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Research Article

Strong Convergence Theorem by Monotone Hybrid Algorithm for Equilibrium Problems, Hemirelatively Nonexpansive Mappings, and Maximal Monotone Operators

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We introduce a new hybrid iterative algorithm for finding a common element of the set of fixed points of hemirelatively nonexpansive mappings and the set of solutions of an equilibrium problem and for finding a common element of the set of zero points of maximal monotone operators and the set of solutions of an equilibrium problem in a Banach space. Using this theorem, we obtain three new results for finding a solution of an equilibrium problem, a fixed point of a hemirelatively nonexpnasive mapping, and a zero point of maximal monotone operators in a Banach space.

1. Introduction

Let E be a Banach space, let C be a closed convex subset of E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find

$$x^* \in C \quad \text{such that } f(x^*, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of such solutions x^* is denoted by $\mathrm{EP}(f)$.

In 2006, Martinez-Yanes and Xu [1] obtained strong convergence theorems for finding a fixed point of a nonexpansive mapping by a new hybrid method in a Hilbert space. In particular, Takahashi and Zembayashi [2] established a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a uniformly convex and uniformly smooth

Banach space. Very recently, Su et al. [3] proved the following theorem by a monotone hybrid method.

Theorem 1.1 (see Su et al. [3]). *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a closed hemirelatively nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that α_n is a sequence in $[0,1]$ such that $\limsup_{n\to\infty}\alpha_n < 1$. Define a sequence x_n in C by the following:*

$$\begin{gathered} x_0 \in C, \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTx_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \end{gathered} \tag{1.2}$$

where J is the duality mapping on E. Then, x_n converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In this paper, motivated by Su et al. [3], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a hemirelatively nonexpansive mapping and for finding a common element of the set of zero points of maximal monotone operators and the set of solutions of an equilibrium problem in a Banach space by using the monotone hybrid method. Using this theorem, we obtain three new strong convergence results for finding a solution of an equilibrium problem, a fixed point of a hemirelatively nonexpnasive mapping, and a zero point of maximal monotone operators in a Banach space.

2. Preliminaries

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. In this case, J is single valued and also one to one.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2. \tag{2.2}$$

Following Alber [4], the generalized projection $\Pi_C : E \to C$ from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C}\phi(y,x) \quad \forall x \in E. \tag{2.3}$$

The generalized projection Π_C from E onto C is well defined and single valued, and it satisfies

$$(\|x\| - \|y\|)^2 \le \phi(y,x) \le (\|x\| + \|y\|)^2 \quad \forall x,y \in E. \tag{2.4}$$

If E is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$ and Π_C is the metric projection of E onto C.

If E is a reflexive strict convex and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $\|x\| = \|y\|$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (2.4), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$, that is, $x = y$.

Let C be a closed convex subset of E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. T is called hemirelatively nonexpansive if $\phi(p,Tx) \le \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.

A point p in C is said to be an asymptotic fixed point of T [5] if C contains a sequence x_n which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A hemirelatively nonexpansive mapping T from C into itself is called relatively nonexpansive [1, 5, 6] if $\widehat{F}(T) = F(T)$.

We need the following lemmas for the proof of our main results.

Lemma 2.1 (see Alber [4]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Then,*

$$\phi(x,\Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y) \quad \forall x \in C,\ y \in E. \tag{2.5}$$

Lemma 2.2 (see Alber [4]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space, let $x \in E$, and let $z \in C$. Then,*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz\rangle \le 0 \quad \forall y \in C. \tag{2.6}$$

Lemma 2.3 (see Kamimura and Takahashi [7]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty}\phi(x_n,y_n) = 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.4 (see Xu [8]). *Let E be a uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function $g : [0,2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|) \quad \forall x,y \in B_r,\ t \in [0,1], \tag{2.7}$$

where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.5 (see Kamimura and Takahashi [7]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \le \phi(x, y) \quad \forall x, y \in B_r. \tag{2.8}$$

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t \to 0} f(tz + (1 - t)x, y) \le f(x, y)$;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex.

