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Research Article

Fixed Points of Maps of a Nonaspherical Wedge

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Let X be a finite polyhedron that has the homotopy type of the wedge of the projective plane and the circle. With the aid of techniques from combinatorial group theory, we obtain formulas for the Nielsen numbers of the selfmaps of X.

1. Introduction

Although compact surfaces were the setting of Nielsen's fixed point theory in 1927 [1], until relatively recently the calculation of the Nielsen number was restricted to maps of very few surfaces. For surfaces with boundary, such calculations were possible on the annulus and Möbius band because they have the homotopy type of the circle. In 1987 [2], Kelly used the commutativity property of the Nielsen number to make calculations for a family of maps of the disc with two holes. We will discuss Kelly's technique in more detail below. The first general algorithm for calculating Nielsen numbers of maps of surfaces with boundary was published by Wagner in 1999 [3]. It applies to many maps and recent research has significantly extended the class of such maps whose Nielsen number can be calculated (see [4–7] and, especially, the survey article [8]). This approach makes use of the fact that a surface with boundary has the homotopy type of a wedge of circles. For the calculation of the Nielsen number, Wagner and her successors employ techniques of combinatorial group theory.

The key properties of surfaces with boundary that are exploited in the Wagner-type calculations are that they have the homotopy type of a wedge and that they are aspherical spaces so their selfmaps are classified up to homotopy by the induced homomorphisms of the fundamental group. The paper [9] studies the fixed point theory of maps of other aspherical spaces that have the homotopy type of a wedge, for instance the wedge of a torus and a circle. The purpose of this paper is to demonstrate that combinatorial group theory furnishes powerful tools for the calculation of Nielsen numbers, even for maps of a nonaspherical space. We investigate a setting that is not aspherical and hence fundamental group information is not sufficient to classify selfmaps up to homotopy. We obtain explicit, easily calculated formulas for the Nielsen numbers of these maps.

Denote the projective plane by P and the circle by C. This paper is concerned with maps of finite polyhedra that have the homotopy type of the wedge $X = P \vee C$. If the polyhedron has no local cut points but is not a surface, then the Nielsen number of a map is the minimum number of fixed points among all the maps homotopic to it [10]. However, since a map of such a polyhedron has the homotopy type of a map of X and the Nielsen number is a homotopy type invariant, we will assume that we are concerned only with maps of X itself. We identify P and C with their images in X and denote their intersection by x_0. We need to consider only selfmaps of X and their homotopies that preserve x_0. The fundamental group of X at x_0 is the free product of a group of order two, whose generator we denote by a, and, choosing an orientation for C, the infinite cyclic group generated by b. To simplify notation, throughout the paper we denote the fundamental group homomorphism induced by a map by the same letter as the map because it will be clear from the context whether it represents the map or the homomorphism. Since all maps from P to C are homotopic to the constant map, we may assume that f_P, the restriction of $f : X \to X$ to P, maps P to itself.

The paper is organized as follows. We will describe in the next section a standard form for the map f in which the fixed point set is minimal on P and on C the fixed point set consists of x_0 together with a fixed point for each appearance of b or b^{-1} in the fundamental group element $f(b)$. In Section 3 we calculate the Nielsen numbers $N(f)$ of the maps for which $f(a) = 1$ by proving that, in that case, $N(f)$ equals the Nielsen number of a certain selfmap of C obtained from f and therefore $N(f)$ is determined by the degree of that map. In Section 4 we obtain formulas for the Nielsen numbers of almost all maps for which $f(a) = a$. The formulas depend on integers obtained from the word $f(b)$ in the fundamental group of X. However, the nonaspherical nature of X, which makes fundamental group information insufficient to determine the homotopy class of a map, requires us to find two different formulas for each word $f(b)$. One formula calculates $N(f)$ in the case that f_P is homotopic to the identity map whereas the other applies when f_P belongs to one of the infinite number of homotopy classes that do not contain the identity map. Section 5 then considers the two exceptional cases that are not calculated in Section 4. We demonstrate there that even if the induced fundamental group homomorphisms in these cases vary only slightly from those of Section 4, their Nielsen numbers can differ by an arbitrarily large amount. Section 6 presents the proof of a technical lemma from Section 4.

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2. The Standard Form of f

Given a map $f : (X, x_0) \to (X, x_0)$ where $X = P \vee C$, we write

$$f(b) = a^{\epsilon_1} b^{k_1} a b^{k_2} \cdots a b^{k_m} a^{\epsilon_2}, \tag{2.1}$$

where $\epsilon_i = 0, 1$ and $k_j \neq 0$ for all j.

Let $f_C : C \to X$ denote the restriction of f to C. By the simplicial approximation theorem, we may homotope f_C to a map with the property that the inverse image of x_0 is a finite union of points and arcs. A further homotopy reduces the inverse image of x_0 to a finite set and we view C as the union of arcs whose endpoints are mapped to x_0. We then homotope the map restricted to each arc, relative to the endpoints, so that it is a loop in X that is an embedding except at the endpoints and it represents either a, b or b^{-1}. If the restriction of the map to adjacent arcs corresponds to any of aa, bb^{-1} or $b^{-1}b$, we can homotope the map to a map constant at x_0 on both intervals and then shrink the intervals. We will continue to denote the map by $f_C : C \to X$. Starting with $x_0 = v_0$ and moving along the circle clockwise until we come to a point of $f_C^{-1}(x_0)$ which we call v_1, we denote the arc in C from v_0 to v_1 by J_1. Continuing in this manner, we obtain arcs $J_1, \ldots, J_n$ where the endpoints of J_n are v_n and v_0. As a final step, we homotope the map so that it is constant at x_0 on arcs J_0 and J_{n+1} that form a neighborhood of x_0 in C. Thus we have constructed a map, still written $f_C : C \to X$, that is constant on J_0 and J_{n+1} and, otherwise, its restriction to an arc is a loop representing a, b or b^{-1} according to the form of $f(b)$ above, in the order of the orientation of C.

Given a map $f : X \to X$, we may deform f by a homotopy so that f_P, its restriction to P, maps P to itself. We will make use of the constructions of Jiang in [11] to deform f so that f_P has a minimal fixed point set. If $f(a) = f_P(a) = 1$, then f_P belongs to one of two possible homotopy classes and, in both cases, Jiang constructs homotopies of f_P to a map with a single fixed point, which we may take to be x_0. Let $\widetilde{f}_P : S^2 \to S^2$ denote a lift of f_P to the universal covering space, then the degree of $\widetilde{f}_P$ is determined up to sign and we denote its absolute value by $d(f_P)$. If $f(a) = f_P(a) = a$, the homotopy class of f_P is determined by $d(f_P)$, which must be an odd natural number. If f_P is a deformation, that is, it is homotopic to the identity map, then $d(f_P) = 1$ and Jiang constructs a map homotopic to f_P with a single fixed point, which we again take to be x_0. For the remaining cases, where $d(f_P) \geq 3$, the Nielsen number $N(f_P) = 2$ and Jiang constructs maps homotopic to f_P with two fixed points. We take one of those fixed points to be x_0 and denote the other fixed point by y_0.

We also homotope f so that f_C, its restriction to C, is in the form described above. The map thus obtained we call the *standard form* of f and denote it also by $f : X \to X$. We note that, for each b in $f(b)$ there is exactly one fixed point of f in C, of index -1, and for each b^{-1} in $f(b)$ there is one fixed point, of index 1. The fixed points x_0 and y_0 are of index 1, see [11]. For the rest of the paper, all maps $f : X \to X$ will be assumed to be in standard form.

Our tools for calculating the Nielsen numbers come from Wagner's paper [3] which we will describe in the specific setting of selfmaps of X. Let x_p be a fixed point of f in C which is distinct from x_0, then x_p lies in an arc corresponding to an element b or b^{-1} in $f(b)$; we write $x_p \in b$ or $x_p \in b^{-1}$. We identify this element by writing $f(b) = V_p b \overline{V}_p$ or $f(b) = V_p b^{-1} \overline{V}_p$. The *Wagner tails* $W_p, \overline{W}_p \in \pi_1(X, x_0)$ of the fixed point x_p are defined by $W_p = V_p$ and $\overline{W}_p = \overline{V}_p^{-1}$ if $x_p \in b$ and by $W_p = V_p b^{-1}$ and $\overline{W}_p = \overline{V}_p^{-1} b$ if $x_p \in b^{-1}$.

