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Research Article

An Iterative Method for Generalized Equilibrium Problems, Fixed Point Problems and Variational Inequality Problems

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We introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of generalized equilibrium problems, the set of common fixed points of infinitely many nonexpansive mappings, and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in Hilbert spaces. We give some strong-convergence theorems under mild assumptions on parameters. The results presented in this paper improve and generalize the main result of Yao et al. (2007).

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. Let $\Psi : C \to H$ be a nonlinear mapping. The generalized equilibrium problem (GEP) for $\Phi : C \times C \to \mathbb{R}$ and $\Psi : C \to H$ is to find $u \in C$ such that

$$\Phi(u,v) + \langle \Psi u, v-u \rangle \geq 0 \quad \forall v \in C. \tag{1.1}$$

The set of solutions for the problem (1.1) is denoted by Ω, that is,

$$\Omega = \{u \in C : \Phi(u,v) + \langle \Psi u, v-u \rangle \geq 0,\ \forall v \in C\}. \tag{1.2}$$

If $\Psi = 0$ in (1.1), then GEP(1.1) reduces to the classical equilibrium problem (EP) and Ω is denoted by EP(Φ), that is,

$$\mathrm{EP}(\Phi) = \{u \in C : \Phi(u, v) \geq 0, \forall v \in C\}. \tag{1.3}$$

If $\Phi = 0$ in (1.1), then GEP(1.1) reduces to the classical variational inequality and Ω is denoted by VI(Ψ, C), that is,

$$\mathrm{VI}(\Psi, C) = \{u^* \in C : \langle \Psi u^*, v - u^* \rangle \geq 0, \forall v \in C\}. \tag{1.4}$$

It is well known that GEP(1.1) contains as special cases, for instance, optimization problems, Nash equilibrium problems, complementarity problems, fixed point problems, and variational inequalities (see, e.g., [1–6] and the reference therein).

A mapping $A : C \to H$ is called α-inverse-strongly monotone [7], if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \tag{1.5}$$

for all $x, y \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping $S : C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \tag{1.6}$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in C : x = S(x)\}$. If $C \subset H$ is bounded, closed and convex and S is a nonexpansive mappings of C into itself, then $F(S)$ is nonempty (see [8]).

In 1997, Flåm and Antipin [9] introduced an iterative scheme of finding the best approximation to initial data when EP(Φ) is nonempty and proved a strong convergence theorem. In 2003, Iusem and Sosa [10] presented some iterative algorithms for solving equilibrium problems in finite-dimensional spaces. They have also established the convergence of the algorithms. Recently, Huang et al. [11] studied the approximate method for solving the equilibrium problem and proved the strong convergence theorem for approximating the solutions of the equilibrium problem.

On the other hand, for finding an element of $F(S) \cap \mathrm{VI}(A, C)$, Takahashi and Toyoda [12] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \quad n = 0, 1, 2, \ldots, \tag{1.7}$$

where $x_0 \in C$, P_C is metric projection of H onto C, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. Further, Iiduka and Takahashi [13] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A x_n), \tag{1.8}$$

where $u, x_0 \in C$, and proved the strong convergence theorems for iterative scheme (1.8) under some conditions on parameters. In 2007, S. Takahashi and W. Takahashi [14] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of an equilibrium problem and the set of fixed points of a nonexpansive mapping in Hilbert spaces. They also proved a strong convergence theorem which is connected with Combettes and Hirstoaga's result [3] and Wittmann's result [15]. Tada and W. Takahashi [16] introduced the Mann type iterative algorithm for finding a common element of the set of solutions of the EP(Φ) and the set of common fixed points of nonexpansive mapping and obtained the weak convergence of the Mann type iterative algorithm. Yao et al. [17] introduced an iteration process for finding a common element of the set of solutions of the EP(Φ) and the set of common fixed points of infinitely many nonexpansive mappings in Hilbert spaces. They proved a strong-convergence theorem under mild assumptions on parameters. Very recently, Moudafi [18] proposed an iterative algorithm for finding a common element of $\Omega \cap F(S)$, where $\Psi : C \to H$ is an α-inverse-strongly monotone mapping, and obtained a weak convergence theorem. There are some related works, we refer to [19–22] and the references therein.

Inspired and motivated by the works mentioned above, in this paper, we introduce an iterative process for finding a common element of the set of common fixed points of infinitely many nonexpansive mappings, the set of solutions of GEP(1.1), and the solution set of the variational inequality problem for an α-inverse-strongly monotone mapping in real Hilbert spaces. We give some strong-convergence theorems under mild assumptions on parameters. The results presented in this paper improve and generalize the main result of Yao et al. [17].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \forall y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle P_C x - P_C y, x - y \rangle \geq \left\| P_C x - P_C y \right\|^2 \tag{2.2}$$

for all $x, y \in H$. Furthermore, $P_C(x) \in C$ is characterized by the following properties:

$$\begin{gathered} \langle x - P_C x, y - P_C x \rangle \leq 0, \\ \|x - P_C x\|^2 + \left\| y - P_C x \right\|^2 \leq \left\| x - y \right\|^2 \end{gathered} \tag{2.3}$$

for all $x \in H$ and $y \in C$. It is easy to see that

$$u \in \mathrm{VI}(A, C) \Longleftrightarrow u = P_C(u - \lambda A u), \tag{2.4}$$

where $\lambda > 0$ is a parameter in $\mathbb{R}$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $p \in T(x)$ and $q \in T(y)$ imply $\langle x - y, p - q\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mappings. It is known that a monotone mapping T is maximal if and only if for $(x, p) \in H \times H$, $\langle x - y, p - q\rangle \geq 0$ for all $(y, q) \in G(T)$ implies $p \in T(x)$. Let $A : C \to H$ be a monotone, L-Lipschitz continuous mappings and let $N_C u$ be the normal cone to C at $u \in C$, that is, $N_C u = \{w \in H : \langle u - v, w\rangle \geq 0, \forall v \in C\}$. Define

$$Tu = \begin{cases} Au + N_C u, & u \in C, \\ \emptyset, & u \notin C. \end{cases} \tag{2.5}$$

Then T is the maximal monotone and $0 \in Tu$ if and only if $u \in \mathrm{VI}(A, C)$; see [23].