Lemma 2.6 (see Blum and Oettli [9]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), let $r > 0$, and let $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0 \quad \forall y \in C. \tag{2.9}$$

Lemma 2.7 (see Takahashi and Zembayashi [10]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $x \in E$, for $r > 0$. Define a mapping $T_r : E \to 2^C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0 \ \forall y \in C \right\} \quad \forall x \in E. \tag{2.10}$$

Then, the following holds:

(1) T_r is single valued;

(2) T_r is a firmly nonexpansive-type mapping [11], that is, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.11}$$

(3) $F(T_r) = \widehat{F}(T_r) = \mathrm{Ep}(f)$;

(4) $\mathrm{Ep}(f)$ is closed and convex.

Lemma 2.8 (see Takahashi and Zembayashi [10]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4). Then, for $r > 0$ and $x \in E$, and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x). \tag{2.12}$$

Lemma 2.9 (see Su et al. [3]). *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a hemirelatively nonexpansive mapping from C into itself. Then, $F(T)$ is closed and convex.*

Recall that an operator T in a Banach space is called closed, if $x_n \to x$, $Tx_n \to y$, then $Tx = y$.

3. Strong convergence theorem

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $T : C \to C$ be a closed hemirelatively nonexpansive mapping such that $F(T) \cap \mathrm{EP}(f) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following:*

$$\begin{aligned}
x_0 &\in C, \text{ chosen arbitrarily},\\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTz_n),\\
z_n &= J^{-1}(\beta_n J x_n + (1-\beta_n) JTx_n),\\
u_n \in C \text{ such that } & f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,\\
C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\},\\
C_0 &= \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\},\\
Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\},\\
Q_0 &= C,\\
x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0),
\end{aligned} \tag{3.1}$$

for every $n \in N \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ such that $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap \mathrm{EP}(f)} x_0$, where $\Pi_{F(T)\cap \mathrm{EP}(f)}$ is the generalized projection of E onto $F(T) \cap \mathrm{EP}(f)$.

Proof. First, we can easily show that C_n and Q_n are closed and convex for each $n \geq 0$.

Next, we show that $F(T) \cap \mathrm{EP}(f) \subset C_n$ for all $n \geq 0$. Let $u \in F(T) \cap \mathrm{EP}(f)$. Putting $u_n = T_{r_n} y_n$ for all $n \in N$, from Lemma 2.8, we have T_{r_n} relatively nonexpansive. Since T_{r_n} are relatively nonexpansive and T is hemirelatively nonexpansive, we have

$$\begin{aligned}
\phi(u, z_n) &= \phi(u, J^{-1}(\beta_n J x_n + (1-\beta_n) JTx_n))\\
&= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1-\beta_n) JTx_n\rangle + \|\beta_n Jx_n + (1-\beta_n) JTx_n\|^2\\
&\leq \|u\|^2 - 2\beta_n\langle u, Jx_n\rangle - 2(1-\beta_n)\langle u, JTx_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|Tx_n\|^2\\
&= \beta_n \phi(u, x_n) + (1-\beta_n)\phi(u, Tx_n)\\
&\leq \phi(u, x_n),\\
\phi(u, u_n) &= \phi(u, T_{r_n} y_n) \leq \phi(u, y_n) \leq \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, z_n) \leq \phi(u, x_n).
\end{aligned} \tag{3.2}$$

Hence, we have

$$F(T) \cap \mathrm{EP}(f) \subset C_n \quad \forall n \geq 0. \tag{3.3}$$

Next, we show that $F(T) \cap \mathrm{EP}(f) \subset Q_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have

$$F(T) \cap \mathrm{EP}(f) \subset Q_0 = C. \tag{3.4}$$

Suppose that $F(T) \cap \mathrm{EP}(f) \subset Q_n$, by Lemma 2.2, we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1} \rangle \geq 0 \quad \forall z \in C_n \cap Q_n. \tag{3.5}$$

