ is the Sugeno-Weber family of t-norms and $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements from $(0,\ 1]$ such that $\lim_{n\to\infty} x_n = 1$ then we have the following equivalence:

$$\sum_{i=1}^{\infty}(1-x_i) < \infty \Longleftrightarrow \lim_{n\to\infty}(T_\lambda^{SW})_{i=n}^{\infty} x_i = 1. \tag{1.12}$$

Proposition 1.6. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and t-norm T is of H-type. Then*

$$\lim_{n\to\infty}\mathbf{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty}\mathbf{T}_{i=n}^{\infty} x_{n+i} = 1. \tag{1.13}$$

Definition 1.7. A *Menger Probabilistic Quasimetric space* (briefly, Menger PQM space) is a triple $(X,\mathcal{F},T)$, where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X\times X$ into D^+, such that, if $F_{p,q}$ denotes the value of $\mathcal{F}$ at the pair (p,q), then the following conditions hold, for all p,q,r in X,

(PQM1) $F_{p,q}(t) = F_{q,p}(t) = \varepsilon_0(t)$ for all $t>0$ if and only if $p=q$;

(PQM2) $F_{p,q}(t+s) \geq T(F_{p,r}(t), F_{r,q}(s))$ for all $p,q,r\in X$ and $t,s\geq 0$.

Definition 1.8. Let $(X,\mathcal{F},T)$ be a Menger PQM space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X if, for every $\epsilon>0$ and $\lambda>0$, there exists positive integer N such that $F_{x_n,x}(\epsilon) > 1-\lambda$ whenever $n\geq N$.

(2) A sequence $\{x_n\}_n$ in X is called Cauchy sequence [15] if, for every $\epsilon>0$ and $\lambda>0$, there exists positive integer N such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$ whenever $n\geq m\geq N$ $(m\geq n\geq N)$.

(3) A Menger PQM space $(X,\mathcal{F},T)$ is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

In 1998, Jungck and Rhoades [16] introduced the following concept of weak compatibility.

Definition 1.9. Let A and S be mappings from a Menger PQM space $(X, \mathcal{F}, T)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

2. The Main Result

Throughout this section, a binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm and satisfies the condition

$$\lim_{n\to\infty} T_{i=n}^{\infty}\left(1 - a^i(t)\right) = 1, \tag{2.1}$$

where $a : \mathbb{R}^+ \to (0,1)$. It is easy to see that this condition implies $\lim_{n\to\infty} a^n(t) = 0$.

Lemma 2.1. *Let $(X, \mathcal{F}, T)$ be a Menger PQM space. If the sequence $\{x_n\}$ in X is such that for every $n \in \mathbb{N}$,*

$$F_{x_n,x_{n+1}}(t) \geq 1 - a^n(t)(1 - F_{x_0,x_1}(t)) \tag{2.2}$$

for very $t > 0$, where $a : \mathbb{R}^+ \to (0,1)$ is a monotone increasing functions.Then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $m > n$ and $x_n, x_m \in X$, we have

$$\begin{aligned}
F_{x_n,x_m}(t) &\geq T\left(T^{m-2}\left(F_{x_n,x_{n+1}}\left(\frac{t}{m-n}\right), \dots, F_{x_{m-2},x_{m-1}}\left(\frac{t}{m-n}\right)\right), F_{x_{m-1},x_m}\left(\frac{t}{m-n}\right)\right) \\
&\geq T^{m-1}\left(1 - a^n\left(\frac{t}{m-n}\right)\left(1 - F_{x_0,x_1}\left(\frac{t}{m-n}\right)\right), 1 - a^{n+1}\left(\frac{t}{m-n}\right)\right. \\
&\quad \times \left(1 - F_{x_0,x_1}\left(\frac{t}{m-n}\right)\right), \dots, 1 - a^{m-1}\left(\frac{t}{m-n}\right)\left.\left(1 - F_{x_0,x_1}\left(\frac{t}{m-n}\right)\right)\right) \\
&\geq T^{m-1}\left(1 - a^n\left(\frac{t}{m-n}\right), 1 - a^{n+1}\left(\frac{t}{m-n}\right), \dots, 1 - a^{m-1}\left(\frac{t}{m-n}\right)\right) \\
&\geq T^{m-1}\left(1 - a^n(t), 1 - a^{n+1}(t), \dots, 1 - a^{m-1}(t)\right) \\
&= T_{i=n}^{m-1}\left(1 - a^i(t)\right) \\
&\geq T_{i=n}^{\infty}\left(1 - a^i(t)\right) \\
&> 1 - \lambda
\end{aligned} \tag{2.3}$$

for each $0 < \lambda < 1$ and $t > 0$. Hence sequence $\{x_n\}$ is Cauchy sequence. □