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself and let $\{\pi_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 1$, define a mapping S_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \pi_n T_n U_{n,n+1} + (1 - \pi_n)I, \\ U_{n,n-1} &= \pi_{n-1} T_{n-1} U_{n,n} + (1 - \pi_{n-1})I, \\ &\ \vdots \\ U_{n,k} &= \pi_k T_k U_{n,k+1} + (1 - \pi_k)I, \\ U_{n,k-1} &= \pi_{k-1} T_{k-1} U_{n,k} + (1 - \pi_{k-1})I, \\ &\ \vdots \\ U_{n,2} &= \pi_2 T_2 U_{n,3} + (1 - \pi_2)I, \\ S_n = U_{n,1} &= \pi_1 T_1 U_{n,2} + (1 - \pi_1)I. \end{aligned} \tag{2.6}$$

Such a mapping S_n is called the S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\pi_n, \pi_{n-1}, \ldots, \pi_1$ see [24]. It is obvious that S_n is nonexpansive and if $x = T_n x$ then $x = U_{n,k} = S_n x$.

Lemma 2.1 (see [24]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{\pi_n\}_{n=1}^{\infty}$ be a sequence in $(0, \sigma]$ for some $\sigma \in (0, 1)$. Then, for every $x \in C$ and $k \in \mathbb{N} = \{1, 2, \ldots\}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.2 (see [17]). It can be known from Lemma 2.1 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$, there exists $n_0 \geq 1$ such that for all $n > n_0$

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \epsilon. \tag{2.7}$$

Using Lemma 2.1, one can define a mapping S of C into itself as follows:

$$Sx = \lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x \tag{2.8}$$

for every $x \in C$. Such a mapping S is called the S-mapping generated by $T_1, T_2, \ldots$ and $\pi_1, \pi_2, \ldots$. Since S_n is nonexpansive, $S : C \to C$ is also nonexpansive. If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 2.2 that for an arbitrary $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$

$$\|S_n x_n - Sx_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D}\|U_{n,1}x - U_1 x\| \leq \epsilon. \tag{2.9}$$

This implies that

$$\lim_{n\to\infty}\|S_n x_n - Sx_n\| = 0. \tag{2.10}$$

Since T_i and $U_{n,i}$ are nonexpansive, we deduce that, for each $n \geq 1$,

$$\begin{aligned}
\|S_{n+1}x_n - S_n x_n\| &= \|\pi_1 T_1 U_{n+1,2}x_n - \pi_1 T_1 U_{n,2}x_n\| \\
&\leq \pi_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\
&= \pi_1\|\pi_2 T_2 U_{n+1,3}x_n - \pi_2 T_2 U_{n,3}x_n\| \\
&\leq \pi_1\pi_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\
&\ \ \vdots \\
&\leq \prod_{i=1}^{n}\pi_i\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&\leq M\prod_{i=1}^{n}\pi_i
\end{aligned} \tag{2.11}$$

for some constant $M \geq 0$.

Lemma 2.3 (see [24]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty}F(T_n) \neq \emptyset$, and let $\{\pi_n\}_{n=1}^{\infty}$ be a sequence in $(0,\sigma]$ for some $\sigma \in (0,1)$. Then, $F(S) = \bigcap_{n=1}^{\infty}F(T_n)$.*

For solving the generalized equilibrium problem, we assume that the bifunction $\Phi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(a1) $\Phi(u,u) = 0$ for all $u \in C$;

(a2) Φ is monotone, that is, $\Phi(u,v) + \Phi(v,u) \leq 0$ for all $u, v \in C$;

(a3) for each $u, v, w \in C$, $\lim_{t\downarrow 0}\Phi(tw + (1-t)u, v) \leq \Phi(u,v)$;

(a4) for each $u \in C$, $v \mapsto \Phi(u,v)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.4 (see [1]). *Let C be a nonempty closed convex subset of H, and let Φ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (a1)–(a4). Let $r > 0$ and $x \in H$. Then, there exists $u \in C$ such that*

$$\Phi(u, v) + \frac{1}{r}\langle v - u, u - x\rangle \geq 0 \quad \forall v \in C. \tag{2.12}$$

The following lemma was also given in [3].

Lemma 2.5 (see [3]). *Assume that $\Phi : C \times C \rightarrow \mathbb{R}$ satisfies (a1)–(a4). For $r > 0$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \left\{u \in C : \Phi(u, v) + \frac{1}{r}\langle v - u, u - x\rangle \geq 0, \forall v \in C\right\} \tag{2.13}$$

for all $x \in H$. Then, the following hold:

(b1) *T_r is single-valued;*

(b2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(b3) *$F(T_r) = EP(\Phi)$;*

(b4) *$EP(\Phi)$ is closed and convex.*

Remark 2.6. Replacing x with $x - r\Psi x \in H$ in (2.12), then there exists $u \in C$ such that

$$\Phi(u, v) + \langle \Psi x, v - u\rangle + \frac{1}{r}\langle v - u, u - x\rangle \geq 0 \quad \forall v \in C. \tag{2.14}$$

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.7 (see [25]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in Banach space E, and let $\{\beta_n\}$ be a sequence in $[0, 1]$. Suppose*

$$x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n \tag{2.15}$$

for all integers $n \geq 1$. If

$$\begin{gathered} 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1, \\ \limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0, \end{gathered} \tag{2.16}$$

then $\lim_{n \to \infty} \|z_n - x_n\| = 0$.

