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Research Article

Generalized Nonlinear Variational Inclusions Involving (A,η)-Monotone Mappings in Hilbert Spaces

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A new class of generalized nonlinear variational inclusions involving (A,η)-monotone mappings in the framework of Hilbert spaces is introduced and then based on the generalized resolvent operator technique associated with (A,η)-monotonicity, the approximation solvability of solutions using an iterative algorithm is investigated. Since (A,η)-monotonicity generalizes A-monotonicity and H-monotonicity, results obtained in this paper improve and extend many others.

1. Introduction and preliminaries

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, engineering sciences, and so on. There exists a vast literature [1–6] on the approximation solvability of nonlinear variational inequalities as well as nonlinear variational inclusions using projection-type methods, resolvent-operator-type methods, or averaging techniques. In most of the resolvent operator methods, the maximal monotonicity has played a key role, but more recently introduced notions of A-monotonicity [4] and H-monotonicity [1, 2] have not only generalized the maximal monotonicity, but gave a new edge to resolvent operator methods.

Recently, Verma [5] generalized the recently introduced and studied notion of A-monotonicity to the case of (A,η)-monotonicity. Furthermore, these developments added a new dimension to the existing notion of the maximal monotonicity and its applications to several other fields such as convex programming and variational inclusions.

In this paper, we explore the approximation solvability of a generalized class of nonlinear variational inclusion problems based on (A,η)-resolvent operator techniques.

Now, we explore some basic properties derived from the notion of (A,η)-monotonicity.

Let H denote a real Hilbert space with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\eta : H\times H :\to H$ be a single-valued mapping. The mapping η is called τ-Lipschitz continuous if there is a constant $\tau>0$ such that $\|\eta(u,v)\| \le \|\tau y - v\|$ for all $u,v\in H$.

Definition 1.1. Let $\eta : H\times H\to H$ be a single-valued mapping and $M : H\to 2^H$ be a multivalued mapping on H.

(i) The mapping M is said to be (r,η)-strongly monotone if

$$\langle u^* - v^*, \eta(u,v)\rangle \ge r\|u-v\|, \quad \forall (u,u^*),\ (v,v^*)\in \operatorname{Graph}(M), \tag{1.1}$$

(ii) the mapping M is said to be (m,η)-relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, \eta(u,v)\rangle \ge -m\|u-v\|^2, \quad \forall (u,u^*),\ (v,v^*)\in \operatorname{Graph}(M). \tag{1.2}$$

Definition 1.2 [3]. A mapping $M : H\to 2^H$ is said to be maximal (m,η)-relaxed monotone if

(i) M is (m,η)-relaxed monotone,

(ii) for $(u,u^*)\in H\times H$ and $\langle u^*-v^*,\eta(u,v)\rangle \ge -m\|u-v\|^2$, for all $(v,v^*)\in \operatorname{Graph}(M)$, and $u^*\in M(u)$.

Definition 1.3 [3]. Let $A : H\to H$ and $\eta : H\times H\to H$ be two single-valued mappings. The mapping $M : H\to 2^H$ is said to be (A,η)-monotone if

(i) M is (m,η)-relaxed monotone,

(ii) $R(A+\rho M) = H$ for $\rho>0$.

Note that, alternatively, the mapping $M : H\to 2^H$ is said to be (A,η)-monotone if

(i) M is (m,η)-relaxed monotone,

(ii) $A+\rho M$ is η-pseudomonotone for $\rho>0$.

Remark 1.4. The (A,η)-monotonicity generalizes the notion of the A-monotonicity introduced by Verma [4] and the H-monotonicity introduced by Fang and Huang [1, 2].

Definition 1.5. Let $A : H\to H$ be an (r,η)-strong monotone mapping and $M : H\to H$ be an (A,η)-monotone mapping. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H\to H$ is defined by $J^{A,\eta}_{M,\rho}(u) = (A+\rho M)^{-1}(u)$ for all $u\in H$.

Definition 1.6. The mapping $T : H\times H$ is said to be relaxed (α,β)-cocoercive with respect to A in the first argument if there exist two positive constants α, β such that

$$\langle T(x,u)-T(y,u), Ax-Ay\rangle \ge (-\alpha)\|T(x,u)-T(y,u)\|^2 + \beta\|x-y\|^2, \quad \forall x,y,u\in H. \tag{1.3}$$

Proposition 1.7 [5]. *Let $\eta : H\times \to H$ be a single-valued mapping, $A : H\to H$ be an (r,η)-strongly monotone mapping and $M : H\to 2^H$ an (A,η)-monotone mapping. Then the mapping $(A+\rho M)^{-1}$ is single-valued.*