We will use the following results of Wagner.

Lemma 2.1 (see [3, Lemma 1.3]). *For any fixed point x_p of f on C,*

$$f(b) = W_p b \overline{W}_p^{-1}. \tag{2.2}$$

Lemma 2.2 (see [3, Lemma 1.5]). *If x_p and x_q are fixed points of $f : X \to X$ on C, then x_p and x_q are in the same fixed point class if and only if there exists $z \in \pi_1(X, x_0)$ such that*

$$z = W_p^{-1} f(z) W_q. \tag{2.3}$$

Wagner's Lemma 1.5 concerns the case $Y \vee C$ where Y is a wedge of circles. However, the same proof establishes the statement of Lemma 2.2 for $X = P \vee C$. When (2.3) holds, we will say that x_p and x_q are f-Nielsen equivalent by z or, when the context is clear, more briefly that x_p and x_q are *equivalent*.

3. The $f(a) = 1$ Case

If Y is an aspherical polyhedron and a map $f : Y \vee C \to Y \vee C$ induces a homomorphism of the fundamental group that is trivial on the $\pi_1(Y, x_0)$ factor of $\pi_1(Y \vee C, x_0)$, then f is homotopic to the map $f_C\pi$ where $\pi : X \to C$ is the retraction sending Y to x_0. Therefore, by the commutativity property of the Nielsen number, $N(f) = N(f_C\pi) = N(\pi f_C)$. Since $\pi f_C : C \to C$, its Nielsen number is easily calculated. This is the technique that Kelly used, with $Y = C$, in [2] to construct his examples. If Y is not aspherical, then a map f that induces a homomorphism that is trivial on the $\pi_1(Y, x_0)$ factor need not be homotopic to $f_C\pi$. However, when $Y = P$, we will prove that it is still true that $N(f) = N(\pi f_C)$.

We note that since, in the $f(a) = 1$ case, all fixed points of f lie in C, then the fixed point sets of f and of πf_C consist of the same points. Moreover, the fixed point index of each fixed point is the same whether we view it as a fixed point of f or of πf_C. We will demonstrate that the fixed point classes f and of πf_C are also the same, and thus the Nielsen numbers are equal.

Since C is a circle with fundamental group generated by b, the condition corresponding to Wagner's for x_p and x_q to be in the same fixed point class of $\pi f_C : C \to C$ in [3, Lemma 1.5] is that there exist an integer r such that

$$b^r = \pi(W_p)^{-1} \pi f_C(b^r) \pi(W_q). \tag{3.1}$$

That is, there exists $z \in \pi_1(X, x_0)$ such that

$$\pi(z) = \pi(W_p)^{-1} \pi f_C(\pi(z)) \pi(W_q). \tag{3.2}$$

Although Wagner's paper [3] assumes reduced form for map and $\pi f_C(b)$ may not be in reduced form, in fact that condition is not used in the proof of [3, Lemma 1.5] so the existence of z satisfying (3.2) is still equivalent to the statement that x_p and x_q are in the same fixed point class of πf_C. Corresponding to the previous terminology, in this case we will say that x_p and x_q are πf_C-Nielsen equivalent by $\pi(z)$.

We have

$$f(b) = a^{e_1} b^{k_1} a b^{k_2} \cdots a b^{k_m} a^{e_2}, \tag{3.3}$$

where $\epsilon_i = 0, 1$ and $k_j \neq 0$ for all j. Let k be the sum of the k_j from 1 to m. Similarly, for an element $z \in \pi_1(X, x_0)$, we write

$$z = a^{\eta_1} b^{\ell_1} a b^{\ell_2} \cdots a b^{\ell_n} a^{\eta_2} \tag{3.4}$$

where, as before, $\eta_i = 0, 1$ and $\ell_j \neq 0$ for all j. Let ℓ be the sum of all the ℓ_j from 1 to n. The retraction $\pi : X \to C$ induces $\pi : \pi_1(X, x_0) \to \pi_1(C, x_0)$ such that $\pi(a) = 1$ and $\pi(b) = b$ and thus $\pi(f(b)) = b^k$ and $\pi(z) = b^\ell$. For fixed points x_p, x_q, define $g = W_p^{-1} W_q$, then $\pi(g) = b^v$ for some integer v.

Lemma 3.1. *If $f(a) = 1$, then the following are equivalent:*

(1) *x_p and x_q are f-Nielsen equivalent by z,*

(2) *x_p and x_q are πf_C-Nielsen equivalent by $\pi(z)$,*

(3) *$\ell = k\ell + v$.*

Proof. (1)⇒(2) If x_p and x_q are f-Nielsen equivalent by z, there exists $z \in \pi_1(X, x_0)$ such that

$$z = W_p^{-1} f(z) W_q \tag{3.5}$$

so

$$\pi(z) = \pi(W_p)^{-1} \pi f(z) \pi(W_q). \tag{3.6}$$

Every element of finite order in the fundamental group of X is a conjugate of an element of finite order in a or in b. Therefore, $f_P(a) = 1$ implies that $f(a) = 1$ so we have $f(z) = f_C(\pi(z))$ and thus

$$\pi(z) = \pi(W_p)^{-1} \pi f_C(\pi(z)) \pi(W_q). \tag{3.7}$$

As we noted above, (3.7) implies that x_p and x_q are πf_C-Nielsen equivalent by $\pi(z)$.

(2)⇒(3) If x_p and x_q are πf_C-Nielsen equivalent by $\pi(z)$, then we have (3.7). Since $\pi(z) = b^\ell$, we see that

$$\begin{aligned} b^\ell &= \pi\left(W_p^{-1} f(z) W_q\right) \\ &= \pi\left(W_p\right)^{-1} \pi\left(f(b)\right)^\ell \pi\left(W_p\right) \pi(g) \\ &= \pi(f(b))^\ell \pi(g) = \left(b^k\right)^\ell b^v. \end{aligned} \tag{3.8}$$

and conclude that $\ell = k\ell + v$.

(3)⇒(1) Suppose that $\ell = k\ell + v$. Since $f(a) = 1$, then $f(g) = f(b)^v$. If $k = 1$, it must be that $v = 0$. So, if we let $z = g$, then $f(z) = f(g) = f(b)^v = 1$ and thus

$$W_p^{-1} f(z) W_q = W_p^{-1} W_q = g = z, \tag{3.9}$$

that is, x_p and x_q are f-Nielsen equivalent by this z. If $k \neq 1$, we define $U_p = b(\overline{W_p})^{-1}$ and, again using the hypothesis $f(a) = 1$, we can write $f(U_p) = f(b)^r$ for some integer r. That hypothesis also implies that

$$f(f(b)) = f\left(a^{\epsilon_1} b^{k_1} a b^{k_2} \cdots a b^{k_m} a^{\epsilon_2}\right) = f(b)^k. \tag{3.10}$$

Now writing $f(b) = W_p b(\overline{W_p})^{-1} = W_p U_p$, we see that

$$U_p W_p = U_p(W_p U_p) U_p^{-1} = U_p f(b) U_p^{-1}. \tag{3.11}$$

If we let $z = (U_p W_p)^\ell g$ then, since $k\ell + v = \ell$, we have

$$\begin{aligned} f(z) &= f\left((U_p W_p)^\ell g\right) = f\left((U_p f(b) U_p^{-1})^\ell g\right) \\ &= f\left(U_p f(b)^\ell U_p^{-1} g\right) \\ &= f(b)^r \left(f(b)^k\right)^\ell f(b)^{-r} f(b)^v \\ &= f(b)^{k\ell+v} = f(b)^\ell = (W_p U_p)^\ell. \end{aligned} \tag{3.12}$$

Therefore,

$$W_p^{-1} f(z) W_q = W_p^{-1} (W_p U_p)^\ell (W_p g) = (U_p W_p)^\ell g = z \tag{3.13}$$

which again means that x_p and x_q are f-Nielsen equivalent by z. □

Since Lemma 3.1 has demonstrated that the fixed point classes of f and of πf_C are identical and the Nielsen number of a map of the circle is determined by its degree, we have