Theorem 2.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger PQM space and let $f, g, h : X \to X$ be maps that satisfy the following conditions:*

(a) $g(X) \cup h(X) \subseteq f(X)$;

(b) *the pairs (f, g) and (f, h) are weak compatible, $f(X)$ is closed subset of X;*

(c) $\min\{F_{g(x),h(y)}(t), F_{h(x),g(y)}(t)\} \geq 1 - a(t)(1 - F_{f(x),f(y)}(t))$ *for all $x, y \in X$ and every $t > 0$, where $a : \mathbb{R}^+ \to (0,1)$ is a monotone increasing function.*

If

$$\lim_{n \to \infty} T_{i=n}^{\infty}\left(1 - a^i(t)\right) = 1, \tag{2.4}$$

then f, g, and h have a unique common fixed point.

Proof. Let $x_0 \in X$. By (a), we can find x_1 such that $f(x_1) = g(x_0)$ and $h(x_1) = f(x_2)$. By induction, we can define a sequence $\{x_n\}$ such that $f(x_{2n+1}) = g(x_{2n})$ and $h(x_{2n+1}) = f(x_{2n+2})$. By induction again,

$$\begin{aligned} F_{f(x_{2n}),f(x_{2n+1})}(t) &= F_{h(x_{2n-1}),g(x_{2n})}(t) \\ &\geq \min\left\{F_{h(x_{2n-1}),g(x_{2n})}(t), F_{g(x_{2n-1}),h(x_{2n})}(t)\right\} \\ &\geq 1 - a(t)\left(1 - F_{f(x_{2n-1}),f(x_{2n})}(t)\right). \end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned} F_{f(x_{2n-1}),f(x_{2n})}(t) &= F_{g(x_{2n-2}),h(x_{2n-1})}(t) \\ &\geq \min\left\{F_{h(x_{2n-2}),g(x_{2n-1})}(t), F_{g(x_{2n-2}),h(x_{2n-1})}(t)\right\} \\ &\geq 1 - a(t)\left(1 - F_{f(x_{2n-2}),f(x_{2n-1})}(t)\right). \end{aligned} \tag{2.6}$$

Hence, it follows that

$$\begin{aligned} F_{f(x_n),f(x_{n+1})}(t) &\geq 1 - a(t)\left(1 - F_{f(x_{n-1}),f(x_n)}(t)\right) \\ &\geq 1 - a(t)\left(1 - \left(1 - a(t)\left(1 - F_{f(x_{n-2}),f(x_{n-1})}(t)\right)\right)\right) \\ &= 1 - a^2(t)\left(1 - F_{f(x_{n-2}),f(x_{n-1})}(t)\right) \\ &\vdots \\ &\geq 1 - a^n(t)\left(1 - F_{f(x_0),f(x_1)}(t)\right). \end{aligned} \tag{2.7}$$

for $n = 1, 2, \ldots$.

Now by Lemma 2.1, $\{f(x_n)\}$ is a Cauchy sequence. Since the space $f(X)$ is complete, there exists a point $y \in X$ such that

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_{2n}) = \lim_{n \to \infty} h(x_{2n+1}) = y \in f(X). \tag{2.8}$$

It follows that, there exists $v \in X$ such that $f(v) = y$. We prove that $g(v) = h(v) = y$. From (c), we get

$$
\begin{aligned}
F_{g(x_{2n}),h(v)}(t) &\geq \min\{F_{g(x_{2n}),h(v)}(t), F_{h(x_{2n}),g(v)}(t)\} \\
&\geq 1 - a(t)\left(1 - F_{f(x_{2n}),f(v)}(t)\right)
\end{aligned}
\tag{2.9}
$$

as $n \longrightarrow \infty$, we have

$$
F_{y,h(v)}(t) \geq 1 - a(t)\left(1 - F_{y,y}(t)\right) = 1 \tag{2.10}
$$

which implies that, $h(v) = y$. Moreover,

$$
\begin{aligned}
F_{g(v),h(x_{2n+1})}(t) &\geq \min\{F_{g(v),h(x_{2n+1})}(t), F_{h(v),g(x_{2n+1})}(t)\} \\
&\geq 1 - a(t)\left(1 - F_{f(v),f(x_{2n+1})}(t)\right)
\end{aligned}
\tag{2.11}
$$