Lemma 2.8 (see [26]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 1, \tag{2.17}$$

where $\{\alpha_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we deal with an iterative scheme by the approximation method for finding a common element of the set of common fixed points of infinitely many nonexpansive mappings, the set of solutions of GEP(1.1), and the solution set of the variational inequality problem for an α-inverse-strongly monotone mapping in real Hilbert spaces.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Φ *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying (a1)–(a4),* $\Psi : C \to H$ *an inverse-strongly monotone mapping with constant* $\phi > 0$, $A : C \to H$ *an inverse-strongly monotone mapping with constant* $\varrho > 0$, $f : C \to C$ *a contraction mapping with constant* $\alpha \in [0,1)$. *Let* $S_n : C \to C$ *be a S-mapping generated by* $T_1, T_2, \ldots$ *and* $\pi_1, \pi_2, \ldots$ *and* $\bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(A,C) \neq \emptyset$, *where sequence* $\{T_n\}$ *is nonexpansive and* $\{\pi_n\}$ *is a sequence in* $(0,\sigma]$ *for some* $\sigma \in (0,1)$. *For* $x_1 \in C$, *suppose that* $\{x_n\}$, $\{y_n\}$, *and* $\{u_n\}$ *are generated by*

$$
\begin{gathered}
\Phi(u_n, v) + \langle \Psi x_n, v - u_n\rangle + \frac{1}{r_n}\langle v - u_n, u_n - x_n\rangle \geq 0, \quad \forall v \in C, \\
y_n = P_C(u_n - \lambda_n A u_n), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n
\end{gathered} \tag{3.1}
$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are three sequences in* $[0,1]$, $\{\lambda_n\}$ *is a sequence in* $(0,b]$ *for some* $0 < b < 2\varrho$ *and* $\{r_n\} \subset (0,d]$ *for some* $0 < d < 2\phi$ *satisfying*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\liminf_{n\to\infty}\lambda_n > 0$ *and* $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$;

(v) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then $\{x_n\}$, $\{y_n\}$, *and* $\{u_n\}$ *converge strongly to the point* $z_0 \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(A,C)$, *where* $z_0 = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap\Omega\cap VI(A,C)} f(z_0)$.

Proof. For any $x, y \in C$ and $r \in (0, 2\phi)$, we have

$$
\begin{aligned}
\|(I - r\Psi)x - (I - r\Psi)y\|^2 &= \|(x-y) - r(\Psi x - \Psi y)\|^2 \\
&= \|x-y\|^2 - 2r\langle x - y, \Psi x - \Psi y\rangle + r^2\|\Psi x - \Psi y\|^2 \\
&\leq \|x-y\|^2 + r(r - 2\phi)\|\Psi x - \Psi y\|^2 \\
&\leq \|x-y\|^2,
\end{aligned} \tag{3.2}
$$

which implies that $I - r\Psi$ is nonexpansive. Remark 2.6 implies that the sequences $\{u_n\}$ and $\{x_n\}$ are well defined. In view of the iterative sequence (3.1), we have

$$\begin{aligned} 0 &\leq \Phi(u_n, v) + \langle \Psi x_n, v - u_n \rangle + \frac{1}{r_n}\langle v - u_n, u_n - x_n \rangle \\ &= \Phi(u_n, v) + \frac{1}{r_n}\langle v - u_n, u_n - (x_n - r_n \Psi x_n) \rangle. \end{aligned} \tag{3.3}$$

It follows from Lemma 2.5 that $u_n = T_{r_n}(x_n - r_n\Psi x_n)$ for all $n \geq 1$. Let $z^* \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \cap \mathrm{VI}(A, C)$. For each $n \geq 1$, we have $z^* = S_n(z^*) = T_{r_n}(z^* - r_n\Psi z^*)$. By Lemma 2.5,

$$\begin{aligned} \|u_n - z^*\|^2 &= \|T_{r_n}(x_n - r_n\Psi x_n) - T_{r_n}(z^* - r_n\Psi z^*)\|^2 \\ &\leq \langle u_n - z^*, (x_n - r_n\Psi x_n) - (z^* - r_n\Psi z^*) \rangle \\ &= \frac{1}{2}\Big(\|u_n - z^*\|^2 + \|(x_n - r_n\Psi x_n) - (z^* - r_n\Psi z^*)\|^2 \\ &\qquad - \|(u_n - z^*) - ((x_n - r_n\Psi x_n) - (z^* - r_n\Psi z^*))\|^2 \Big) \end{aligned} \tag{3.4}$$

and so (3.2) implies that

$$\begin{aligned} \|u_n - z^*\|^2 &\leq \|(x_n - r_n\Psi x_n) - (z^* - r_n\Psi z^*)\|^2 - \|(u_n - x_n) - r_n(\Psi z^* - \Psi x_n)\|^2 \\ &\leq \|x_n - z^*\|^2 - \|(u_n - x_n) - r_n(\Psi z^* - \Psi x_n)\|^2 \\ &\leq \|x_n - z^*\|^2. \end{aligned} \tag{3.5}$$