Theorem 3.2. *Let $\pi : \pi_1(X, x_0) \to \pi_1(C, x_0)$ be induced by retraction. If $f : X \to X$ is a map such that $f(a) = 1$ and $\pi(f(b)) = b^k$, then*

$$N(f) = N\left(\pi f_C\right) = |1 - \deg\left(\pi f_C\right)| = |1 - k|. \tag{3.14}$$

4. The $f(a) = a$ Case

Let $f : (X, x_0) \to (X, x_0)$ be a map, where $X = P \vee C$, such that $f(a) = a$. We will use Lemma 2.2 to calculate the Nielsen number of most such maps. We write

$$f(b) = a^{\epsilon_1} b^{k_1} a b^{k_2} \cdots a b^{k_m} a^{\epsilon_2}, \tag{4.1}$$

where $\epsilon_i = 0, 1$ and $k_j \neq 0$ for all j. Suppose that $\epsilon_2 = 1$. Then there is a map $h : (X, x_0) \to (X, x_0)$ that induces the homomorphism $h(\cdot) = a f(\cdot) a$, that is, $h(a) = a$ and

$h(b) = a^{\epsilon_1+1}b^{k_1}ab^{k_2}\cdots ab^{k_m}$. (2.3) of Lemma 2.2 is satisfied for f if and only if it is satisfied for h. Thus, we can assume that $\epsilon_2 = 0$ in $f(b)$ and we write

$$f(b) = a^{\epsilon}b^{k_1}ab^{k_2}\cdots ab^{k_m} = a^{\epsilon}cdc^{-1}, \tag{4.2}$$

where $\epsilon = 0, 1$ and either $d = a$ or d is cyclically reduced, which means that dd is a reduced word. Then, for some integers r and t,

$$c = b^{k_1}ab^{k_2}\cdots b^{k_r}ab^{t}, \qquad d = b^{k_{r+1}-t}a\cdots ab^{k_{m-r}+t}, \tag{4.3}$$

where t may be zero. If $t \neq 0$, then either $k_{r+1} = t$ or $k_{m-r} = -t$. Let $r = 0$ when $c = b^t$.

Now suppose that fixed points x_p and x_q are equivalent by

$$z = a^{\eta_1}b^{\ell_1}ab^{\ell_2}\cdots ab^{\ell_n}a^{\eta_2}, \tag{4.4}$$

where $\eta_i = 0, 1$ and $\ell_j \neq 0$ for all j. Let L denote the sum of the $|\ell_i|$ from 1 to n and let

$$R = W_p^{-1}f(z)W_q = W_p^{-1}a^{\eta_1}\left(a^{\epsilon}cdc^{-1}\right)^{\ell_1}a\cdots a\left(a^{\epsilon}cdc^{-1}\right)^{\ell_n}a^{\eta_2}W_q \tag{4.5}$$

be the right-hand side of the (2.3) of Lemma 2.2.

Denote the length of a word w in $\pi_1(X, x_0)$ by $|w|$, where the unit element is of length zero.

Lemma 4.1. *Suppose x_p and x_q are equivalent fixed points of f. If $\epsilon = 0$ and $d \neq a$, then $W_p = \overline{W}_q$ or $\overline{W}_p = W_q$.*

Proof. Suppose that $\epsilon = 0$ and $d \neq a$. Then

$$R = W_p^{-1}a^{\eta_1}cd^{\ell_1}c^{-1}a\cdots acd^{\ell_n}c^{-1}a^{\eta_2}W_q. \tag{4.6}$$

Case 1. $\eta_1 = 1$ and $\eta_2 = 1$.

Since $\epsilon = 0$ so that $f(b)$ starts and ends with b or b^{-1}, it follows that one of those elements ends W_p^{-1} and one of them starts W_q. Since $\eta_1 = \eta_2 = 1$, we see that R is reduced (c may be 1) and therefore

$$\begin{aligned} |R| &= \left|W_p\right| + \left|W_q\right| + (n+1)|a| + 2n|c| + L|d| \\ &> (n+1) + L \quad \left(\text{because } \left|W_p\right| + \left|W_q\right| > 0\right) \\ &= |z|. \end{aligned} \tag{4.7}$$

This is a contradiction and thus there is no solution in this case.

Case 2. $\eta_1 = 0$ and $\eta_2 = 1$. ($\eta_1 = 1$ and $\eta_2 = 0$ is similar.)

If there is no cancellation between W_p^{-1} and d^{ℓ_1}, then we can see that the solution z does not exist as in Case 1. Suppose there is a cancellation between W_p^{-1} and d^{ℓ_1}. Suppose $\ell_1 < 0$

and write $d = d_1 d_2$ where d_2^{-1} is the part of d^{-1} that is cancelled by W_p^{-1}, then $W_p^{-1} = \widehat{W}_p^{-1} d_2 c^{-1}$. By Lemma 2.1,

$$cdc^{-1} = f(b) = W_p b \overline{W}_p^{-1} = cd_2^{-1} \widehat{W}_p b \overline{W}_p^{-1} \tag{4.8}$$

so $d = d_1 d_2 = d_2^{-1} d_0 d_2$, for some word d_0, which contradicts the assumption that d is cyclically reduced. Thus $\ell_1 > 0$ so we may write $z = bz'$ and we have

$$\begin{aligned} bz' &= W_p^{-1} f(bz') W_q \\ &= W_p^{-1} f(b) f(z') W_q \\ &= W_p^{-1}(W_p b \overline{W}_p^{-1}) f(z') W_q \quad \text{(by Lemma 2.1)} \\ &= W_p^{-1}(W_p b \overline{W}_p^{-1}) cd^{(\ell_1-1)} c^{-1} a \cdots acd^{\ell_n} c^{-1} a W_q. \end{aligned} \tag{4.9}$$

and thus

$$z' = \overline{W}_p^{-1} cd^{(\ell_1-1)} c^{-1} a \cdots acd^{\ell_n} c^{-1} a W_q. \tag{4.10}$$

We have shown that ℓ_1 cannot be negative and, if $\ell_1 = 1$ then z' begins with $\overline{W}_p^{-1} a$ which cannot be reduced since $\epsilon = 0$ implies that $\overline{W}_p^{-1}$ ends with either b or b^{-1}. So suppose $\ell_1 > 1$ and $\overline{W}_p^{-1}$ cancels part of $d^{(\ell_1-1)}$. Then $\overline{W}_p^{-1}$ must end with c^{-1} to cancel c and, since $\overline{W}_p^{-1}$ is either $\overline{V}_p$ or $b^{-1}\overline{V}_p$, further cancellation would cancel parts of dd. But d is cyclically reduced and therefore we conclude that there is no further cancellation. Thus, as in Case 1, there are no solutions z' to this equation.

Case 3. $\eta_1 = 0$ and $\eta_2 = 0$.

If $n \geq 2$, then an argument similar to that of Case 2 applies. Thus we may assume that $n = 1$, which implies that $z = b$ or $z = b^{-1}$. Suppose that $z = b$, then

$$b = W_p^{-1} f(b) W_q = W_p^{-1}(W_p b \overline{W}_p^{-1}) W_q = b \overline{W}_p^{-1} W_q. \tag{4.11}$$

and so $\overline{W}_p = W_q$. Similarly, if $z = b^{-1}$, then $W_p = \overline{W}_q$. □

Lemma 4.2. *Suppose x_p and x_q are equivalent fixed points of f. If $\epsilon = 1$ and $d \neq a$, then $W_p = \overline{W}_q$ or $\overline{W}_p = W_q$.*

The proof of Lemma 4.2 is similar to that of Lemma 4.1, but it requires the analysis of a greater number of cases, so we postpone it to Section 6.