as $n \longrightarrow \infty$, we have

$$
F_{g(v),y}(t) \geq 1 - a(t)\left(1 - F_{y,y}(t)\right) = 1 \tag{2.12}
$$

which implies that $g(v) = y$. Since, the pairs (f, g) and (f, h) are weak compatible, we have $f(g(v)) = g(f(v))$, hence it follows that $f(y) = g(y)$. Similarly, we get $f(y) = h(y)$. Now, we prove that $g(y) = y$. Since, from (c) we have

$$
\begin{aligned}
F_{g(y),h(x_{2n+1})}(t) &\geq \min\{F_{g(y),h(x_{2n+1})}(t), F_{h(y),g(x_{2n+1})}(t)\} \\
&\geq 1 - a(t)\left(1 - F_{f(y),f(x_{2n+1})}(t)\right)
\end{aligned}
\tag{2.13}
$$

as $n \longrightarrow \infty$, we have

$$
\begin{aligned}
F_{g(y),y}(t) &\geq 1 - a(t)\left(1 - F_{f(y),y}(t)\right) \\
&= 1 - a(t)\left(1 - F_{g(y),y}(t)\right) \\
&\geq 1 - a(t)\left(1 - \left(1 - a(t)\left(1 - F_{g(y),y}(t)\right)\right)\right) \\
&= 1 - a^2(t)\left(1 - F_{g(y),y}(t)\right) \\
&\vdots \\
&\geq 1 - a^n(t)\left(1 - F_{g(y),y}(t)\right) \longrightarrow 1.
\end{aligned}
\tag{2.14}
$$

It follows that $g(y) = y$. Therefore, $h(y) = f(y) = g(y) = y$. That is y is a common fixed point of f, g, and h.

If y and z are two fixed points common to f, g, and h, then

$$\begin{aligned} F_{y,z}(t) &= F_{g(y),h(z)}(t) \\ &\geq \min\{F_{g(y),h(z)}(t), F_{h(y),g(z)}(t)\} \\ &\geq 1 - a(t)(1 - F_{f(y),f(z)}(t)) \\ &= 1 - a(t)(1 - F_{y,z}(t)) \\ &\geq 1 - a(t)(1 - (1 - a(t)(1 - F_{y,z}(t)))) \\ &\vdots \\ &\geq 1 - a^n(t)(1 - F_{y,z}(t)) \longrightarrow 1 \end{aligned} \tag{2.15}$$

as $n \to \infty$, which implies that $y = z$ and so the uniqueness of the common fixed point. □

Corollary 2.3. *Let $(X, \mathcal{F}, T)$ be a complete Menger PQM space and let $f, g : X \to X$ be maps that satisfy the following conditions:*

(a) $g(X) \subseteq f(X)$;

(b) *the pair (f, g) is weak compatible, $f(X)$ is closed subset of X;*

(c) $F_{g(x),g(y)}(t) \geq 1 - a(t)(1 - F_{f(x),f(y)}(t))$ *for all $x, y \in X$ and $t > 0$, where $a : \mathbb{R}^+ \to (0,1)$ is monotone increasing function.*

If

$$\lim_{n \to \infty} T_{i=n}^{\infty}\left(1 - a^i(t)\right) = 1, \tag{2.16}$$

then f and g have a unique common fixed point.

Proof. It is enough, set $h = g$ in Theorem 2.2. □

Corollary 2.4. *Let $(X, \mathcal{F}, T)$ be a complete Menger PQM space and let $f_1, f_2, \ldots, f_n, g : X \to X$ be maps that satisfy the following conditions:*

(a) $g(X) \subseteq f_1 f_2 \cdots f_n(X)$;

(b) *the pair $(f_1 f_2 \cdots f_n, g)$ is weak compatible, $f_1 f_2 \cdots f_n(X)$ is closed subset of X;*

(c) $F_{g(x),g(y)}(t) \geq 1 - a(t)(1 - F_{f_1 f_2 \cdots f_n(x), f_1 f_2 \cdots f_n(y)}(t))$ *for all $x, y \in X$ and $t > 0$, where $a : \mathbb{R}^+ \to (0,1)$ is monotone increasing function;*