As $F(T) \cap \mathrm{EP}(f) \subset C_n \cap Q_n$, by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T) \cap \mathrm{EP}(f)$. This, together with the definition of Q_{n+1}, implies that $F(T) \cap \mathrm{EP}(f) \subset Q_{n+1}$. So, $\{x_n\}$ is well defined.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $C_n \cap Q_n \subset C_{n-1} \cap Q_{n-1}$ for all $n \geq 1$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \quad \forall n \geq 0. \tag{3.6}$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. In addition, from the definition of Q_n and Lemma 2.2, $x_n = \Pi_{Q_n} x_0$. Therefore, for each $u \in F(T) \cap \mathrm{EP}(f)$, we have

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, x_n) \leq \phi(u, x_0). \tag{3.7}$$

Therefore, $\phi(x_n, x_0)$ and $\{x_n\}$ are bounded. This, together with (3.6), implies that the limit of $\{\phi(x_n, x_0)\}$ exists. From Lemma 2.1, we have, for any positive integer m,

$$\phi(x_{n+m}, x_n) = \phi(x_{n+m}, \Pi_{Q_n} x_0) \leq \phi(x_{n+m}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) = \phi(x_{n+m}, x_0) - \phi(x_n, x_0) \quad \forall n \geq 0. \tag{3.8}$$

Therefore,

$$\lim_{n \to \infty} \phi(x_{n+m}, x_n) = 0. \tag{3.9}$$

From (3.9), we can prove that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $\widehat{x} \in C$ such that $\{x_n\}$ converges strongly to $\widehat{x}$.

Since $x_{n+1} \in C_n$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{3.10}$$

Therefore, we have

$$\phi(x_{n+1}, u_n) \longrightarrow 0. \tag{3.11}$$

From Lemma 2.3, we have

$$\lim_{n\to\infty}\|x_{n+1}-u_n\| = \lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.12}$$

So, we have

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.13}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_n-Ju_n\| = 0. \tag{3.14}$$

Let $r = \sup_{n\in N}\{\|x_n\|, \|Tx_n\|\}$. Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore, from Lemma 2.4, there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$, such that

$$\|\alpha x^* + (1-\alpha)y^*\|^2 \le \alpha\|x^*\|^2 + (1-\alpha)\|y^*\|^2 - \alpha(1-\alpha)g(\|x^*-y^*\|) \tag{3.15}$$

for $x^*, y^* \in B_r$, and $\alpha \in [0,1]$. So, we have that for $u \in F(T)\cap \mathrm{EP}(f)$,

$$\begin{aligned}
\phi(u,z_n) &= \phi\left(u, J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n)\right)\\
&= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1-\beta_n)JTx_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JTx_n\|^2\\
&\le \phi(u,x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|),\\
\phi(u,u_n) &\le \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,z_n)\\
&\le \phi(u,x_n) - (1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|).
\end{aligned} \tag{3.16}$$

Therefore, we have

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JTx_n\|) \le \phi(u,x_n) - \phi(u,u_n). \tag{3.17}$$

Since

$$\phi(u,x_n) - \phi(u,u_n) = \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \le \|x_n-u_n\|(\|x_n\|+\|u_n\|) + 2\|u\|\|Jx_n-Ju_n\|, \tag{3.18}$$

we have

$$\lim_{n\to\infty}\phi(u,x_n) - \phi(u,u_n) = 0. \tag{3.19}$$

From $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty} g(\|Jx_n - JTx_n\|) = 0. \tag{3.20}$$

Therefore, from the property of g, we have

$$\lim_{n\to\infty}\|Jx_n - JTx_n\| = 0. \tag{3.21}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.22}$$

Since T is a closed operator and $x_n \to \hat{x}$, then $\hat{x}$ is a fixed point of T.

On the other hand,

$$\phi(u_n, y_n) = \phi(T_{r_n}y_n, y_n) \le \phi(u, y_n) - \phi(u, T_{r_n}y_n) \le \phi(u, x_n) - \phi(u, T_{r_n}y_n) = \phi(u, x_n) - \phi(u, u_n). \tag{3.23}$$

So, we have from (3.19) that

$$\lim_{n\to\infty}\phi(u_n, y_n) = 0. \tag{3.24}$$

From Lemma 2.3, we have that

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.25}$$

From $x_n \to \hat{x}$ and $\|x_n - u_n\| \to 0$, we have $y_n \to \hat{x}$.