Suppose x_p, x_q are fixed points of f with $x_p \in b$ and $x_q \in b$, then $W_p = W_q$ implies $x_p = x_q$ because f is in standard form; the same is true in the case $x_p \in b^{-1}$ and $x_q \in b^{-1}$. In these cases, $\overline{W}_p = \overline{W}_q$ also implies $x_p = x_q$. On the other hand, if $x_p \in b^{-1}$ and $x_q \in b$ or $x_p \in b$ and $x_q \in b^{-1}$, then $W_p \neq W_q$ and $\overline{W}_p \neq \overline{W}_q$. Thus, in our setting, the only ways that two distinct fixed points x_p and x_q of f can be *directly related* in the sense of [3, page 47] are if $W_p = \overline{W}_q$ or if $W_q = \overline{W}_p$. The point of Lemmas 4.1 and 4.2 is that, if two fixed points in C are equivalent, then they *must* be directly related rather than related by intermediate fixed points. It is this property that permits the calculations of Nielsen numbers that occupy the rest of this section.

We continue to assume that f is in standard form and $f(a) = a$. If f_P is a deformation, then x_0 is the only fixed point of f on P. Otherwise, there is another fixed point of f on P denoted by y_0 and both x_0 and y_0 are of index 1, see [11]. We again write $x_p \in b$ or $x_p \in b^{-1}$ depending on whether f maps the arc containing x_p to b or to b^{-1}. The fixed points of f on C are $x_0, x_1, x_2, \ldots, x_{K-1}, x_K$, ordered so that x_1 lies in the arc corresponding to the first appearance of b or b^{-1} in $f(b)$. Moreover, for w a subword of $f(b)$, we write $x_p \in w$ if x_p lies in an arc corresponding to an element of w. Let K_d denote the number of fixed points x_p such that $x_p \in d$.

Lemma 4.3. *Suppose f_p is not a deformation and, if $\epsilon = 1$, suppose also that $d \neq a$. If $\epsilon = 1$ and $x_1 \in b$, then y_0 and x_1 are equivalent. Otherwise, y_0 is not equivalent to any other fixed point of f.*

Proof. Let $x_j \in C$ be a fixed point of f and let γ^+ and γ^- denote the arcs of C going from x_0 to x_j in the clockwise and counterclockwise directions, respectively. Then $f(\gamma^+) = W\gamma^+$ and $f(\gamma^-) = \overline{W}\gamma^-$, where W and $\overline{W}$ are the Wagner tails of x_j. The fixed points y_0 and x_j are equivalent if and only if there is a path β in X from y_0 to x_j such that the loops $\gamma^+\beta^{-1}f(\beta)(\gamma^+)^{-1}$ and $\gamma^-\beta^{-1}f(\beta)(\gamma^-)^{-1}$ represent the identity element of $\pi_1(X, x_0)$. Using a homotopy, we may assume that β is of the form $\alpha z\gamma^+$ or $\alpha z\gamma^-$ where α is a path in P from y_0 to x_0 and z is a loop in X based at x_0. Since, by [11], the fixed points y_0 and x_0 are not f_P-Nielsen equivalent, then $[\alpha^{-1}f(\alpha)] = a$, the only nonidentity element of $\pi_1(P, x_0)$.

If $\beta = \alpha z\gamma^+$, then y_0 and x_j are equivalent by β if and only if

$$\begin{aligned}
1 &= \left[\gamma^+\beta^{-1}f(\beta)(\gamma^+)^{-1}\right] \\
&= \left[\gamma^+(\gamma^+)^{-1}z^{-1}\alpha^{-1}f(\alpha)f(z)W\gamma^+(\gamma^+)^{-1}\right] \\
&= z^{-1}af(z)W
\end{aligned} \tag{4.12}$$

which is equivalent to $az = f(z)W$, for some z which we now view as an element of $\pi_1(X, x_0)$. If $\beta = \alpha z\gamma^-$ then, similarly, y_0 and x_j are equivalent by β if and only if $az = f(z)\overline{W}$.

There is no solution z to $az = f(z)W$ or $az = f(z)\overline{W}$ for which $\epsilon = 0$ since az starts with a^{η_1+1} but $f(z)W$ and $f(z)\overline{W}$ will start with a^{η_1}. If $\epsilon = 1$, and $\ell_1 < 0$, then there is no solution either since, again, az starts with a^{η_1+1} and $f(z)W$ starts with a^{η_1}. If $\epsilon = 1$, $\ell_1 > 0$ and $k_1 < 0$, then there is no solution since az starts with $a^{\eta_1+1}b$ but $f(z)W$ starts with $a^{\eta_1+1}b^{-1}$. If $\epsilon = 1$, $\ell_1 = 0$ and $k_1 < 0$, then there is no solution since $az = a^{\eta_1+1}$ but $f(z)W$ contains at least one b or b^{-1}. So suppose that $\epsilon = 1$, $\ell_1 \geq 0$ and $k_1 > 0$. This means that $x_1 \in b$ with $W = a$ so x_1 is equivalent to y_0 by letting $z = a$. However, no other fixed point is equivalent to y_0 because it would then also be equivalent to x_1 and, in this case, every W starts with a and no $\overline{W}$ starts with a so, since we assumed $d \neq a$, we may conclude from Lemma 4.2 that no such equivalence is possible. □

We now have the tools we will need to calculate the Nielsen number $N(f)$ for almost all maps $f : X \to X$ such that $f(a) = a$. (The remaining cases will be computed in Section 5.) We continue to write $f(b) = a^{\epsilon}cdc^{-1}$ where $\epsilon = 0, 1$.

Theorem 4.4. *If $\epsilon = 0$, $c = 1$, $d \neq a$ and f_P is not a deformation, then*

$$N(f) = \begin{cases} K & \text{if } d \neq b, \quad k_1 > 0, \\ K + 2 & \text{if } k_1 < 0. \end{cases} \tag{4.13}$$

Proof. Since d is cyclically reduced, if $k_1 > 0$ then $k_m > 0$ also and thus, for $x_p = x_j$ where $j = 2, 3, \dots, K-1$, the Wagner tail W_p starts with b and $\overline{W}_p$ starts with b^{-1} so, by Lemma 4.1, no two of the fixed points $x_2, \dots, x_{K-1}$ are equivalent. However, x_1 and x_K are equivalent to x_0 so, since y_0 is an essential fixed point class by Lemma 4.3, there are K essential fixed point classes. If $k_1 < 0$ none of the fixed points on C are equivalent to each other, nor is y_0 equivalent to any of them. □

In standard form, each $b^{k_j} \subseteq f(b)$ is represented by $|k_j|$ consecutive arcs in C and there is a first arc and a last arc with respect to the orientation of C, which correspond to the first and last appearance, respectively, of b or b^{-1} in b^{k_j}. We will refer to the fixed points in these arcs as the *first* and *last* fixed points in b^{k_j}.

We say that a fixed point x_p *cancels* a fixed point x_q if x_p and x_q are equivalent and one is of index 1 and the other is of index -1.

Theorem 4.5. *If $\epsilon = 0$, $d \neq a$, $c \neq 1$ but $t = 0$ and f_P is not a deformation, then*

$$N(f) = \begin{cases} K_d + 2r - 1 & \text{if } d \neq b, \quad k_{r+1} > 0, \\ K_d + 2r & \text{otherwise}. \end{cases} \tag{4.14}$$

Proof. If $x_p \in b^{k_j} \subseteq c$ and $k_j > 0$ then, if x_p is not the first fixed point, it cancels one $x_q \in b^{-k_j} \subseteq c^{-1}$ because $W_p = \overline{W}_q$. The only fixed point of b^{-k_j} not so cancelled is the first one. If $k_j < 0$, then all but the last fixed point of b^{k_j} cancels a fixed point of b^{-k_j} with only the last fixed point not cancelled. One of x_1 and x_K is cancelled by x_0 but each remaining uncancelled fixed point in c and c^{-1} is an essential fixed point class. Thus, including y_0, there are $2r$ fixed point classes outside of d. Let $x_p \in b^{k_{r+1}}$ such that $V_p = c$ and $x_q \in b^{k_{m-r}}$ such that $\overline{V}_q = c^{-1}$. Then x_p and x_q are equivalent if and only if $k_{r+1} > 0$ since that implies $k_{m-r} > 0$ and thus to $W_p = \overline{W}_q = c$. We conclude that the number of essential fixed point classes in d is $K_d - 1$ if $d \neq b$ and $k_{r+1} > 0$ and K_d otherwise. □