(d)

$$\begin{gathered}
g(f_2\cdots f_n)=(f_2\cdots f_n)g,\\
g(f_3\cdots f_n)=(f_3\cdots f_n)g,\\
\vdots\\
gf_n=f_ng,\\
f_1(f_2\cdots f_n)=(f_2\cdots f_n)f_1,\\
f_1f_2(f_3\cdots f_n)=(f_3\cdots f_n)f_1f_2,\\
\vdots\\
f_1\cdots f_{n-1}(f_n)=(f_n)f_1\cdots f_{n-1}.
\end{gathered}\tag{2.17}$$

If

$$\lim_{n\to\infty}T_{i=n}^{\infty}\left(1-a^i(t)\right)=1,\tag{2.18}$$

then $f_1,f_2,\ldots,f_n,g$ *have a unique common fixed point.*

Proof. By Corollary 2.3, if set $f_1f_2\cdots f_n=f$ then f,g have a unique common fixed point in X. That is, there exists $x\in X$, such that $f_1f_2\cdots f_n(x)=g(x)=x$. We prove that $f_i(x)=x$, for $i=1,2,\ldots$. From (c), we have

$$F_{g(f_2\cdots f_nx),g(x)}(t)\geq 1-a(t)\left(1-F_{f_1f_2\cdots f_n(f_2\cdots f_nx),f_1f_2\cdots f_n(x)}(t)\right).\tag{2.19}$$

By (d), we get

$$F_{f_2\cdots f_n(x),x}(t)\geq 1-a(t)\left(1-F_{f_2\cdots f_n(x),x}(t)\right)\tag{2.20}$$

Hence, $f_2\cdots f_n(x)=x$. Thus , $f_1(x)=f_1f_2\cdots f_n(x)=x$.

Similarly, we have $f_2(x)=\cdots f_n(x)=x$. □

Corollary 2.5. *Let* $(X,\ \mathcal{F},\ T)$ *be a complete PQM space and let* $f,g,h:X\to X$ *satisfy conditions (a), (b), and (c) of Theorem 2.2. If* T *is a t-norm of* H*-type then there exists a unique common fixed point for the mapping* $f,g,$ *and* h.

Proof. By Proposition 1.6 all the conditions of the Theorem 2.2 are satisfied. □

Corollary 2.6. *Let* $(X,\ \mathcal{F},\ T_\lambda^D)$ *for some* $\lambda>0$ *be a complete PQM space and let* $f,g,h:X\to X$ *satisfy conditions (a), (b), and (c) of Theorem 2.2. If* $\sum_{i=1}^{\infty}(a^i(t))^\lambda<\infty$ *then there exists a unique common fixed point for the mapping* $f,g,$ *and* h.

Proof. From equivalence (1.10) we have

$$\sum_{i=1}^{\infty}\left(a^{i}(t)\right)^{\lambda}<\infty \Longleftrightarrow \lim_{n\to\infty}(T_{\lambda}^{D})_{i=n}^{\infty}\left(1-a^{i}(t)\right)=1. \tag{2.21}$$

□

Corollary 2.7. *Let* $(X, \mathcal{F}, T_{\lambda}^{AA})$ *for some* $\lambda>0$ *be a complete PQM space and let* $f, g, h : X \to X$ *satisfy conditions (a), (b), and (c) of Theorem 2.2. If* $\sum_{i=1}^{\infty}(a^{i}(t))^{\lambda}<\infty$ *then there exists a unique common fixed point for the mapping* f, g*, and* h*.*

Proof. From equivalence (1.11) we have

$$\sum_{i=1}^{\infty}\left(a^{i}(t)\right)^{\lambda}<\infty \Longleftrightarrow \lim_{n\to\infty}(T_{\lambda}^{AA})_{i=n}^{\infty}\left(1-a^{i}(t)\right)=1. \tag{2.22}$$

□

Corollary 2.8. *Let* $(X, \mathcal{F}, T_{\lambda}^{SW})$ *for some* $\lambda>-1$ *be a complete PQM space and let* $f, g, h : X \to X$ *satisfy conditions (a), (b), and (c) of Theorem 2.2. If* $\sum_{i=1}^{\infty}(a^{i}(t))<\infty$ *then there exists a unique common fixed point for the mapping* f, g*, and* h*.*

Proof. From equivalence (1.12) we have

$$\sum_{i=1}^{\infty}\left(a^{i}(t)\right)<\infty \Longleftrightarrow \lim_{n\to\infty}(T_{\lambda}^{SW})_{i=n}^{\infty}\left(1-a^{i}(t)\right)=1. \tag{2.23}$$

□

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