2. Results on algorithmic convergence analysis

Let $N : H \times H \to H$, $g : H \to H$, $\eta : H \times H \to H$ be three nonlinear mappings and $M : H \to 2^H$ be an (A,η)-monotone mapping. Then the nonlinear variational inclusion (NVI) problem: determine an element $u \in H$ for a given element $f \in H$ such that

$$f \in N(u,u) + M[g(u)]. \tag{2.1}$$

A special cases of the NVI (2.1) problem is to find an element $u \in H$ such that

$$0 \in N(u,u) + M[g(u)]. \tag{2.2}$$

If $g = I$ in (2.1), then NVI (2.1) reduces to the following nonlinear variational inclusion problem: determine an element $u \in H$ for a given element $f \in H$ such that

$$f \in N(u,u) + M(u). \tag{2.3}$$

The solvability of the NVI problem (2.1) depends on the equivalence between (2.1) and the problem of finding the fixed point of the associated generalized resolvent operator. Note that, if M is (A,η)-monotone, then the corresponding generalized resolvent operator $J_{M,\rho}^{A,\eta}$ is defined by $J_{M,\rho}^{A,\eta}(u) = (A + \rho M)^{-1}(u)$ for all $u \in H$, where $\rho > 0$ and A is an (r,η)-strongly monotone mapping.

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n, \quad \forall n \geq n_0, \tag{2.4}$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}\lambda_n = \infty$, $b_n = \circ(\lambda_n)$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. *Let H be a real Hilbert space and $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping. Let $A : H \to H$ be a (r,η)-strongly monotone and $M : H \to 2^H$ be (A,η)-monotone. Then the generalized resolvent operator $J_{M,\rho}^{A,\eta} : H \to H$ is $\tau/(r - \rho m)$-Lipschitz continuous, that is,*

$$\left\|J_{M,\rho}^{A,\eta}(x) - J_{M,\rho}^{A,\eta}(y)\right\| \leq \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in H. \tag{2.5}$$

Lemma 2.3. *Let H be a real Hilbert space, $A : H \to H$ be (r,η)-strongly monotone and $M : H \to 2^H$ be (A,η)-monotone. Let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping. Then the following statements are mutually equivalent:*

(i) *An element $u \in H$ is a solution to the NVI (2.1).*

(ii) $g(u) = J_{M,\rho}^{A,\eta}[Ag(u) - \rho N(u,u) + \rho f]$.

From Lemma 2.3, we have the following:

$$u = u - g(u) + J_{M,\rho}^{A,\eta}(Ag(u) - \rho N(u,u) + \rho f), \tag{2.6}$$

where u is a solution to the NVI problem (2.1). Let S be a nonexpansive mapping on H. If u is also a fixed point of S, we have

$$u = S\{u - g(u) + J_{M,\rho}^{A,\eta}(Ag(u) - \rho N(u,u) + \rho f)\}. \tag{2.7}$$

Next, we consider the following algorithms and denote the solution to the NVI problem (2.1) by Ω_1, the NVI problem (2.3) by Ω_2, respectively.

ALGORITHM 2.4. *For any $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative processes*

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n S\{u_n - g(u_n) + J_{M,\rho}^{A,\eta}(Ag(u_n) - \rho N(u_n,u_n) + \rho f)\}, \tag{2.8}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and S is a nonexpansive mapping on H.

If $S = g = I$ and $\{\alpha_n\} = 1$ in Algorithm 2.4, then we have the following algorithm.

ALGORITHM 2.5. *For any $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative processes*

$$u_{n+1} = J_{M,\rho}^{A,\eta}(Au_n - \rho N(u_n,u_n) + \rho f). \tag{2.9}$$

We remark that Algorithm 2.5 gives the approximate solution to the NVI problem (2.3).

Now, we are in the position to prove our main results.

THEOREM 2.6. *Let H be a real Hilbert space, $A : H \times H$ be (r,η)-strongly monotone and s-Lipschitz continuous and $M : H \to 2^H$ be (A,η)-monotone. Let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping and $N : H \times H \to H$ be relaxed (α_1,β_1)-cocoercive (with respect to Ag) and μ_1-Lipschitz coninuous in the first variable and N be ν_1-Lipschitz continuous in the second variable. Let $g : H \to H$ be relaxed (α_2,β_2)-cocoercive and μ_2-Lipschitz continuous on H, $S : H \to H$ be a nonexpansive mapping and $\{u_n\}$ be a sequence generated by Algorithm 2.4. Suppose the following conditions are satisfied:*

(i) *$\alpha_n \subset (0,1)$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;*

(ii) *$\tau(\theta_1 + \rho\nu_1) < (r - \rho m)(1 - \theta_2)$, where $\theta_1 = \sqrt{\mu_2^2 s^2 - 2\rho\beta_1 + 2\rho\alpha_1\mu_1^2 + \rho^2\mu_1^2}$ and $\theta_2 = \sqrt{1 + 2\mu_2^2\alpha_2 - 2\beta_2 + \mu_2^2}$.*

Then the sequence $\{u_n\}$ converges strongly to $u^* \in F(S) \cap \Omega_1$.