For $z^* \in \mathrm{VI}(A, C)$, we have $z^* = P_C(z^* - \lambda_n A z^*)$ from (2.4). Since P_C is a nonexpansive mapping and A is an inverse-strongly monotone mapping with constant $\varrho > 0$, by (3.1), we have

$$\begin{aligned} \|y_n - z^*\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(z^* - \lambda_n A z^*)\|^2 \\ &\leq \|(u_n - \lambda_n A u_n) - (z^* - \lambda_n A z^*)\|^2 \\ &\leq \|u_n - z^*\|^2 + \lambda_n(\lambda_n - 2\varrho)\|A u_n - A z^*\|^2 \\ &\leq \|u_n - z^*\|^2. \end{aligned} \tag{3.6}$$

Thus, (3.5) and (3.6) imply that

$$\|y_n - z^*\| \leq \|u_n - z^*\| \leq \|x_n - z^*\|, \tag{3.7}$$

and so

$$\begin{aligned}
\|x_{n+1} - z^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n - z^*\| \\
&\le \alpha_n \|f(x_n) - z^*\| + \beta_n \|x_n - z^*\| + \gamma_n \|S_n y_n - z^*\| \\
&\le \alpha_n \left(\|f(x_n) - f(z^*)\| + \|f(z^*) - z^*\|\right) + \beta_n \|x_n - z^*\| + \gamma_n \|S_n y_n - S_n z^*\| \\
&\le \alpha_n \left(\alpha \|x_n - z^*\| + \|f(z^*) - z^*\|\right) + \beta_n \|x_n - z^*\| + \gamma_n \|y_n - z^*\| \\
&\le \alpha_n \left(\alpha \|x_n - z^*\| + \|f(z^*) - z^*\|\right) + \beta_n \|x_n - z^*\| + \gamma_n \|x_n - z^*\| \\
&= [1 - \alpha_n(1-\alpha)] \|x_n - z^*\| + \alpha_n (1-\alpha) \frac{\|f(z^*) - z^*\|}{1-\alpha} \\
&\le \max\left\{ \|x_1 - z^*\|, \frac{\|f(z^*) - z^*\|}{1-\alpha} \right\}.
\end{aligned} \tag{3.8}$$

This implies that $\{x_n\}$ is bounded. Therefore, $\{u_n\}$, $\{y_n\}$, $\{\Psi x_n\}$, $\{Au_n\}$, and $\{S_n y_n\}$ are also bounded.

From $u_n = T_{r_n}(x_n - r_n \Psi x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1} \Psi x_{n+1})$, we have

$$\Phi(u_n, v) + \langle \Psi x_n, v - u_n \rangle + \frac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \ge 0 \quad \forall v \in C, \tag{3.9}$$

$$\Phi(u_{n+1}, v) + \langle \Psi x_{n+1}, v - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle v - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0 \quad \forall v \in C. \tag{3.10}$$

Putting $v = u_{n+1}$ in (3.9) and $v = u_n$ in (3.10), we get

$$\begin{gathered}
\Phi(u_n, u_{n+1}) + \langle \Psi x_n, u_{n+1} - u_n \rangle + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \ge 0, \\
\Phi(u_{n+1}, u_n) + \langle \Psi x_{n+1}, u_n - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0.
\end{gathered} \tag{3.11}$$

Adding the above two inequalities, the monotonicity of Φ implies that

$$\langle \Psi x_{n+1} - \Psi x_n, u_n - u_{n+1} \rangle + \left\langle u_n - u_{n+1}, \frac{u_{n+1} - x_{n+1}}{r_{n+1}} - \frac{u_n - x_n}{r_n} \right\rangle \ge 0, \tag{3.12}$$

and so

$$\begin{aligned}
0 &\le \left\langle u_n - u_{n+1}, r_n(\Psi x_{n+1} - \Psi x_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n) \right\rangle \\
&= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right) u_{n+1} + (x_{n+1} - r_n \Psi x_{n+1}) - (x_n - r_n \Psi x_n) - x_{n+1} + \frac{r_n}{r_{n+1}} x_{n+1} \right\rangle \\
&= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) + (x_{n+1} - r_n \Psi x_{n+1}) - (x_n - r_n \Psi x_n) \right\rangle.
\end{aligned} \tag{3.13}$$

It follows from (3.2) that

$$\|u_{n+1}-u_n\|^2 \le \|u_{n+1}-u_n\|\left\{\left|1-\frac{r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\right\}, \tag{3.14}$$

and hence

$$\|u_{n+1}-u_n\| \le \left|1-\frac{r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|. \tag{3.15}$$

From (3.1),

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|P_C(u_{n+1}-\lambda_{n+1}Au_{n+1})-P_C(u_n-\lambda_n Au_n)\| \\
&\le \|(u_{n+1}-\lambda_{n+1}Au_{n+1})-(u_n-\lambda_n Au_n)\| \\
&\le \|(u_{n+1}-\lambda_{n+1}Au_{n+1})-(u_n-\lambda_n Au_{n+1})\|+|\lambda_{n+1}-\lambda_n|\|Au_n\| \\
&\le \|u_{n+1}-u_n\|+|\lambda_{n+1}-\lambda_n|\|Au_n\|.
\end{aligned} \tag{3.16}$$

Putting

$$z_n = \frac{\alpha_n}{1-\beta_n}f(x_n)+\frac{\gamma_n}{1-\beta_n}S_n y_n, \tag{3.17}$$

we have

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n. \tag{3.18}$$

Obviously, we get

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \left\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})+\frac{\gamma_{n+1}}{1-\beta_{n+1}}S_{n+1}y_{n+1}-\left(\frac{\alpha_n}{1-\beta_n}f(x_n)+\frac{\gamma_n}{1-\beta_n}S_n y_n\right)\right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)\| \\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|S_{n+1}y_{n+1}-S_n y_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|S_n y_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\alpha\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)\| \\
&\quad+\left|\frac{\alpha_n}{1-\beta_n}-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|\|S_n y_n\|+\left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)\|S_{n+1}y_{n+1}-S_n y_n\|.
\end{aligned} \tag{3.19}$$