Theorem 4.6. *If $\epsilon = 0$, $d \neq a$ and $t \neq 0$, and f_P is not a deformation, then*

$$N(f) = \begin{cases} K_d + 2r & \text{if } k_{r+1} - t > 0 \text{ or } k_{n-r} + t > 0, \\ K_d + 2r + 2 & \text{if } k_{r+1} - t < 0 \text{ or } k_{n-r} + t < 0. \end{cases} \tag{4.15}$$

Proof. If $k_{r+1} - t > 0$ then, since c ends with b^t and d begins with $b^{k_{r+1}-t}$, a negative t would produce cancellations in the reduced word $f(b)$, so we have $0 < t < k_{r+1}$. Since d is cyclically reduced, it must be that $k_{n-1} + t = 0$. As in the previous proof, there are r fixed points in each of c and c^{-1} that do not cancel, x_0 is cancelled by x_1 but y_0 is an essential fixed point class. Similarly, in each of b^t and b^{-t} there is one fixed point that is not cancelled. However, there exist $x_p \in d$ and $x_q \in c^{-1}$ such that $W_p = \overline{W}_q = c$ and they cancel each other, so $N(f) = K_d + 2r$. If $k_{r+1} - t < 0$ then there is one uncancelled fixed point in each of b^t and b^{-t}, and no fixed point in d is cancelled, so $N(f) = K_d + 2r + 2$. The other cases are symmetric to these. □

In each of Theorems 4.4, 4.5, and 4.6, we assume that f_P is not a deformation, so y_0 is an essential fixed point class of f. If $\epsilon = 0$ and $d \neq a$ but f_P *is* a deformation, let $h : (X, x_0) \to (X, x_0)$ be a map such that $h(x) = f(x)$ for all $x \in C$ but the restriction of h to P is not a deformation though it induces a homomorphism mapping a to itself. Then $N(f) = N(h) - 1$ by Lemma 4.3 and $N(h)$ can be calculated by the previous theorems. We note that, since f and h induce the same fundamental group homomorphism, this difference in the Nielsen numbers reflects the nonaspherical nature of X. This completes the calculation of $N(f)$ in the case that $\epsilon = 0$ and $d \neq a$.

Theorem 4.7. *Suppose $\epsilon = 1$ and $d \neq a$. If f_P is not a deformation, then*

$$N(f) = \begin{cases} 1 & \text{if } c = 1, \quad d = b, \\ K + 2 & \text{if } k_1 < 0, \quad k_m < 0, \\ K - 2 & \text{if } k_1 > 0, \quad k_m > 0, \\ K & \text{if } k_1 \cdot k_m < 0. \end{cases} \tag{4.16}$$

If f_P is a deformation, then

$$N(f) = \begin{cases} K + 1 & \text{if } k_m < 0, \\ K - 1 & \text{if } k_m > 0. \end{cases} \tag{4.17}$$

Proof. By Lemma 4.2, no two among the fixed points $x_1, \ldots, x_{K-1}$ can be equivalent because, for each one, W_p begins with a and $\overline{W}_p$ does not. Suppose $k_1 < 0$ and $k_m < 0$. If f_P is not a deformation then, using Lemma 4.3, we see that each of $y_0, x_0, x_1, \ldots, x_K$ is an essential fixed point class so $N(f) = K + 2$ whereas, if f_P is a deformation, then $N(f) = K + 1$. If $k_1 > 0$ and $k_m > 0$, then x_K cancels x_0. If f_P is not a deformation then, by Lemma 4.3, y_0 cancels x_1 so $N(f) = K - 2$ except when $K = 1$. However, if f_P is a deformation, then x_1 is an essential fixed point class so $N(f) = K - 1$. If $k_1 < 0$ and $k_m > 0$ then x_K cancels x_0 whereas if y_0 is fixed by f, then it is an essential fixed point class so $N(f) = K$ if f_P is not a deformation and $N(f) = K - 1$ if it is. Finally, suppose $k_1 > 0$ and $k_m < 0$. If f_P is not a deformation, then y_0 cancels x_1 by Lemma 4.3 so $N(f) = K$. If f_P is a deformation, then each of $x_0, x_1, \ldots, x_K$ is an essential fixed point class and $N(f) = K + 1$. □

5. The Exceptional Cases

The only cases remaining occur when $f(a) = a$ and $f(b) = a^\epsilon cac^{-1}$ for $\epsilon = 0, 1$.

We will make use of the following result concerning Wagner tails.

Lemma 5.1. *Let x_p and x_q be fixed points of f in $C - \{x_0\}$. If one of $W_p^{-1}W_q, \overline{W}_p^{-1}W_q, W_p^{-1}\overline{W}_q$ or $\overline{W}_p^{-1}\overline{W}_q$ is in the kernel of f, then x_p is equivalent to x_q.*

Proof. Let W_{pq} denote the word in the hypotheses that is in the kernel of f. If $W_{pq} = W_p^{-1}W_q$ let $z = W_{pq}$, if $W_{pq} = \overline{W_p}^{-1}W_q$ let $z = W_{pq}b^{-1}$, if $W_{pq} = W_p^{-1}\overline{W_q}$ let $z = bW_{pq}$ and if $W_{pq} = \overline{W_p}^{-1}\overline{W_q}$ let $z = bW_{pq}b^{-1}$. Using Lemma 2.1, we verify that $W_p^{-1}f(z)W_q = z$, so x_p is equivalent to x_q by Lemma 2.2. □

If $\epsilon = 0$, so $f(a) = a$ and $f(b) = cac^{-1}$, then the kernel of f is the normal closure of the subgroup of G generated by b^2. Let $h : G \to H = G/\ker(f)$ be the quotient homomorphism, then there is a homomorphism $\overline{f} : H \to H$ such that $hf = \overline{f}h$. Setting $h(a) = \overline{a}$ and $h(b) = h(b^{-1}) = \overline{b}$, we note that

$$\overline{f}(\overline{b}) = \overline{a}^{\eta}\overline{b}\overline{a}\cdots\overline{a}\overline{b}\overline{a}^{\eta}, \tag{5.1}$$

where $\eta = 0$ or 1. Let U denote the number of appearances of $\overline{b}$ in $\overline{f}(\overline{b})$.

Theorem 5.2. *Suppose $f(a) = a$ and*

$$f(b) = cac^{-1} = b^{k_1}a\cdots ab^{k_r}ab^{-k_r}a\cdots ab^{-k_1}. \tag{5.2}$$

If f_P is not a deformation, then

$$N(f) = \begin{cases} 2 & \text{if } U = 0, \\ U & \text{if } U \neq 0. \end{cases} \tag{5.3}$$

and, if f_P is a deformation, then

$$N(f) = \begin{cases} U - 1 & \text{if } \eta = 0, \quad U \neq 0, \\ U + 1 & \text{otherwise}. \end{cases} \tag{5.4}$$

Proof. As in the proof of Theorem 4.5, if $k_j > 0$ then each fixed point x_p of $b^{k_j} \subseteq c$ except the first one cancels a fixed point $x_q \in b^{-k_j}$ because $W_p = \overline{W}_q$, leaving only the first fixed point of b^{-k_j} uncancelled in this way. If $k_j < 0$, it is the last fixed point of b^{k_j} and the last of b^{-k_j} that are the only fixed points that are not cancelled in this way. However, further cancellations take place. If k_j is even, let x_p and x_q be the uncancelled fixed points of b^{k_j} and b^{-k_j} respectively. Then $W_p^{-1}W_q = b^{|k_j|}$ is in the kernel of f so the fixed points cancel by Lemma 5.1.

Suppose that k_i and k_j, for $i < j \leq r$, are odd numbers and

$$g = ab^{k_{i+1}}a\cdots ab^{k_{j-1}}a \tag{5.5}$$

is in the kernel of f, and thus in the kernel of h as well. Let $x_p \in b^{k_i}, x_{p'} \in b^{k_j}, x_q \in b^{-k_i}$ and $x_{q'} \in b^{-k_j}$ be fixed points in $C - \{x_0\}$ that were not cancelled in the previous step. If $k_i \cdot k_j < 0$, then x_p cancels $x_{p'}$ and x_q cancels $x_{q'}$ whereas if $k_i \cdot k_j > 0$, then x_p cancels $x_{q'}$ and x_q cancels $x_{p'}$. We will demonstrate these cancellations only in the case $k_i > 0$ and $k_j < 0$ because the other three cases are similar. Since g is in the kernel of f, then $W_p^{-1}W_{p'} = b^{k_i}gb^{k_j}$ and $\overline{W_q}^{-1}\overline{W_{q'}} = g$ are also in the kernel, so p and p' cancel, as do q and q', by Lemma 5.1.