Proof. Let $u^* \in C$ be the common element of $F(S) \cap \Omega_1$. Then we have

$$u^* = (1-\alpha_n)u^* + \alpha_n S\{u^* - g(u^*) + J_{M,\rho}^{A,\eta}(Ag(u^*) - \rho N(u^*,u^*) + \rho f)\}. \tag{2.10}$$

It follows that

$$\begin{aligned} \|u_{n+1} - u^*\| \le{}& (1-\alpha_n)\|u_n - u^*\| + \alpha_n\|u_n - u^* - [g(u_n) - g(u^*)]\| \\ &+ \frac{\tau\alpha_n}{r - \rho m}\|Ag(u_n) - Ag(u^*) - \rho[N(u_n,u_n) - N(u^*,u_n)] \\ &\quad - \rho[N(u^*,u_n) - N(u^*,u^*)]\|. \end{aligned} \tag{2.11}$$

It follows from relaxed (α_1,β_1)-cocoercive monotonicity and μ_1-Lipschitz continuity of N in the first variable, the s-Lipschitz continuity of A and the μ_2-Lipschitz continuity of g that

$$\begin{aligned}&\|Ag(u_n)-Ag(u^*)-\rho(N(u_n,u_n)-N(u^*,u_n))\|^2\\&\quad=\|Ag(u_n)-Ag(u^*)\|^2-2\rho\langle N(u_n,u_n)-N(u^*,u_n),Ag(u_n)-Ag(u^*)\rangle\\&\qquad+\rho^2\|N(u_n,u_n)-N(u^*,u_n)\|^2\le\theta_1^2\|u_n-u^*\|^2,\end{aligned}\tag{2.12}$$

where $\theta_1=\sqrt{\mu_2^2s^2-2\rho\beta_1+2\rho\alpha_1\mu_1^2+\rho^2\mu_1^2}$. Observe that the ν_1-Lipschitz continuity of N in the second argument yields that

$$\|N(u^*,u_n)-N(u^*,u)\|\le\nu_1\|u_n-u^*\|.\tag{2.13}$$

Now, we consider the second term of the right side of (2.11). It follows from the relaxed (α_2,β_2)-cocoercive monotonicity and μ_2-Lipschitz continuity of g that

$$\begin{aligned}&\|u_n-u^*-g(u_n)-g(u^*)\|^2\\&\quad=\|u_n-u^*\|^2-2\langle g(u_n)-g(u^*),u_n-u^*\rangle+\|g(u_n)-g(u^*)\|^2\\&\quad\le\|u_n-u^*\|^2-2\Big[-\alpha_2\|g(u_n)-g(u^*)\|^2+\beta_2\|u_n-u^*\|^2\Big]+\|g(u_n)-g(u^*)\|^2\\&\quad\le\theta_2^2\|u_n-u^*\|^2,\end{aligned}\tag{2.14}$$

where $\theta_2=\sqrt{1+2\mu_2^2\alpha_2-2\beta_2+\mu_2^2}$. Substituting (2.12), (2.13), and (2.14) into (2.11), we arrive at

$$\begin{aligned}&\|u_{n+1}-u\|\\&\quad\le(1-\alpha_n)\|u_n-u^*\|+\alpha_n\theta_2\|u_n-u^*\|+\frac{\tau\alpha_n}{r-\rho m}\theta_1\|u_n-u^*\|+\frac{\tau\alpha_n\rho\nu_1}{r-\rho m}\|u_n-u^*\|\\&\quad=\left[1-\alpha_n\left(1-\theta_2-\frac{\tau}{r-\rho m}\theta_1-\frac{\tau\rho\nu_1}{r-\rho m}\right)\right]\|u_n-u^*\|.\end{aligned}\tag{2.15}$$

Using the conditions (i)-(ii) and applying Lemma 2.1 to (2.15), we can obtain the desired conclusion. This completes the proof. □

Remark 2.7. Theorem 2.6 mainly improves the results of Verma [5, 6].

Corollary 2.8. *Let H be a real Hilbert space, $A:H\times H$ be (r,η)-strongly monotone, and s-Lipschitz continuous and $M:H\to 2^H$ be (A,η)-monotone. Let $\eta:H\times H\to H$ be a τ-Lipschitz continuous nonlinear mapping and $N:H\times H\to H$ be relaxed (α_1,β_1)-cocoercive (with respect to A) and μ_1-Lipschitz coninuous in the first variable and N be ν_1-Lipschitz continuous in the second variable. Let $\{u_n\}$ be a sequence generated by Algorithm 2.5. Suppose the following condition is satisfied: $\tau(\theta_1+\rho\nu_1)<r-\rho m$, where $\theta_1=\sqrt{\mu_2^2s^2-2\rho\beta_1+2\rho\alpha_1\mu_1^2+\rho^2\mu_1^2}$, then the sequence $\{u_n\}$ converges strongly to $u^*\in\Omega_2$.*

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