From (3.25), we have

$$\lim_{n\to\infty}\|Ju_n - Jy_n\| = 0. \tag{3.26}$$

From $r_n \ge a$, we have

$$\lim_{n\to\infty}\frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.27}$$

By $u_n = T_{r_n}y_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0 \quad \forall y \in C. \tag{3.28}$$

From (A2), we have that

$$\frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge -f(u_n, y) \ge f(y, u_n) \quad \forall y \in C. \tag{3.29}$$

From (3.27) and (A4), we have

$$f(y,\widehat{x}) \le 0 \quad \forall y \in C. \tag{3.30}$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)\widehat{x}$. We have $f(y_t,\widehat{x}) \le 0$. So, from (A1), we have

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t,\widehat{x}) \le tf(y_t, y). \tag{3.31}$$

Dividing by t, we have

$$f(y_t, y) \ge 0 \quad \forall y \in C. \tag{3.32}$$

Letting $t \to 0$, from (A3), we have

$$f(\widehat{x}, y) \ge 0 \quad \forall y \in C. \tag{3.33}$$

Therefore, $\widehat{x} \in \mathrm{EP}(f)$. Finally, we prove that $\widehat{x} = \Pi_{F(T)\cap \mathrm{EP}(f)}x_0$. From Lemma 2.1, we have

$$\phi\left(\widehat{x}, \Pi_{F(T)\cap \mathrm{EP}(f)}x_0\right) + \phi\left(\Pi_{F(T)\cap \mathrm{EP}(f)}x_0, x_0\right) \le \phi(\widehat{x}, x_0). \tag{3.34}$$

Since $x_{n+1} = \Pi_{C_n\cap Q_n}x_0$ and $\widehat{x} \in F(T)\cap \mathrm{EP}(f) \subset C_n \cap Q_n$, for all $n \ge 0$, we get from Lemma 2.1 that

$$\phi\left(\Pi_{F(T)\cap \mathrm{EP}(f)}x_0, x_{n+1}\right) + \phi(x_{n+1}, x_0) \le \phi\left(\Pi_{F(T)\cap \mathrm{EP}(f)}x_0, x_0\right). \tag{3.35}$$

By the definition of $\phi(x, y)$, it follows that $\phi(\widehat{x}, x_0) \le \phi(\Pi_{F(T)\cap \mathrm{EP}(f)}x_0, x_0)$ and $\phi(\widehat{x}, x_0) \ge \phi(\Pi_{F(T)\cap \mathrm{EP}(f)}x_0, x_0)$, whence $\phi(\widehat{x}, x_0) = \phi(\Pi_{F(T)\cap \mathrm{EP}(f)}x_0, x_0)$. Therefore, it follows from the uniqueness of $\Pi_{F(T)\cap \mathrm{EP}(f)}x_0$ that $\widehat{x} = \Pi_{F(T)\cap \mathrm{EP}(f)}x_0$. This completes the proof. □

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, and let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4). Define a sequence $\{x_n\}$ in C by the following:*

$$\begin{gathered}
x_0 \in C, \ \text{chosen arbitrarily}, \\
u_n \in C \ \text{such that} \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \ge 0, \quad \forall y \in C, \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \le \phi(z, x_n)\}, \\
C_0 = \{z \in C : \phi(z, u_0) \le \phi(z, x_0)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\
Q_0 = C, \\
x_{n+1} = \Pi_{C_n\cap Q_n}(x_0),
\end{gathered} \tag{3.36}$$

for every $n \in N \cup \{0\}$, *where* J *is the duality mapping on* E *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{\mathrm{EP}(f)} x_0$.