After all the cancellations, let $x_p \in b^{k_i}, x_q \in b^{k_j}$ be adjacent fixed points in $C - \{x_0\}$ among those that remain. Writing

$$f(b) = g_1 b^{k_i} g_2 b^{k_j} g_3, \tag{5.6}$$

it must be that k_i and k_j are odd and $h(g_2) \neq 1$. Therefore

$$\overline{f}(\overline{b}) = \overline{f}h(b) = hf(b) = h\left(g_1 b^{k_i} g_2 b^{k_j} g_3\right) = h(g_1)\overline{b}\overline{a}\overline{b}h\left(g_3\right) \tag{5.7}$$

so that x_p and x_q contribute two copies of $\overline{b}$ to $\overline{f}(\overline{b})$. We conclude that there are U fixed points remaining in $C - \{x_0\}$.

None of the remaining fixed points in $C - \{x_0\}$ are equivalent. Let $x_s \in b^{k_s}$ and $x_t \in b^{k_t}$ be two such fixed points, so $U \geq 2$. We claim that there is no solution to the equation

$$\overline{z} = h\left(W_s^{-1}\right)\overline{f}(\overline{z})h\left(W_t\right) \tag{5.8}$$

for any $\overline{z} = h(z)$, which implies that x_s and x_t are not equivalent since (2.3) of Lemma 2.2 then has no solution. We first show that $\overline{z} = 1$ is not a solution to (5.8) because $W_s^{-1}W_t$ is not in the kernel of h. Let

$$g_{st} = ab^{k_{s+1}}a \cdots ab^{k_{t-1}}a \tag{5.9}$$

then g_{st} cannot be in the kernel of h since, otherwise, x_s and x_t would have been eliminated previously. If $k_s < 0$ and $k_t > 0$, then $W_s^{-1}W_t = g_{st}$ whereas if $k_s > 0$ and $k_t < 0$ then $W_s^{-1}W_t = b^{k_s}g_{st}b^{k_t}$ which also cannot be in the kernel of h since k_s and k_t are odd. If $k_s k_t > 0$ then, if $W_s^{-1}W_t$ is in the kernel of h, there must exist u with $s < u < t$ and k_u odd, and both g_{su} and g_{ut} are in the kernel of h. But that would have eliminated these fixed points, so we have proved that $\overline{z} = 1$ is not a solution to (5.8). The argument that there is no solution $\overline{z}$ to (5.8) with $|\overline{z}| \geq 1$ depends on word length considerations like those in the proofs of Lemmas 4.1 and 4.2, which we therefore omit, and we conclude that none of the remaining fixed points in $C - \{x_0\}$ are equivalent.

Suppose $U \neq 0$ and let x_v and x_w be the first and last uncancelled fixed points in $C - \{x_0\}$, respectively. Assume that $\eta = 0$, then either x_v or x_w is cancelled by x_0. The reason is that, since $f(b) = cac^{-1}$, it must be that $x_v \in b^{k_v}$ implies that $x_w \in b^{-k_v}$. If $k_v > 0$, then $f(W_v) = h(W_v) = 1$ so x_v is cancelled by x_0 because $W_0^{-1}f(W_v)W_v = W_v$ so (2.3) of Lemma 2.2 is satisfied with $z = W_v$. Similarly, if $k_v < 0$, then x_w is cancelled by x_0 because $f(\overline{W}_w) = 1$ and therefore (2.3) is satisfied by setting $z = b\overline{W}_w$. On the other hand, if $\eta = 1$, then x_0 is not equivalent to any of the remaining fixed point in C because, under this condition, there is no solution to (5.8) above when $W_s = 1$ or $W_t = 1$. Thus, if f_P is a deformation so there are no fixed points other than x_0 on P in the standard form of f, we see that $N(f) = U - 1$ if $\eta = 0$ and $N(f) = U + 1$ if $\eta = 1$. If $U = 0$, then x_0 is the only uncancelled fixed point and $N(f) = 1$.

Now suppose f_p is not a deformation so the standard form of f has a fixed point y_0 in $P - \{x_0\}$. If $U = 0$ then y_0 and x_0 are the only fixed point that do not cancel, so $N(f) = 2$. If $\eta = 0$, then y_0 is not equivalent to any other fixed point by the following argument. Let W and $\overline{W}$ denote the Wagner tails of x_j. As in the proof of Lemma 4.3, y_0 and x_j are equivalent if and only if $az = f(z)W$ or $az = f(z)\overline{W}$ for some z and therefore, in the quotient group $G/\ker(f)$, we would have $\overline{a}\,\overline{z} = \overline{f}(\overline{z})h(W)$ or $\overline{a}\,\overline{z} = \overline{f}(\overline{z})h(\overline{W})$. Since $\eta = 0$, there is no such z because $\overline{a}\,\overline{z}$ starts with $\overline{a}^{\eta_1+1}$ but $\overline{f}(\overline{z})$ starts with a^{η_1}. Since we have seen that one of x_v or x_w is cancelled by x_0, we conclude that $N(f) = U$. If $\eta = 1$, then, in contrast to Lemma 4.3, y_0 does cancel a fixed point in C. Let z be the Wagner tail W_v of x_v then, since $h(z) = \overline{a}$, we see that $f(z) = a$ so $f(z)W_v = aW_v$ and therefore y_0 cancels x_v. Thus we again conclude that $N(f) = U$. □

Example 5.3. Let $c = (b^2a)^r b^{-1}$ and define maps $f, g : X \to X$ such that $f(a) = g(a) = a$ but f_P and g_P are not deformations, $f(b) = cac^{-1}$ and $g(b) = cabc^{-1}$. Then $\overline{f}(\overline{b}) = \overline{b}\overline{a}\overline{b}$ so, by Theorem 5.2, $N(f) = U = 2$. On the other hand, by Theorem 4.6, $N(g) = 2r+1$. Thus, the class of maps in Theorem 5.2 are truly very exceptional in their fixed point behavior compared to those of Section 4.

In the final case, where $\epsilon = 1$ so $f(a) = a$ and $f(b) = acac^{-1}$, the kernel of f is the normal closure of the subgroup of G generated by $(ab)^2$. Let H again be the quotient group of G by the normal closure of b^2. Define $k : G \to H$ by $k(a) = \overline{a}$ and $k(b) = \overline{a}\overline{b}$, then there is a homomorphism $\overline{f} : H \to H$ such that $kf = \overline{f}k$ given by $\overline{f}(\overline{a}) = \overline{a}$ and

$$\overline{f}(\overline{b}) = \overline{f}k(ab) = kf(ab) = k\left(cac^{-1}\right) = \overline{a}^{\eta}\overline{b}\overline{a}\cdots\overline{a}\overline{b}\overline{a}^{\eta} \tag{5.10}$$

where $\eta = 0$ or 1. Let V denote the number of appearances of $\overline{b}$ in $\overline{f}(\overline{b})$.