From (2.11) and (3.16), we have

$$\begin{aligned}\|S_{n+1}y_{n+1} - S_n y_n\| &\le \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_n y_n\| \\ &\le \|y_{n+1} - y_n\| + M\prod_{i=1}^{n}\pi_i \\ &\le \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|Au_n\| + M\prod_{i=1}^{n}\pi_i\end{aligned} \tag{3.20}$$

for some constant $M \ge 0$. Combining (3.15), (3.19), and (3.20), we deduce

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\quad\le \frac{\alpha_{n+1}(\alpha - 1)}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|(\|f(x_n)\| + \|S_n y_n\|) \\ &\quad\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\left(M\prod_{i=1}^{n}\pi_i + |\lambda_{n+1} - \lambda_n|\|Au_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|u_{n+1} - x_{n+1}\|\right).\end{aligned} \tag{3.21}$$

It is easy to check that

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right| = 0, \qquad \lim_{n\to\infty}\prod_{i=1}^{n}\pi_i = 0, \qquad \lim_{n\to\infty}|r_{n+1} - r_n| = 0, \tag{3.22}$$

and so

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.23}$$

Thus, by Lemma 2.7, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$. It then follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|z_n - x_n\| = 0. \tag{3.24}$$

By (3.15) and (3.16), we have

$$\lim_{n\to\infty}\|y_{n+1} - y_n\| = \lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.25}$$

Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n$, we get

$$\begin{aligned}\left\|x_{n+1} - \frac{\beta_n x_n + \gamma_n S_n y_n}{1-\alpha_n}\right\| &= \left\|\alpha_n f(x_n) + \left(1 - \frac{1}{1-\alpha_n}\right)(\beta_n x_n + \gamma_n S_n y_n)\right\| \\ &= \alpha_n \left\|f(x_n) - \frac{\beta_n x_n + \gamma_n S_n y_n}{1-\alpha_n}\right\| \longrightarrow 0.\end{aligned} \tag{3.26}$$

On the other hand, for $\alpha_n + \beta_n + \gamma_n = 1$,

$$x_n - \frac{\beta_n x_n + \gamma_n S_n y_n}{1-\alpha_n} = \frac{\gamma_n}{1-\alpha_n}(x_n - S_n y_n). \tag{3.27}$$

It follows that

$$\begin{aligned}\gamma_n \|x_n - S_n y_n\| &= (1-\alpha_n)\left\|x_n - \frac{\beta_n x_n + \gamma_n S_n y_n}{1-\alpha_n}\right\| \\ &\leq (1-\alpha_n)\left(\|x_n - x_{n+1}\| + \left\|x_{n+1} - \frac{\beta_n x_n + \gamma_n S_n y_n}{1-\alpha_n}\right\|\right) \longrightarrow 0.\end{aligned} \tag{3.28}$$

It is easy to see that $\liminf_{n\to\infty}\gamma_n > 0$ and hence $\lim_{n\to\infty}\|x_n - S_n y_n\| = 0$.

From (3.5) and (3.6), we obtain

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\leq \alpha_n \left\|f(x_n) - z^*\right\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \left\|S_n y_n - z^*\right\|^2 \\ &\leq \alpha_n \left\|f(x_n) - z^*\right\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \left\|y_n - z^*\right\|^2 \\ &\leq \alpha_n \left\|f(x_n) - z^*\right\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \left\{\|u_n - z^*\|^2 + \lambda_n(\lambda_n - 2\varrho)\|Au_n - Az^*\|^2\right\} \\ &\leq \alpha_n \left\|f(x_n) - z^*\right\|^2 + \|x_n - z^*\|^2 + \gamma_n \lambda_n(\lambda_n - 2\varrho)\|Au_n - Az^*\|^2.\end{aligned} \tag{3.29}$$

Since $\liminf\, r_n > 0$, without loss of generality, we may assume that there exists a real number $a > 0$ such that $r_n > a$ for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}&\gamma_n a(2\varrho - b)\|Au_n - Az^*\|^2 \\ &\quad\leq \alpha_n \left\|f(x_n) - z^*\right\|^2 + \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 \\ &\quad\leq \alpha_n \left\|f(x_n) - z^*\right\|^2 + \|x_n - x_{n+1}\|(\|x_n - z^*\| + \|x_{n+1} - z^*\|).\end{aligned} \tag{3.30}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$ and $\{x_n\}$ is bounded, (3.30) implies that $\|Au_n - Az^*\| \to 0$ as $n \to \infty$. From (2.2), we have

$$\begin{aligned}
\|y_n - z^*\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(z^* - \lambda_n Az^*)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*), y_n - z^* \rangle \\
&= \frac{1}{2}\Big\{ \|(u_n - \lambda_n Au_n) - (z^* - \lambda_n Az^*)\|^2 + \|y_n - z^*\|^2 \\
&\qquad - \|(u_n - y_n) - \lambda_n (Au_n - Az^*)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|u_n - z^*\|^2 + \|y_n - z^*\|^2 \\
&\qquad - \Big(\|u_n - y_n\|^2 - 2\lambda_n \langle u_n - y_n, Au_n - Az^* \rangle + \lambda_n^2 \|Au_n - Az^*\|^2 \Big) \Big\},
\end{aligned} \tag{3.31}$$