Proof. Putting $T = I$ in Theorem 3.1, we obtain Corollary 3.2. □

Corollary 3.3. *Let* E *be a uniformly convex and uniformly smooth real Banach space, let* C *be a nonempty closed convex subset of* E, *and let* $T : C \to C$ *be a closed hemirelatively nonexpansive mapping. Define a sequence* $\{x_n\}$ *in* C *by the following:*

$$\begin{gathered}
x_0 \in C, \text{ chosen arbitrarily}, \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\
z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T x_n), \\
u_n = \Pi_C y_n, \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \Phi(z, u_n) \le \phi(z, x_n)\}, \\
C_0 = \{z \in C : \phi(z, u_0) \le \phi(z, x_0)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\
Q_0 = C, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{3.37}$$

for every $n \in N \cup \{0\}$, *where* J *is the duality mapping on* E, $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $[0, 1]$ *such that* $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x_0$.

Proof. Putting $f(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all n in Theorem 3.1, we obtain Corollary 3.3. □

Corollary 3.4. *Let* E *be a uniformly convex and uniformly smooth real Banach space, let* C *be a nonempty closed convex subset of* E, *let* f *be a bifunction from* $C \times C$ *to* R *satisfying (A1)–(A4), and let* $T : C \to C$ *be a closed relatively nonexpansive mapping such that* $F(T) \cap \mathrm{EP}(f) \neq \emptyset$. *Define a sequence* $\{x_n\}$ *in* C *by the following:*

$$\begin{gathered}
x_0 \in C, \text{ chosen arbitrarily}, \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\
z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T x_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \ge 0, \quad \forall y \in C, \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \le \phi(z, x_n)\}, \\
C_0 = \{z \in C : \phi(z, u_0) \le \phi(z, x_0)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\
Q_0 = C, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{3.38}$$

for every $n \in N \cup \{0\}$, *where* J *is the duality mapping on* E, $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$ *and* $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap \mathrm{EP}(f)} x_0$.

Proof. Since every relatively nonexpansive mapping is a hemirelatively one, Corollary 3.4 is implied by Theorem 3.1. □

Remark 3.5 (see Rockafellar [12]). Let E be a reflexive, strictly convex, and smooth Banach space and let A be a monotone operator from E to E^*. Then, A is maximal if and only if $R(J+rA) = E^*$ for all $r > 0$.

Let E be a reflexive, strictly convex, and smooth Banach space and let A be a maximal monotone operator from E to E^*. Using Remark 3.5 and strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rAx_r$. If $J_r x = x_r$, then we can define a single-valued mapping $J_r : E \to D(A)$ by $J_r = (J+rA)^{-1}J$, and such a J_r is called the resolvent of A. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$ and J_r is relatively nonexpansive mapping (see [2] for more details). Using Theorem 3.1, we can consider the problem of strong convergence concerning maximal monotone operators in a Banach space.

Theorem 3.6. *Let* E *be a uniformly convex and uniformly smooth real Banach space, let* C *be a nonempty closed convex subset of* E, *let* f *be a bifunction from* $C \times C$ *to* R *satisfying (A1)–(A4), and let* J_r *be a resolvent of* A *and a closed mapping such that* $A^{-1}0 \cap \mathrm{EP}(f) \neq \emptyset$, *where* $r > 0$. *Define a sequence* $\{x_n\}$ *in* C *by the following:*

$$\begin{gathered} x_0 \in C, \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JJ_r x_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_r x_n), \\ u_n \in C \text{ such that } f(u_n,y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{z \in C_{n-1}\cap Q_{n-1} : \phi(z,u_n) \leq \phi(z,x_n)\}, \\ C_0 = \{z \in C : \phi(z,u_0) \leq \phi(z,x_0)\}, \\ Q_n = \{z \in C_{n-1}\cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n\cap Q_n}(x_0), \end{gathered} \tag{3.39}$$

for every $n \in N \cup \{0\}$, *where* J *is the duality mapping on* E, $\{\alpha_n\}$ *is a sequences in* $[0,1]$ *such that* $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$ *and* $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$, *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{A^{-1}0\cap \mathrm{EP}(f)} x_0$.

Proof. Since J_r is a closed relatively nonexpansive mapping and $A^{-1}0 = F(J_r)$, from Corollary 3.4, we obtain Theorem 3.6. □

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