Theorem 5.4. *Suppose $f(a) = a$ and $f(b) = acac^{-1}$. If f_P is not a deformation, then*

$$N(f) = \begin{cases} 2 & \text{if } V = 0, \\ V & \text{if } V \neq 0. \end{cases} \tag{5.11}$$

and, if f_P is a deformation, then

$$N(f) = \begin{cases} V - 1 & \text{if } \eta = 0, \quad V \neq 0, \\ V + 1 & \text{otherwise}. \end{cases} \tag{5.12}$$

Proof. Let $\varphi, \psi : X \to X$ be maps such that $\varphi_P = \psi_P = id_P$ and φ_C and ψ_C are maps in standard form representing homomorphisms such that $\varphi(b) = ab$ and $\psi(b) = cac^{-1}$ so $f = \psi \circ \varphi$. Let $e = \varphi \circ \psi$, then $N(f) = N(e)$ by the commutativity property of the Nielsen number. We note that $e(a) = a$ and $e(b) = \varphi \circ \psi(b) = \varphi(cac^{-1}) = \varphi(c)a\varphi(c)^{-1}$ so e or a map $e' : X \to X$ that induces $e'(a) = a$ and $e'(b) = ae(b)a$ satisfies the hypotheses of Theorem 5.2. Since (2.3) of Lemma 2.2 is satisfied for e if and only if it is satisfied for e', we may assume that we can apply Theorem 5.2 to e. Thus if e_P is not a deformation, then $N(f) = 2$ if $U = 0$, and $N(f) = U$ if $U \neq 0$, and if e_P is a deformation, then $N(f) = U - 1$ if $\eta = 0$ and $U \neq 0$, and $N(f) = U + 1$ otherwise, where U is the number of appearances of $\overline{b}$ in $\overline{e}(\overline{b})$, where $he = \overline{e}h$ for $h : G \to g/\ker(e)$. Since $\varphi_P = \psi_P = id_P$, then f_P is a deformation if and only if e_P is a deformation. Noting that $k = h \circ \varphi$, we have $\overline{f} = \overline{e}$ so $V = U$ and the conclusion of the theorem follows. □

Example 5.5. Let $c = (ba)^r b^{-1}$ for $r \geq 1$ and define maps $f, g : X \to X$ such that $f(a) = g(a) = a$ but f_P and g_P are not deformations, $f(b) = acac^{-1}$ and $g(b) = acabc^{-1}$. Then, by Theorem 5.4, $N(f) = V = 2$ if r is even and $N(f) = 4$ if r is odd. On the other hand, $N(g) = 2r + 2$ by Theorem 4.7 and we find that the maps of Theorem 5.4 also have very different fixed point behavior compared to the maps of Section 4.

6. Proof of Lemma 4.2

Suppose x_p and x_q are equivalent fixed points of f where $f(b) = acdc^{-1}$ and $d \neq a$. Lemma 4.2 asserts that either $W_p = \overline{W}_q$ or $\overline{W}_p = W_q$. We now present the proof of this assertion.

In the notation introduced at the beginning of Section 4, we write

$$z = a^{\eta_1} b^{\ell_1} ab^{\ell_2} \cdots ab^{\ell_n} a^{\eta_2}, \tag{6.1}$$

$$\begin{aligned} R = W_p^{-1} f(z) W_q &= W_p^{-1} a^{\eta_1} \left(acdc^{-1}\right)^{\ell_1} a\left(acdc^{-1}\right)^{\ell_2} \cdots a\left(acdc^{-1}\right)^{\ell_n} a^{\eta_2} W_q \\ &= W_p^{-1} a^{\lambda_1} cd^{\delta_1} \left(c^{-1}acd^{\delta_1}\right)^{|\ell_1|-1} g_1 i d^{\delta_2} \left(c^{-1}acd^{\delta_2 a}\right)^{|\ell_2|-1} g_2 \\ &\quad \cdots g_{n-1} d^{\delta_n} \left(c^{-1}acd^{\delta_n}\right)^{|\ell_n|-1} c^{-1} a^{\lambda_2} W_q, \end{aligned} \tag{6.2}$$

where $\lambda_i = 0, 1$,

$$\delta_i = \begin{cases} 1 & \text{if } \ell_i > 0, \\ -1 & \text{if } \ell_i < 0 \end{cases}, \qquad g_i = \begin{cases} 1 & \text{if } \ell_i \cdot \ell_{i+1} > 0, \\ c^{-1}ac & \text{if } \ell_i \cdot \ell_{i+1} < 0. \end{cases} \tag{6.3}$$

Let G be the sum of the $|g_i|$.

Case 1. There are no cancellations between W_p^{-1} and the first d^{δ_1} nor between the last d^{δ_n} and W_q. As in Case 1 of Lemma 4.1, we will prove that there are no equivalent fixed points x_p and x_q for which W_p^{-1} and W_q possess these noncancellation properties.

Subcase 1.1. $|d| \geq 3$. Then,

$$\begin{aligned} |R| &= \left|W_p^{-1} a^{\lambda_1} c\right| + \left|c^{-1} a^{\lambda_2} W_q\right| + 2(L-n)|c| + (L-n)|a| + G + L|d| \\ &\geq 3L \quad (\text{because } |d| \geq 3) \\ &\geq n + 1 + L \quad (\text{because } L \geq n \geq 1) \\ &\geq \eta_1 + \eta_2 + n - 1 + L \\ &= |z|. \end{aligned} \tag{6.4}$$

Since $R = z$, all equalities must hold, and thus we have

$$\eta_1 = \eta_2 = 1, \quad L = n = 1, \quad W_p^{-1} a^{\lambda_1} c = 1, \quad c^{-1} a^{\lambda_2} W_q = 1, \qquad G = 0. \tag{6.5}$$

This implies that $z = aba$ or $z = ab^{-1}a$, and $R = d$ or $R = d^{-1}$. Since $z = R$, we conclude that $d = aba$ or $d = ab^{-1}a$, which is contrary to the hypothesis that d is cyclically reduced.

Subcase 1.2. $|d| = 2$. We first consider the case of $|c| \geq 1$ and then the case of $|c| = 0$. For $|c| \geq 1$, we have

$$\begin{aligned}
|R| &= \left|W_p^{-1}a^{\lambda_1}c\right| + \left|c^{-1}a^{\lambda_2}W_q\right| + 2(L-n)|c| + (L-n)|a| + G + L|d| \\
&= \left|W_p^{-1}a^{\lambda_1}c\right| + \left|c^{-1}a^{\lambda_2}W_q\right| + (2|c|+1)(L-n) + G + 2L \\
&\geq \left|W_p^{-1}a^{\lambda_1}c\right| + \left|c^{-1}a^{\lambda_2}W_q\right| + (2|c|+1)(L-n) + G + n + L \\
&\geq \eta_1 + \eta_2 + n - 1 + L \quad \text{(because of the } \textit{claim} \text{ below)} \\
&= |z|.
\end{aligned} \tag{6.6}$$

Claim 1. $|W_p^{-1}a^{\lambda_1}c| + |c^{-1}a^{\lambda_2}W_q| + (2|c|+1)(L-n) + G \geq \eta_1 + \eta_2 - 1$. The inequality is obvious except for the case of $L = n$, $G = 0$ and $\eta_1 = \eta_2 = 1$. In that case, we know that all the ℓ_i have the same sign and therefore either λ_1 or λ_2 is equal to zero. Since $|c| \geq 1$, if $\lambda_1 = 0$ then $|W_p^{-1}a^{\lambda_1}c| > 0$ and if $\lambda_2 = 0$ then $|c^{-1}a^{\lambda_2}W_q| > 0$. Since $R = z$, the equalities above must hold and so we have

$$L = n, \qquad \left|W_p^{-1}a^{\lambda_1}c\right| + \left|c^{-1}a^{\lambda_2}W_q\right| + G = \eta_1 + \eta_2 - 1. \tag{6.7}$$

Since $\eta_i \leq 1$, this implies that $G = 0$ and thus all ℓ_i have the same sign. Suppose that $\eta_1 = 0$ and $\eta_2 = 1$. (The case of $\eta_1 = 1$ and $\eta_2 = 0$ is similar.) Since all ℓ_i have the same sign, $a^{\lambda_1} = a^{\lambda_2}$ and therefore $W_p = a^{\lambda_1}c = a^{\lambda_2}c = W_q$. If $V_p \neq V_q$, that would imply either that $x_p \in b$ and $x_q \in b^{-1}$ or $x_p \in b^{-1}$ and $x_q \in b$ in adjacent arcs in C, contrary to the assumption that $f(b)$ is reduced. Thus $V_p = V_q$ which, since f is in standard form, would imply $x_p = x_q$, a contradiction. Now suppose that $\eta_1 = 1$ and $\eta_2 = 1$. All the ℓ_i have the same sign; suppose it is negative and thus all $\ell_i = -1$. Then $|W_p^{-1}ac| + |c^{-1}W_q| = 1$ where $\epsilon_1 = 1$ implies that $W_q \neq c$ so $W_q = 1, c = b$ or b^{-1} and $W_p^{-1}ac = 1$ so $R = (d^{-1})^n c^{-1}$. If $c = b$ then either $d = ba$ and thus $z = R = (ab^{-1})^n b^{-1}$ or $d = ab^{-1}$ and thus $z = R = (ba)^n b^{-1}$, both of which contradict the assumption that $\eta_2 = 1$. If $c = b^{-1}$, substituting $d = ab$ or $d = b^{-1}a$ again leads to a contradiction, now to $\eta_2 = 1$. If all $\ell_i = 1$ then, similarly, all cases lead to a contradiction to the assumption that $\eta_1 = 1$.