and so

$$\|y_n - z^*\|^2 \le \|u_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n \langle u_n - y_n, Au_n - Az^* \rangle - \lambda_n^2 \|Au_n - Az^*\|^2. \tag{3.32}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z^*\| &\le \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \|y_n - z^*\|^2 \\
&\le \alpha_n \|f(x_n) - z^*\|^2 + \beta_n \|x_n - z^*\|^2 \\
&\quad + \gamma_n \Big(\|u_n - z^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n \langle u_n - y_n, Au_n - Az^* \rangle - \lambda_n^2 \|Au_n - Az^*\|^2 \Big) \\
&\le \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - z^*\|^2 - \gamma_n \|u_n - y_n\|^2 \\
&\quad + 2\gamma_n \lambda_n \langle u_n - y_n, Au_n - Az^* \rangle - \gamma_n \lambda_n^2 \|Au_n - Az^*\|^2
\end{aligned} \tag{3.33}$$

which implies that

$$\begin{aligned}
\gamma_n \|u_n - y_n\|^2 &\le \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 \\
&\quad + 2\gamma_n \lambda_n \langle u_n - y_n, Au_n - Az^* \rangle \\
&\le \alpha_n \|f(x_n) - z^*\|^2 + \|x_n - x_{n+1}\| (\|x_n - z^*\| + \|x_{n+1} - z^*\|) \\
&\quad + 2\gamma_n \lambda_n \|Au_n - Az^*\| \|u_n - y_n\|.
\end{aligned} \tag{3.34}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, $\|Au_n - Az^*\| \to 0$, and the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded, it follows from (3.34) that $\lim_{n\to\infty}\|u_n - y_n\| = 0$. On the other hand, from (3.5), we have

$$\begin{aligned}
\|x_{n+1} - z^*\|^2 &\le \alpha_n \left\| f(x_n) - z^* \right\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \left\| y_n - z^* \right\|^2 \\
&\le \alpha_n \left\| f(x_n) - z^* \right\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \|u_n - z^*\|^2 \\
&\le \alpha_n \left\| f(x_n) - z^* \right\|^2 + \beta_n \|x_n - z^*\|^2 \\
&\quad + \gamma_n \left(\|x_n - z^*\|^2 + r_n (r_n - 2\phi) \|\Psi x_n - \Psi z^*\|^2 \right) \\
&\le \alpha_n \left\| f(x_n) - z^* \right\|^2 + \|x_n - z^*\|^2 - \gamma_n r_n (2\phi - r_n) \|\Psi x_n - \Psi z^*\|^2.
\end{aligned} \tag{3.35}$$

The same as in (3.30), we have $\|\Psi x_n - \Psi z^*\| \to 0$ as $n \to \infty$. Likewise, using (3.5), we find

$$\begin{aligned}
\|x_{n+1} - z^*\|^2 &\le \alpha_n \left\| f(x_n) - z^* \right\|^2 + \beta_n \|x_n - z^*\|^2 + \gamma_n \|u_n - z^*\|^2 \\
&\le \alpha_n \left\| f(x_n) - z^* \right\|^2 + \beta_n \|x_n - z^*\|^2 \\
&\quad + \gamma_n \left(\|x_n - z^*\|^2 - \|u_n - x_n\|^2 + 2r_n \langle x_n - u_n, \Psi x_n - \Psi z^* \rangle - r_n^2 \|\Psi x_n - \Psi z^*\|^2 \right) \\
&\le \alpha_n \left\| f(x_n) - z^* \right\|^2 + \|x_n - z^*\|^2 + 2\gamma_n r_n \langle x_n - u_n, \Psi x_n - \Psi z^* \rangle - \gamma_n \|u_n - x_n\|^2.
\end{aligned} \tag{3.36}$$

The same as in (3.34), we have $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Since

$$\begin{aligned}
\|x_n - S_n x_n\| &\le \|x_n - S_n y_n\| + \|S_n y_n - S_n u_n\| + \|S_n u_n - S_n x_n\| \\
&\le \|x_n - S_n y_n\| + \|y_n - u_n\| + \|u_n - x_n\|,
\end{aligned} \tag{3.37}$$

we get $\|x_n - S_n x_n\| \to 0$ as $n \to \infty$. From (2.10) and

$$\begin{aligned}
\|S_n y_n - y_n\| &\le \|S_n y_n - S_n x_n\| + \|S_n x_n - x_n\| + \|x_n - y_n\| \\
&\le \|S_n x_n - x_n\| + 2\|x_n - y_n\| \\
&\le \|S_n x_n - x_n\| + 2(\|x_n - u_n\| + \|u_n - y_n\|) \longrightarrow 0,
\end{aligned} \tag{3.38}$$

we get $\lim_{n\to\infty}\|Sy_n - y_n\| = 0$.