Suppose that $|c| = 0$. Since $|d| = 2$, we have $d = b^2$ or $d = b^{-2}$, which is a subword of $R = z$ and thus $L \geq n + 1$. Therefore,

$$\begin{aligned}
|R| &= \left|W_p^{-1}a^{\lambda_1}c\right| + \left|c^{-1}a^{\lambda_2}W_q\right| + (L-n)|a| + G + L|d| \\
&= \left|W_p^{-1}a^{\lambda_1}c\right| + \left|c^{-1}a^{\lambda_2}W_q\right| + (L-n) + G + 2L \\
&\geq \left|W_p^{-1}a^{\lambda_1}c\right| + \left|c^{-1}a^{\lambda_2}W_q\right| + 2 + n + L \quad \text{(because } L \geq n+1\text{)} \\
&> \eta_1 + \eta_2 + n - 1 + L \\
&= |z|
\end{aligned} \tag{6.8}$$

This is a contradiction so, if there are no such cancellations, there cannot be a solution to (2.3) of Lemma 2.2.

Subcase 1.3. $d = b$ or b^{-1}. If $c = 1$, then $f(b) = b$ or $f(b) = b^{-1}$ and the lemma is obviously true. Thus we assume that $|c| \geq 1$. We will consider only the case $d = b$ because the other is similar. Since we suppose x_p and x_q equivalent, we are assuming there exists z such that $z = R$. But then, as in Subcase 1.2, we will show that, for any choice of z we have $|R| = |W_p^{-1}f(z)W_q| > |z|$.

We will do it by dividing z into subwords such that their image under f is reduced in R and of greater word length. We first consider each subword ab^{ℓ_i} of z for which $|\ell_i| \geq 2$. Then $f(ab^{\ell_i}) = a(acbc^{-1})^{\ell_i}$ contains the subword $b^{\delta_i}(c^{-1}acb^{\delta_i})^{\ell_i}$ that is reduced in R and

$$\left|b^{\delta_i}\left(c^{-1}acb^{\delta_i}\right)^{\ell_i}\right| \geq 4\left(\left|\ell_i\right| - 1\right) + 1 = \left|\ell_i\right| + \left(3\left|\ell_i\right| - 3\right) > \left|\ell_i\right| + 1 = \left|ab^{\ell_i}\right| \tag{6.9}$$

because $|\ell_i| \geq 2$, so $|f(ab^{\ell_i})| > |ab^{\ell_i}|$.

Now consider a subword of z of the form $b^{\ell_j}ab^{\ell_{j+1}}a\cdots ab^{\ell_{j+r}}ab^{\ell_{j+r+1}}$ where $\ell_{j+1} = \cdots = \ell_{j+r} = 1$ but $\ell_j \neq 1$ and $\ell_{j+r+1} \neq 1$. Suppose $\ell_j < 0$ (or $j = 0$) and $\ell_{j+r+1} < 0$ (or $j + r = n$). Then $f(ab^{\ell_{j+1}}a)\cdots ab^{\ell_{j+r}} = f(aba\cdots ab)$ contains a subword ab^rc^{-1} which is reduced in R. In the case $r = 1$ we have $|ab| < |cbc^{-1}|$, so we consider $r \geq 2$. Since we are assuming that $z = R$, it must be that $b^{\ell_k} = b^r$ for some k. Since $r \geq 2$, it follows that $f(ab^{\ell_k})$ contains a subword $b(c^{-1}acb)^{r-1}$ that is reduced in R and

$$\begin{aligned}
\left|ab^{\ell_{j+1}}a\cdots ab^{\ell_{j+r}}\right| + \left|ab^{\ell_k}\right| &= 2r + (r+1) < (r+2) + (4r-3) \\
&\leq \left|cb^rc^{-1}\right| + \left|b\left(c^{-1}acb\right)^{r-1}\right| \\
&\leq \left|f\left(ab^{\ell_{j+1}}a\cdots ab^{\ell_{j+r}}\right)\right| + \left|f\left(ab^{\ell_k}\right)\right|.
\end{aligned} \tag{6.10}$$

If, instead, $\ell_j \geq 2$ and $\ell_{j+r+1} < 0$ (or $j + r = n$), then

$$f\left(ab^{\ell_j}ab^{\ell_{j+1}}a\cdots ab^{\ell_{j+r}}\right) = acb\left(c^{-1}acb\right)^{\ell_j - 1}b^rc^{-1} \tag{6.11}$$

contains $cb^{r+1}c^{-1}$ as a subword that is reduced in R. The assumption that $z = R$ then implies that $b^{\ell_k} = b^{r+1}$ for some k. The length of the image under f of the subword of z consisting of ab^{ℓ_k} and $ab^{\ell_j}ab^{\ell_{j+1}}a\cdots ab^{\ell_{j+r}}$ is greater than that of the word itself. The same holds for the appropriate choice of subwords of z when $\ell_j \neq 1$ and $\ell_{j+r+1} \geq 2$.

Suppose instead that we consider a subword of z of the form $b^{\ell_j}ab^{\ell_{j+1}}a\cdots ab^{\ell_{j+r}}ab^{\ell_{j+r+1}}$ where now $\ell_{j+1} = \cdots = \ell_{j+r} = -1$ but $\ell_j \neq -1$ and $\ell_{j+r+1} \neq -1$. An analysis like that just presented again leads to the conclusion that f increases the word length of subwords of z. Thus we have established that $|z| < |R|$ and consequently there are no solutions to (2.3) of Lemma 2.2.

Case 2. Suppose there is a cancellation between W_p^{-1} and the first d^{δ_1} but no cancellation between the last d^{δ_n} and W_q. If $\ell_1 > 0$ and $\eta_1 = 1$ or $\ell_1 < 0$ and $\eta_1 = 0$, then R begins with $W_p^{-1}c$ and no such cancellation is possible. If $\ell_1 < 0$ and $\eta_1 = 1$, an argument like that of Case 2 of Lemma 4.1 shows that a cancellation would contradict the assumption that d is cyclically reduced. Thus, we can conclude that $\ell_1 > 0$ and $\eta_1 = 0$ so $z = bz'$ and so, similarly to Lemma 4.1,

$$z' = \overline{W}_p^{-1}\left(acdc^{-1}\right)^{\ell_1 - 1}a\left(acdc^{-1}\right)^{\ell_2}\cdots a\left(acdc^{-1}\right)^{\ell_n}a^{\eta_2}W_q. \tag{6.12}$$

There are no further cancellations and thus, as in the previous case, there is no solution to (2.3) unless $z' = 1$ so that $z = b$ and $\overline{W}_p = W_q$.

Case 3. Suppose there is no cancellation between W_p^{-1} and the first d^{δ_1} but there is cancellation between the last d^{δ_n} and W_q. An argument similar to that of Case 2 demonstrates that $z = z'b^{-1}$ but then a solution is possible only if $z' = 1$ and thus that $\overline{W}_q = W_p$.

Case 4. Suppose that there is a cancellation between W_p^{-1} and the first d^{δ_1} and also between W_q and the last d^{δ_n}. Following Cases 2 and 3, we conclude that $z = bz'b^{-1}$ and that

$$z' = \overline{W}_p^{-1}\left(acdc^{-1}\right)^{\ell_1 - 1} a\left(acdc^{-1}\right)^{\ell_2} \cdots a\left(acdc^{-1}\right)^{\ell_n + 1}\overline{W}_q. \tag{6.13}$$

There are now no cancellations so $z' = 1$ and $\overline{W}_p = \overline{W}_q$. Since f is in standard form, the condition $\overline{W}_p = \overline{W}_q$ also implies that $x_p = x_q$ and thus there is no solution of this type.

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