Next, we show $\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0 \rangle \le 0$, where $z_0 = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap\Omega\cap \mathrm{VI}(A,C)} f(z_0)$. To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0) - z_0, Sy_n - z_0 \rangle = \lim_{i\to\infty}\langle f(z_0) - z_0, Sy_{n_i} - z_0 \rangle. \tag{3.39}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup z$. From $\|Sy_n - y_n\| \to 0$, we obtain $Sy_{n_i} \rightharpoonup z$. We now show that $z \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \cap \mathrm{VI}(A, C)$. Indeed, we observe that $u_n = T_{r_n}(x_n - r_n \Psi x_n)$ and

$$\Phi(u_n, v) + \langle \Psi x_n, v - u_n \rangle + \frac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq 0 \quad \forall v \in C. \tag{3.40}$$

From (a2), we deduce that

$$\langle \Psi x_n, v - u_n \rangle + \frac{1}{r_n} \langle v - u_n, u_n - x_n \rangle \geq -\Phi(u_n, v) = \Phi(v, u_n), \tag{3.41}$$

and hence

$$\langle \Psi x_{n_i}, v - u_{n_i} \rangle + \left\langle v - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq \Phi(v, u_{n_i}). \tag{3.42}$$

Form $\lim_{n\to\infty} \|u_n - y_n\| = 0$, we get $u_{n_i} \rightharpoonup z$. Put $z_t = tv + (1-t)z$ for all $t \in (0, 1]$ and $v \in C$. Consequently, we get $z_t \in C$. From (3.42), it follows that

$$\begin{aligned}
\langle \Psi z_t, z_t - u_{n_i} \rangle &\geq \langle \Psi z_t, z_t - u_{n_i} \rangle - \langle \Psi x_{n_i}, z_t - u_{n_i} \rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + \Phi(z_t, u_{n_i}) \\
&= \langle \Psi z_t - \Psi u_{n_i}, z_t - u_{n_i} \rangle + \langle \Psi u_{n_i} - \Psi x_{n_i}, z_t - u_{n_i} \rangle \\
&\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + \Phi(z_t, u_{n_i}).
\end{aligned} \tag{3.43}$$

From the Lipschitz continuous of Ψ and $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we obtain $\|\Psi u_{n_i} - \Psi x_{n_i}\| \to 0$. Since Ψ is monotone, we know that $\langle \Psi z_t - \Psi u_{n_i}, z_t - u_{n_i} \rangle \geq 0$. Further, $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$. It follows from (a4) that

$$\Phi(z_t, z) \leq \lim_{i\to\infty} \Phi(z_t, u_{n_i}) \leq \lim_{i\to\infty} \langle \Psi z_t, z_t - u_{n_i} \rangle = \langle \Psi z_t, z_t - z \rangle. \tag{3.44}$$

Owing to (a1) and (a4), we get that

$$\begin{aligned}
0 = \Phi(z_t, z_t) &\leq t\Phi(z_t, v) + (1-t)\Phi(z_t, z) \\
&\leq t\Phi(z_t, v) + (1-t)\langle \Psi z_t, z_t - z \rangle \\
&\leq t\Phi(z_t, v) + (1-t)t\langle \Psi z_t, v - z \rangle,
\end{aligned} \tag{3.45}$$

and hence

$$\Phi(z_t, v) + (1-t)\langle \Psi z_t, v - z \rangle \geq 0. \tag{3.46}$$

Letting $t \to 0$, we have

$$\Phi(z, v) + \langle \Psi z, v - z \rangle \geq 0. \tag{3.47}$$

This implies that $z \in \Omega$.

Furthermore, we prove that $z \in F(S) = \bigcap_{n=1}^{\infty} F(T_n)$. Assume $z \neq Sz$, since $\lim_{n\to\infty}\|u_n - x_n\| = 0$, we have $x_{n_i} z$. From Opial's theorem [27], we get

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i} - z\| &< \liminf_{i\to\infty}\|x_{n_i} - Sz\| \\ &\leq \liminf_{i\to\infty}(\|x_{n_i} - Sx_{n_i}\| + \|Sx_{n_i} - Sz\|) \\ &\leq \liminf_{i\to\infty}\|x_{n_i} - z\|.\end{aligned} \tag{3.48}$$

This is a contradiction. Hence, $z \in F(S) = \bigcap_{n=1}^{\infty} F(T_n)$.

Now, we will show that $z \in \mathrm{VI}(A, C)$. Let

$$Tu = \begin{cases} Au + N_C u, & u \in C, \\ \emptyset, & u \text{ is not in } C. \end{cases} \tag{3.49}$$

Then T is a maximal monotone [23]. Let $(u, w) \in G(T)$, since $w - Au \in N_C(u)$ and $y_n \in C$, we have $\langle u - y_n, w - Au \rangle \geq 0$. From $y_n = P_C(u_n - \lambda_n A u_n)$, we have

$$\langle u - y_n, y_n - (u_n - \lambda_n A u_n) \rangle \geq 0. \tag{3.50}$$

This is,

$$\left\langle u - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \right\rangle \geq 0. \tag{3.51}$$

Therefore, we obtain

$$\begin{aligned}\langle u - y_{n_i}, w \rangle &\geq \langle u - y_{n_i}, Au \rangle \\ &\geq \langle u - y_{n_i}, Au \rangle - \left\langle u - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \right\rangle \\ &= \left\langle u - y_{n_i}, Au - Au_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\ &= \langle u - y_{n_i}, Au - Ay_{n_i} \rangle + \langle u - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \left\langle u - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle u - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \left\langle u - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle.\end{aligned} \tag{3.52}$$

Noting that $\lim_{n\to\infty}\|u_n - y_n\| = 0$ and A is Lipschitz continuous, we obtain

$$\langle u - z, w\rangle \geq 0. \tag{3.53}$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and so $z \in \mathrm{VI}(A, C)$. Thus, $z \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \cap \mathrm{VI}(A, C)$. The property of the metric projection implies that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle &= \limsup_{n\to\infty}\langle f(z_0) - z_0, S_n y_n - z_0\rangle \\
&= \limsup_{n\to\infty}\langle f(z_0) - z_0, S y_n - z_0\rangle \\
&= \lim_{i\to\infty}\langle f(z_0) - z_0, S y_{n_i} - z_0\rangle \\
&= \langle f(z_0) - z_0, z - z_0\rangle \leq 0.
\end{aligned} \tag{3.54}$$

From (3.1) we obtain

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n - z_0, x_{n+1} - z_0\rangle \\
&= \alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle + \beta_n\langle x_n - z_0, x_{n+1} - z_0\rangle + \gamma_n\langle S_n y_n - z_0, x_{n+1} - z_0\rangle \\
&\leq \alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\quad + \frac{1}{2}\beta_n\left(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\right) + \frac{1}{2}\gamma_n\left(\left\|S_n y_n - z_0\right\|^2 + \|x_{n+1} - z_0\|^2\right) \\
&\leq \frac{1}{2}(1 - \alpha_n)\left(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\right) + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\quad + \frac{1}{2}\alpha_n\left(\left\|f(x_n) - f(z_0)\right\|^2 + \|x_{n+1} - z_0\|^2\right) \\
&\leq \frac{1}{2}\left(1 - \alpha_n\left(1 - \alpha^2\right)\right)\|x_n - z_0\|^2 + \frac{1}{2}\|x_{n+1} - z_0\|^2 + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle
\end{aligned} \tag{3.55}$$

which implies that

$$\|x_{n+1} - z_0\|^2 \leq \left(1 - \alpha_n\left(1 - \alpha^2\right)\right)\|x_n - z_0\|^2 + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle. \tag{3.56}$$

Setting

$$\delta_n = 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle, \tag{3.57}$$

we have

$$\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n(1 - \alpha^2)} \leq 0. \tag{3.58}$$

Applying Lemma 2.8 to (3.56), we conclude that $\{x_n\}$ converges strongly to z_0. Consequently, $\{u_n\}$ and $\{y_n\}$ converge strongly to z_0. This completes the proof. □

As direct consequences of Theorem 3.1, we have the following two corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction mapping with Lipschitz constant $\alpha \in [0,1)$, and let $A : C \to H$ be an inverse-strongly monotone mapping with constant $\varrho > 0$. Suppose $x_1 \in C$ and $\{x_n\}$ generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C(x_n - \lambda_n A x_n) \tag{3.59}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$, $\{\lambda_n\}$ is a sequence in $(0,b]$ for some $0 < b < 2\varrho$ satisfying conditions (i)–(iv). Then $\{x_n\}$ converges strongly to the point $x^* \in VI(A,C)$, where $x^* = P_{VI(A,C)} f(x^*)$.

Proof. Let $\Phi(u,v) = 0$ and $\Psi x = 0$ for all $u, v, x \in C$ and $r_n = 1$ in Theorem 3.1. Then $u_n = x_n$ for $n = 1, 2, \ldots$. Letting $T_n = I$ (the identity mapping) for all $n \in \mathbb{N}$, then $S_n = I$ for $n = 1, 2, \ldots$. It is easy to see that all conditions of Theorem 3.1 hold. Therefore, we know that the sequence $\{x_n\}$ generated by (3.59) converges strongly to $x^* = P_{\mathrm{VI}(A,C)} f(x^*)$. This completes the proof. □

Remark 3.3. From Corollary 3.2, we can get an iterative scheme for finding the solution of the variational inequality involving the α-inverse-strongly monotone mapping A.

Corollary 3.4 (see [17, Theorem 3.5]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Φ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (a1)–(a4), $f : C \to C$ a contraction mapping with constant $\alpha \in [0,1)$. Let $S_n : C \to C$ be an S-mapping generated by $T_1, T_2, \ldots$ and $\pi_1, \pi_2, \ldots$ and $\bigcap_{n=1}^{\infty} F(T_n) \cap EP(\Phi) \neq \emptyset$, where sequence $\{T_n\}$ is nonexpansive and $\{\pi_n\}$ is a sequence in $(0,\sigma]$ for some $\sigma \in (0,1)$. Suppose $x_1 \in C$ and $\{x_n\}$, $\{u_n\}$ are generated by*

$$\begin{gathered}
\Phi(u_n, v) + \frac{1}{r_n}\langle v - u_n, u_n - x_n\rangle \geq 0 \quad \forall v \in C, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n u_n
\end{gathered} \tag{3.60}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$, and $\{r_n\}$ is a sequence in $(0,+\infty)$ satisfying conditions (i)–(iii) and (v). Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to the point $x^* \in \bigcap_{n=1}^{\infty} F(T_n) \cap EP(\Phi)$, where $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \cap EP(\Phi)} f(x^*)$.

Proof. Let $\lambda_n = 1$ for $n = 1, 2, \ldots$ and $\Psi(x) = 0$ and $A(x) = 0$ for all $x \in C$ in Theorem 3.1. Since $u_n \in C$, we get that $u_n = P_C u_n$. It follows from Theorem 3.1 that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to the point $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(\Phi)} f(x^*)$. This completes the proof. □

Remark 3.5. The main result of Yao et al. [17, Corollary 3.2] improved and extended the corresponding theorems in Combettes and Hirstoaga [3] and S. Takahashi and W. Takahashi [14].

Our Theorem 3.1 improves and extends Theorem 3.5 of Yao et al. [17] in the following aspects:

(1) the equilibrium problem is extended to the generalized equilibrium problem;

(2) our iterative process (3.1) is different from Yao et al. iterative process (3.60) because there are a project operator and an α-inverse-strongly monotone mapping;

(3) our iterative process (3.1) is more general than Yao et al. iterative process (3.60) because it can be applied to solving the problem of finding a common element of the set of solutions of generalized equilibrium problems, the set of common fixed points of infinitely many nonexpansive mappings, and the set of solutions of the variational inequality for α-inverse-strongly monotone mapping.

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